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## DIFFERENTIAL FLATNESS BY PURE PROLONGATION: NECESSARY AND SUFFICIENT CONDITIONS

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Dedicated to Professor Ezra Zeheb on the occasion of his 85th birthday

**Abstract.** In this article, we introduce the notion of *differential flatness by pure prolongation*: loosely speaking, a system admits this property if, and only if, there exists a pure prolongation of finite order such that the prolonged system is feedback linearizable. We obtain Lie-algebraic necessary and sufficient conditions for a general nonlinear multi-input system to satisfy this property. These conditions are comprised of the involutivity and relative invariance of a pair of filtrations of distributions of vector fields. An algorithm computing the minimal prolongation lengths of the input channels that achieve the system linearization, yielding the associated flat outputs, is deduced. Examples that show the efficiency and computational tractability of the approach are then presented.

**Keywords.** Differential flatness; Lie-Bäcklund isomorphism; Lie brackets; distributions of vector fields; Prolongation of vector fields; Dynamic feedback linearization.

**2020 Mathematics Subject Classification.** 34H05, 93C10, 93C35.

### 1. INTRODUCTION

The notions of *static feedback linearization* [17, 14] and *dynamic feedback linearization* of a nonlinear system, whose preliminary results were reported in [6, 7, 34, 1] (see also [35, 36, 21, 11, 2, 12]), were at the origin of a so far uninterrupted thread of studies in nonlinear system theory. In particular, during the last three decades, they gave birth to the concept of *differential flatness*, that plays a prominent part in motion planning and trajectory tracking problems and their applications (see [26, 9, 10] and [23] for a thorough presentation).

Sufficient or necessary conditions in special cases, as well as a general differential flatness characterization, although without an upper bound on the number of iterations of the corresponding algorithm, have been obtained (see e.g. [23, 24] for a historical review). Nevertheless, the question of obtaining *computationally tractable* general necessary and sufficient conditions for dynamic feedback linearization as well as for differential flatness, remains open. In this

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paper, we restrict our study to the class of *differentially flat systems by pure prolongation*, i.e., roughly speaking, the class of  $n$ -dimensional systems with  $m$  inputs

$$\dot{x} = f(x, u_1, \dots, u_m)$$

for which there exists a multi-index  $\mathbf{j} \triangleq (j_1, \dots, j_m)$  such that the prolonged system

$$\begin{aligned} \dot{x} &= f(x, u_1, \dots, u_m), \\ u_i^{(j_i+1)} &= v_i, \quad i = 1, \dots, m, \end{aligned} \tag{1.1}$$

of dimension  $n + \sum_{i=1}^m j_i$ , where we have denoted by  $u_i^{(j_i)}$  the  $j_i$ -th order time derivative of  $u_i$ , is locally static feedback linearizable. This special class of differentially flat systems, initially introduced in the framework of dynamic feedback linearization [6, 7, 34, 1], has also been studied in [21, 35, 36], where the corresponding transformation is called *pure dynamic extension*, in [21] under the name of *linearization by a restricted class of dynamic feedback*, in [11, 12] and in [2], under the name of *linearization by prolongation*. The aim of all these works is to find upper bounds of the prolongation indices  $j_1, \dots, j_m$ , if they exist, and in [2], a constructive algorithm is obtained, yielding a sufficient condition.

To name such dynamic feedback transformations, we have preferred here the expression *pure prolongations* initially introduced in [7], in reference to É. Cartan [5] (see also [34]).

The necessary and sufficient conditions presented in this paper, tackled via the here introduced concept of *equivalence by pure prolongation*, are original to the author's knowledge. Moreover, they are easily checkable by an algorithm that only involves Lie bracket and linear algebra operations, yielding the minimal pure prolongations if they exist.

Our main contributions are:

1. The introduction of the **equivalence relation of systems by pure prolongation**, strictly coarser than the well-known equivalence by diffeomorphism and feedback (see e.g. [17, 15, 16, 30, 22]) and strictly finer than the Lie-Bäcklund equivalence (see e.g. [10, 23]). For this equivalence relation, the set of flat systems by pure prolongation is identified with the equivalence class of 0 (modulo the trivial vector field)(see definition 4.3);
2. **Proposition 3.9**, extending and specifying results implicitly present or in embryonic state in various forms in [7, 11, 2, 12], where we prove that, whatever the prolongation  $\mathbf{j} \triangleq (j_1, \dots, j_m)$ , the filtration of prolonged distributions, built on the successive Lie brackets of the prolonged drift with the prolonged control vector fields, is **decomposable into the direct sum of two filtrations of distributions**. The first one, denoted by  $\{\Delta_k^{(\mathbf{j})}\}_{k \geq 0}$ , is, at each point of the prolonged manifold, generated by vector fields belonging to the original tangent bundle, of dimension  $n + m$  (in a suitably defined vertical bundle, see (3.13)); The second one, denoted by  $\{\Gamma_k^{(\mathbf{j})}\}_{k \geq 0}$ , is generated by the sequence of prolonged control vector fields of decreasing orders up to 1, i.e.  $\Gamma_k^{(\mathbf{j})} = \bigoplus_{i=1, \dots, m} \left\{ \frac{\partial}{\partial u_i^{(j_i-r)}} \mid r = 0, \dots, \min(j_i - 1, k) \right\}$  (see (3.12));
3. **theorem 4.8** giving the **necessary and sufficient conditions**: (i)  $\Delta_k^{(\mathbf{j})}$  must be involutive with locally constant dimension and (ii) invariant by  $\Gamma_k^{(\mathbf{j})}$  for all  $k$ , and (iii)  $\dim \Delta_k^{(\mathbf{j})}$  must be equal to  $n + m$  for all  $k$  large enough (strong controllability);
4. **Formula (4.11) and theorem 4.14** giving the prolongation lengths, if they exist, or otherwise a criterion of non flatness by pure prolongation;

5. **Algorithm 1** whose input comprises the system vector fields and whose output is either the list of minimal prolongation lengths or the claim that the system is not flat by pure prolongation, deduced in a finite number of steps.

The paper is organized as follows: In section 2, we present the necessary recalls of basic results on differential flatness and feedback linearization. Then we introduce and study the purely prolonged distributions and the associated vertical ones in section 3. The equivalence by pure prolongation and the definition of flatness by pure prolongation, followed by its necessary and sufficient conditions and by the pure prolongation algorithm are then presented in section 4. A series of five examples, four with two inputs (sections 5.1, 5.2, 5.3 and 5.4) and one with three inputs (section 5.5) then illustrate our results in section 5. Moreover, the pendulum example (section 5.4), is proven to be non flat by pure prolongation, though known to be differentially flat [10, 23], thus proving that the set of flat systems by pure prolongation is strictly contained in the set of differentially flat ones. The paper ends with concluding remarks and an appendix establishing a comparison formula between prolonged and non prolonged Lie brackets.

## 2. PRELIMINARIES: RECALLS ON DIFFERENTIAL FLATNESS AND FEEDBACK LINEARIZATION

Consider a non-linear system over a smooth  $n$ -dimensional manifold  $X$  given by

$$\dot{x} = f(x, u) \quad (2.1)$$

where  $x$  is the  $n$ -dimensional state vector,  $u \in \mathbb{R}^m$  the input or control vector, with  $m \leq n$ , and  $f$  a  $C^\infty$  vector field in the tangent bundle  $TX$  of  $X$  for each  $u \in \mathbb{R}^m$  and whose dependence on  $u$  is of class  $C^\infty$ .

Before going further, we need the following notations, valid all along this paper:

- Boldface letters  $\mathbf{j}, \mathbf{k}, \dots$ , are systematically used to denote multi-integers, *i.e.*

$$\mathbf{j} \triangleq (j_1, j_2, \dots, j_m), \quad \mathbf{k} \triangleq (k_1, k_2, \dots, k_m), \quad \dots$$

- We denote the minimum of two arbitrary integers  $k$  and  $l$  by  $k \vee l \triangleq \min(k, l)$ , and their maximum by  $k \wedge l \triangleq \max(k, l)$ .

Also, for every  $\mathbf{j} \triangleq (j_1, \dots, j_m) \in \mathbb{N}^m$  and  $\mathbf{k} \triangleq (k_1, \dots, k_m) \in \mathbb{N}^m$ , we use the *componentwise minimum* notation

$$\mathbf{j} \vee \mathbf{k} \triangleq (j_1 \vee k_1, \dots, j_m \vee k_m) = (\min(j_1, k_1), \dots, \min(j_m, k_m)). \quad (2.2)$$

Accordingly, the *componentwise maximum* is denoted by

$$\mathbf{j} \wedge \mathbf{k} \triangleq (j_1 \wedge k_1, \dots, j_m \wedge k_m) = (\max(j_1, k_1), \dots, \max(j_m, k_m)). \quad (2.3)$$

If  $k_1 = \dots = k_m = k \in \mathbb{N}$ , we also use the notations  $\mathbf{j} \wedge k$  and  $\mathbf{j} \vee k$ , with  $k$  in regular math typeface.

- Overlined symbols will denote a collection of successive time derivatives of a time-dependent function as follows. Given a multi-integer  $\mathbf{k} = (k_1, \dots, k_m) \in \mathbb{N}^m$  and a locally defined  $C^\infty$  function  $t \mapsto \xi(t) \in \mathbb{R}^m$ ,
  - $\overline{\xi}^{(\mathbf{k})}$  denotes the vector  $\left( \xi_1, \dot{\xi}_1, \dots, \xi_1^{(k_1)}, \dots, \xi_m, \dot{\xi}_m, \dots, \xi_m^{(k_m)} \right)$  of dimension  $m + |\mathbf{k}|$ , with  $|\mathbf{k}| \triangleq \sum_{i=1}^m k_i$  and  $\xi_i^{(j)} \triangleq \frac{d^j \xi_i}{dt^j}$ ,  $j = 1, \dots, k_i$ ,  $i = 1, \dots, m$ ;

–  $\bar{\xi}$  denotes the infinite sequence

$$\bar{\xi} \triangleq (\xi, \xi, \xi, \dots) \triangleq \left( \xi_i^{(k)} ; i = 1, \dots, m ; k \geq 0 \right) \in \mathbb{R}_\infty^m,$$

where  $\mathbb{R}_\infty^m \triangleq \mathbb{R}^m \times \mathbb{R}^m \times \dots$  is the product of an infinite number of copies of  $\mathbb{R}^m$ .

**2.1. Recalls on Lie-Bäcklund Equivalence.** For a detailed presentation of the topics of this section, the reader may refer e.g. to [10, 31, 23, 24, 32].

We stress that all the results of this paper are only valid in suitably chosen local charts. For global results in particular cases the reader may refer to [18, 19].

**Definition 2.1.** The infinite order jet space prolongation of system (2.1) is given by the pair  $(X \times \mathbb{R}_\infty^m, C_f)$ , where  $X \times \mathbb{R}_\infty^m$  is the product of  $X$  with an infinite number of copies of  $\mathbb{R}^m$ , with coordinates  $(x, \bar{u})$ , endowed with the Cartan field

$$C_f = f(x, u) \frac{\partial}{\partial x} + \sum_{j \geq 0} \sum_{i=1}^m u_i^{(j+1)} \frac{\partial}{\partial u_i^{(j)}}, \quad (2.4)$$

defined on  $\text{TX} \times \text{T}\mathbb{R}_\infty^m$ , the tangent bundle of  $X \times \mathbb{R}_\infty^m$ .

**Definition 2.2** (Lie-Bäcklund equivalence). Consider two systems:

$$\dot{x} = g(x, u) \quad \text{and} \quad \dot{y} = \gamma(y, v) \quad (2.5)$$

and their respective prolongations  $(X \times \mathbb{R}_\infty^m, C_g)$ , with coordinates  $(x, \bar{u})$  and Cartan field

$$C_g = g(x, u) \frac{\partial}{\partial x} + \sum_{j \geq 0} \sum_{i=1}^m u_i^{(j+1)} \frac{\partial}{\partial u_i^{(j)}}, \quad (2.6)$$

and  $(Y \times \mathbb{R}_\infty^\mu, C_\gamma)$ , with coordinates  $(y, \bar{v})$ , and Cartan field

$$C_\gamma = \gamma(y, v) \frac{\partial}{\partial y} + \sum_{j \geq 0} \sum_{i=1}^\mu v_i^{(j+1)} \frac{\partial}{\partial v_i^{(j)}}. \quad (2.7)$$

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We say that they are *Lie-Bäcklund equivalent* at a pair of points  $(x_0, \bar{u}_0)$  and  $(y_0, \bar{v}_0)$  if, and only if, there exists neighborhoods  $\mathcal{N}_{x_0, \bar{u}_0} \subset X \times \mathbb{R}_\infty^m$  and  $\mathcal{N}_{y_0, \bar{v}_0} \subset Y \times \mathbb{R}_\infty^\mu$  and a  $C^\infty$  isomorphism<sup>1</sup>  $\Phi : \mathcal{N}_{y_0, \bar{v}_0} \rightarrow \mathcal{N}_{x_0, \bar{u}_0}$  satisfying  $\Phi(y_0, \bar{v}_0) = (x_0, \bar{u}_0)$ , with  $C^\infty$  inverse  $\Psi$ , such that the respective Cartan fields are  $\Phi$  and  $\Psi$  related, i.e.  $\Phi_* C_\gamma = C_g$  in  $\mathcal{N}_{x_0, \bar{u}_0}$  and  $\Psi_* C_g = C_\gamma$  in  $\mathcal{N}_{y_0, \bar{v}_0}$ .

In other words, the two systems are *Lie-Bäcklund equivalent* at the points  $(x_0, \bar{u}_0)$  and  $(y_0, \bar{v}_0)$  if there exist neighborhoods of these points where every integral curve of the first system is mapped to an integral curve of the second one and conversely, with the same time parameterization. Clearly, this relation is an equivalence relation.

We recall, without proof, a most important result from [26] (see also [9, 10, 23]) giving an interpretation of the Lie-Bäcklund equivalence in terms of diffeomorphism in finite dimension and endogenous dynamic feedback, that will be useful later on.

<sup>1</sup>Recall that a continuous function and, *a fortiori*, differentiable, resp.  $C^\infty$ , depends, by definition of the source and target product topologies, on a finite number of components of its variables, namely  $\Phi$  (resp.  $\Psi$ ) depends on a finite number of components of  $(y, \bar{v})$  (resp.  $(x, \bar{u})$ ) (see e.g. [20, 38, 23]).

**Theorem 2.3** (Martin [26]). *If the two systems (2.5) are Lie-Bäcklund equivalent at a given pair of points, then (i) and (ii) must be satisfied:*

(i)  $m = \mu$ , i.e. they must have the same number of independent inputs;

(ii) there exist

– an endogenous dynamic feedback<sup>2</sup>

$$u = \alpha(x, z, w), \quad \dot{z} = \beta(x, z, w), \quad (2.8)$$

where  $z$  belongs to  $Z$ , a finite dimensional smooth manifold,

– a multi-integer<sup>3</sup>  $\mathbf{r} \triangleq (r_1, \dots, r_m)$ ,

– and a local diffeomorphism  $\chi : Y \times \mathbb{R}^{|\mathbf{r}|} \rightarrow X \times Z$ ,

all defined in a neighborhood of the considered points, such that the extended system

$$\dot{y} = \gamma(y, v), \quad v^{(\mathbf{r})} = w \quad (2.9)$$

and the closed-loop one

$$\dot{x} = g(x, \alpha(x, z, w)), \quad \dot{z} = \beta(x, z, w) \quad (2.10)$$

are  $\chi$ -related for all  $w \in \mathbb{R}^m$ , i.e.

$$(x, z) = \chi(y, v, \dot{v}, \dots, v^{(\mathbf{r}-1)}), \quad (y, v, \dot{v}, \dots, v^{(\mathbf{r}-1)}) = \chi^{-1}(x, z) \quad (2.11)$$

and

$$\hat{g} = \chi_* \hat{\gamma}, \quad \hat{\gamma} = \chi_*^{-1} \hat{g} \quad (2.12)$$

where we have denoted

$$\begin{aligned} \hat{g}(x, z, w) &\triangleq g(x, \alpha(x, z, w)) \frac{\partial}{\partial x} + \beta(x, z, w) \frac{\partial}{\partial z} \\ \hat{\gamma}(y, v, \dot{v}, \dots, v^{(\mathbf{r}-1)}, w) &\triangleq \gamma(y, v) \frac{\partial}{\partial y} + \sum_{i=1}^m \sum_{j=0}^{r_i-1} v_i^{(j+1)} \frac{\partial}{\partial v_i^{(j)}} + w_i \frac{\partial}{\partial v_i^{(r_i)}}. \end{aligned}$$

## 2.2. Recalls on Differential Flatness.

**Definition 2.4.** We say that system (2.1) is *differentially flat* (or, more shortly, *flat*) at the pair of points  $(x_0, \bar{u}_0)$  and  $\bar{y}_0 \in \mathbb{R}_\infty^m$  if and only if, it is *Lie-Bäcklund equivalent to the trivial system*  $(\mathbb{R}_\infty^m, \tau)$  where  $\tau$  is the trivial Cartan field

$$\tau \triangleq \sum_{j \geq 0} \sum_{i=1}^m y_i^{(j+1)} \frac{\partial}{\partial y_i^{(j)}} \quad (2.13)$$

at the considered points.

Otherwise stated, the locally defined flat output  $y = \Psi(x, \bar{u})$  is such that  $(x, \bar{u}) = \Phi(\bar{y}) \triangleq (\Phi_{-1}(\bar{y}), \Phi_0(\bar{y}), \Phi_1(\bar{y}), \dots)$ , with<sup>4</sup>  $\frac{d}{dt} \Phi_{-1}(\bar{y}) \equiv f(\Phi_{-1}(\bar{y}), \Phi_0(\bar{y}))$  for all sufficiently differentiable function  $y : t \in \mathbb{R} \mapsto y(t) \in \mathbb{R}^m$ .

<sup>2</sup>A dynamic feedback is said *endogenous* if, and only if, the closed-loop system and the original one are Lie-Bäcklund equivalent, i.e. if, and only if, the extended state  $z$  can be locally expressed as a smooth function of  $x$ ,  $u$  and a finite number of time derivatives of  $u$  (see [26, 9, 10, 23]).

<sup>3</sup>Recall that we denote by  $v^{(\mathbf{r})} \triangleq (v_1^{(r_1)}, \dots, v_m^{(r_m)}) \triangleq \left( \frac{d^{r_1} v_1}{dt^{r_1}}, \dots, \frac{d^{r_m} v_m}{dt^{r_m}} \right)$ .

<sup>4</sup>The first component of  $\Phi$ , corresponding to the  $x$  component, is denoted by  $\Phi_{-1}$  so that the component  $\Phi_i$  effectively corresponds to the  $i$ th time derivative of  $u$ .

This definition immediately implies that a system is flat if, and only if, there exists a generalized output  $y = \Psi(x, \bar{u})$  of dimension  $m$ , depending at most on a finite number of derivatives of  $u$ , with independent derivatives of all orders, such that  $x$  and  $\bar{u}$  can be expressed in terms of  $y$  and a finite number of its successive derivatives, *i.e.*  $(x, \bar{u}) = \Phi(\bar{y})$ , and such that the system equation  $\frac{d}{dt}\Phi_{-1}(\bar{y}) = f \circ \Phi(\bar{y})$  is identically satisfied for all sufficiently differentiable  $y : \mathbb{R} \rightarrow \mathbb{R}^m$ .

For a flat system, with the notations of theorem 2.3, the vector field  $\gamma$ , or  $\hat{\gamma}$  indifferently, corresponds to the linear system in Brunovský canonical form

$$y_i^{(r_i+1)} = w_i, \quad i = 1, \dots, m, \quad (2.14)$$

and  $C_\gamma$ , defined by (2.7), is the trivial Cartan field  $C_\gamma = \tau$ , with  $\tau$  given by (2.13).

**Remark 2.5.** In view of Definition 2.2, since the trivial Cartan field (2.13) is the infinite jet prolongation of  $\dot{y}_i = u_i$ ,  $i = 1, \dots, m$ , system (2.1) is flat if, and only if, it is Lie-Bäcklund equivalent to  $m$  simple integrators in parallel. We may also remark that, since the Cartan field is equal to 0 (modulo  $\tau$ ), the class of flat systems may be identified with the Lie-Bäcklund equivalence class of 0 (modulo  $\tau$ ).

Theorem 2.3 reads:

**Corollary 2.6.** *If system (2.1) is flat at a given point, there exists an endogenous dynamic feedback of the form (2.8), a multi-integer  $\mathbf{r} \triangleq (r_1, \dots, r_m)$  and a finite dimensional local diffeomorphism  $\chi$  such that systems (2.10), with  $f$  in place of  $g$ , and (2.14) are  $\chi$ -related for all  $w \in \mathbb{R}^m$ .*

Consequently, a flat system can be transformed, by diffeomorphism and feedback of a suitably extended space, in a linear controllable system, which motivates the recalls of the next section.

**2.3. Recalls on Feedback Linearization.** Feedback linearizable systems [17, 15] (see also [16, 30, 25, 22]) constitute a subclass of differentially flat systems. They constitute the equivalence class of linear controllable systems with respect to the following finer equivalence relation, called *equivalence by diffeomorphism and feedback*.

**Definition 2.7.** The two systems given by (2.5) are said equivalent by diffeomorphism and feedback if, and only if, there exists

- a local diffeomorphism  $\varphi$  from a neighborhood of an equilibrium point of  $X$  (which may be chosen, without loss of generality, as the origin  $0 \in X$ ) to a suitable neighborhood of  $0 \in Y$ ,
- and a *static* feedback  $v = \alpha(x, u)$ ,  $\alpha$  being invertible with respect to  $u$  for all  $x$  in the above mentioned neighborhood of the origin, *i.e.*  $\text{rank} \left( \frac{\partial \alpha}{\partial u} \right) (x, u) = m$  for all  $x$  and  $u$  as above,

such that  $\varphi_*^{-1} f(x, u) = \gamma \circ (\varphi \times \alpha)(x, u) = \gamma(y, v)$ ,

Indeed, this equivalence implies that  $n = \dim X = \dim Y$  and that both  $u$  and  $v$  are  $m$ -dimensional.

**Definition 2.8** (Feedback Linearizability). System (2.1) is said *static feedback linearizable* or, shortly, *feedback linearizable* if the context allows, if, and only if, it is equivalent by diffeomorphism and feedback to a linear system in Brunovský controllability canonical form

$$y_i^{(r_i)} = v_i, \quad i = 1, \dots, m, \quad (2.15)$$

where the multi-integer  $\mathbf{r} \triangleq (r_1, \dots, r_m)$ , whose components  $r_i$  are called the *controllability indices* of system (2.15), satisfies  $|\mathbf{r}| \triangleq \sum_{i=1}^m r_i = n = \dim X$ .

Indeed, since the  $n$ -dimensional vector

$$\bar{y}^{(\mathbf{r}-\mathbf{1})} \triangleq (y_1, \dots, y_1^{(r_1-1)}, \dots, y_m, \dots, y_m^{(r_m-1)})$$

and  $x$  are diffeomorphic (again we have noted  $\mathbf{r} - \mathbf{1} \triangleq (r_1 - 1, \dots, r_m - 1)$ ), it is immediate to verify that  $(y_1, \dots, y_m)$  is a flat output and that every feedback linearizable system is differentially flat.

These systems have been first characterized by [17, 14, 15] for control-affine systems, i.e. systems given by  $f(x, u) = f_0(x) + \sum_{i=1}^m u_i f_i(x)$ . More generally, it can be easily proven that systems of the form (2.1) are feedback linearizable if, and only if, the first-order control-affine prolongation

$$\begin{aligned} \dot{x} &= f(x, u), \\ \dot{u}_i &= u_i^{(1)}, \quad i = 1, \dots, m \end{aligned} \tag{2.16}$$

with state  $\bar{x}^{(0)} \triangleq (x, u) = (x_1, \dots, x_n, u_1, \dots, u_m) \in X^{(0)} \triangleq X \times \mathbb{R}^m$  and control vector  $u^{(1)} \triangleq (u_1^{(1)}, \dots, u_m^{(1)}) \in \mathbb{R}^m$ , is feedback linearizable (see e.g. [6, 33, 35, 22]).

Indeed, in the local coordinates<sup>5</sup>  $\bar{x}^{(0)}$  of  $X^{(0)}$ , denoting the associated vector fields by

$$g_0^{(0)}(\bar{x}^{(0)}) \triangleq \sum_{i=1}^n f_i(x, u) \frac{\partial}{\partial x_i}, \quad g_i^{(0)}(\bar{x}^{(0)}) \triangleq \frac{\partial}{\partial u_i}, \quad i = 1, \dots, m, \tag{2.17}$$

defined on the tangent bundle  $\mathrm{TX}^{(0)} = \mathrm{TX} \times \mathrm{T}\mathbb{R}^m$ , system (2.16) reads

$$\dot{\bar{x}}^{(0)} = g_0^{(0)}(\bar{x}^{(0)}) + \sum_{i=1}^m u_i^{(1)} g_i^{(0)}(\bar{x}^{(0)}) \tag{2.18}$$

with the usual abuse of notations identifying a vector field expressed in local coordinates with its associated (Lie derivative) first order partial differential operator.

Until now, for simplicity's sake, a system (2.1) will always be considered in the form (2.18), even if the vector-field  $f$  is already given in control-affine form. For the sake of coherence, we set  $u = (u_1, \dots, u_m) \triangleq u^{(0)} = (u_1^{(0)}, \dots, u_m^{(0)})$ , so that  $\dot{u}^{(0)} = u^{(1)}$ .

**2.4. Recalls on Lie brackets and Distributions.** The Lie bracket  $[\eta, \gamma]$  of two arbitrary vector fields  $\eta$  and  $\gamma$  of  $\mathrm{TX}^{(0)}$  is given, in the  $\bar{x}^{(0)}$ -coordinates, by

$$[\eta, \gamma] \triangleq \sum_{i=1}^{n+m} \sum_{j=1}^{n+m} \left( \eta_j \frac{\partial \gamma_i}{\partial x_j^{(0)}} - \gamma_j \frac{\partial \eta_i}{\partial x_j^{(0)}} \right) \frac{\partial}{\partial x_i^{(0)}},$$

with  $\bar{x}^{(0)} = (x, u^{(0)}) \triangleq (x_1^{(0)}, \dots, x_{n+m}^{(0)})$ .

For iterated Lie brackets, we use the notation  $\mathrm{ad}_\eta \gamma \triangleq [\eta, \gamma]$  and  $\mathrm{ad}_\eta^k \gamma \triangleq [\eta, \mathrm{ad}_\eta^{k-1} \gamma]$  for  $k \geq 1$ , with the convention that  $\mathrm{ad}_\eta^0 \gamma = \gamma$ . A distribution  $\Gamma$  of vector fields on  $\mathrm{TX}^{(0)}$  is a mapping  $\Gamma : \xi \in X^{(0)} \mapsto \Gamma(\xi) \in \mathrm{T}_\xi X^{(0)}$ . If  $\Gamma$  is locally generated by  $p$  vector fields  $\gamma_1, \dots, \gamma_p$ , with  $p$  arbitrary, we

<sup>5</sup>We introduce the superscript  $(0)$  from now on to get ready to work with higher order prolongations (see section 3).

denote by  $\Gamma \triangleq \{\gamma_1, \dots, \gamma_p\}$  the vector space generated by the vectors  $\gamma_1(\xi), \dots, \gamma_p(\xi)$  at every point  $\xi \in X^{(0)}$ . We also note  $\text{ad}_\eta^k \Gamma \triangleq \{\text{ad}_\eta^k \gamma : \gamma \in \Gamma\}$ .

The distribution  $\Gamma$  is said *involutive* if, and only if,  $[\eta, \gamma] \in \Gamma$  for every pair of vector fields  $\eta, \gamma \in \Gamma$ , to which case we note  $[\Gamma, \Gamma] \subset \Gamma$ , or  $\bar{\Gamma} = \Gamma$ , where  $\bar{\Gamma}$  denotes the involutive closure of  $\Gamma$ , i.e. the smallest involutive distribution containing  $\Gamma$ .

Consider the ( $\mathbf{0}$ th-order or non prolonged) filtration of distributions built on the vector fields (2.17)<sup>6</sup>:

$$G_0^{(0)} \triangleq \{g_1^{(0)}, \dots, g_m^{(0)}\}, \quad G_{k+1}^{(0)} \triangleq G_k^{(0)} + \text{ad}_{g_0^{(0)}} G_k^{(0)}, \quad \forall k \geq 0, \quad (2.19)$$

indeed satisfying  $G_0^{(0)} \subset \dots \subset G_k^{(0)} \subset G_{k+1}^{(0)} \subset \dots \subset \text{TX}^{(0)}$ .

**Theorem 2.9** ([17, 15]). *System (2.1), or equivalently system (2.16), is feedback linearizable in a neighborhood of the origin of  $X^{(0)}$  if, and only if, in this neighborhood:*

- (i)  $G_k^{(0)}$  is involutive with constant dimension for all  $k \geq 0$ ,
- (ii) there exists an integer  $k_\star^{(0)} \leq n$  such that  $G_k^{(0)} = G_{k_\star^{(0)}}^{(0)} = \text{TX}^{(0)}$  for all  $k \geq k_\star^{(0)}$ .

Note that, according to (2.17) and (2.19),  $G_0^{(0)} = \{\frac{\partial}{\partial u_1^{(0)}}, \dots, \frac{\partial}{\partial u_m^{(0)}}\}$  is involutive with constant dimension, equal to  $m$ , by construction.

theorem 2.9 provides a construction of flat outputs via Frobenius theorem (see e.g. [8]) and the list of the so-called *Brunovský's controllability indices* [4] as follows:

**Definition 2.10.** Consider the sequence of integers

$$\rho_k^{(0)} \triangleq \dim G_k^{(0)} / G_{k-1}^{(0)} \quad \forall k \geq 1, \quad \rho_0^{(0)} \triangleq \dim G_0^{(0)} = m.$$

The Brunovský controllability indices  $\kappa_k^{(0)}$ 's are defined by

$$\kappa_k^{(0)} \triangleq \#\{l \mid \rho_l^{(0)} \geq k\}, \quad k = 1, \dots, m,$$

where  $\#A$  denotes the number of elements of an arbitrary set  $A$ .

It can be proven (see e.g. [17, 15, 16, 30, 22]) that, for a feedback linearizable nonlinear system (2.1), or (2.16), we have:

- the sequences  $\{\rho_k^{(0)}\}_{k \geq 0}$  and  $\{\kappa_k^{(0)}\}_{k \geq 0}$  are non increasing,
- $\rho_k^{(0)} \leq m$  for all  $k$ ,  $\rho_k^{(0)} = 0$ , for all  $k \geq k_\star^{(0)} + 1$ ,
- $\kappa_1^{(0)} = k_\star^{(0)} + 1$ ,  $\kappa_m^{(0)} \geq 1$ ,
- $\sum_{k=0}^{k_\star^{(0)}} \rho_k^{(0)} = \sum_{k=1}^m \kappa_k^{(0)} = \dim G_{k_\star^{(0)}}^{(0)} = n + m$ .

The list  $\kappa_1^{(0)}, \dots, \kappa_m^{(0)}$  is uniquely defined up to input permutation, invariant by static state feedback and state diffeomorphism, and is indeed equal to the list of controllability indices of the associated linear system (2.15) with  $r_i = \kappa_i^{(0)}$ ,  $i = 1, \dots, m$ .

<sup>6</sup>As before, the superscript  $(0)$  is used to indicate that the distributions  $G_k^{(0)}$  and the related indices  $\rho_k^{(0)}$  and  $\kappa_k^{(0)}$  are built on the non prolonged vector fields (2.17) and to distinguish them from the prolonged distributions  $G_k^{(j)}$  of arbitrary  $\mathbf{j}$ th order,  $\mathbf{j} \in \mathbb{N}^m$ , and related indices,  $\rho_k^{(j)}$  and  $\kappa_k^{(j)}$ , later introduced in sections 3 and 4.

Moreover, for all  $k$  and all  $i = 1, \dots, m$ , and possibly up to a suitable input reordering, we have

$$G_k^{(0)} = \bigoplus_{j=1}^m \{ \text{ad}_{g_0^{(0)}}^l g_j^{(0)} \mid l = 0, \dots, k \vee (\kappa_j^{(0)} - 1) \}, \quad G_{\kappa_1^{(0)} - 1}^{(0)} = G_{\kappa_*^{(0)}}^{(0)} = \text{TX}^{(0)}.$$

Then, flat outputs  $(y_1, \dots, y_m)$  are locally non trivial solutions of the system of PDE's

$$\begin{aligned} L_{\text{ad}_{g_0^{(0)}}^k g_j^{(0)}} y_i &= 0, \quad k = 0, \dots, \kappa_i^{(0)} - 2, \quad j = 1, \dots, m, \\ \text{with } L_{\text{ad}_{g_0^{(0)}}^{\kappa_i^{(0)} - 1} g_i^{(0)}} y_i &\neq 0, \end{aligned} \tag{2.20}$$

for  $i = 1, \dots, m$ , where we have denoted by  $L_\eta \varphi$  the Lie derivative of a vector function  $\varphi$  along the vector field  $\eta$ . These solutions are such that the mapping

$$\bar{x}^{(0)} \mapsto (y_1, \dots, y_1^{(\kappa_1^{(0)} - 1)}, \dots, y_m, \dots, y_m^{(\kappa_m^{(0)} - 1)})$$

is a local diffeomorphism.

**Remark 2.11.** Recall from [6] that, for single input systems, differential flatness and feedback linearizability are equivalent.

### 3. SYSTEM PURE PROLONGATION

**3.1. Purely Prolonged Distributions.** We now introduce higher order prolongations of the vector fields defined by (2.17), called *pure prolongations* [7] (see also [35, 36, 2, 12]).

Given a multi-integer  $\mathbf{j} \triangleq (j_1, \dots, j_m) \in \mathbb{N}^m$ , we note, as before,  $|\mathbf{j}| \triangleq \sum_{i=1}^m j_i$  and the prolonged state:

$$\bar{x}^{(\mathbf{j})} \triangleq (x, \bar{u}^{(\mathbf{j})}) \triangleq (x_1, \dots, x_n, u_1^{(0)}, \dots, u_1^{(j_1)}, \dots, u_m^{(0)}, \dots, u_m^{(j_m)}),$$

with  $u_i^{(0)} \triangleq u_i$ ,  $i = 1, \dots, m$ .

Let  $X^{(\mathbf{j})} \triangleq X \times \mathbb{R}^{m+|\mathbf{j}|}$  be the associated  $\mathbf{j}$ -th order jet manifold of dimension  $n + m + |\mathbf{j}|$  with coordinates  $\bar{x}^{(\mathbf{j})}$ .

The pure prolongation of order  $\mathbf{j}$  of system (2.16), or otherwise said, of the vector fields (2.17), in the tangent bundle  $\text{TX}^{(\mathbf{j})} = \text{TX} \times \text{T}\mathbb{R}^{m+|\mathbf{j}|}$ , is defined by

$$\begin{aligned} g_0^{(\mathbf{j})}(\bar{x}^{(\mathbf{j})}) &= \sum_{i=1}^n f_i(x, u) \frac{\partial}{\partial x_i} + \sum_{i=1}^m \sum_{k=0}^{j_i-1} u_i^{(k+1)} \frac{\partial}{\partial u_i^{(k)}} \\ g_i^{(\mathbf{j})}(\bar{x}^{(\mathbf{j})}) &\triangleq g_i^{(j_i)}(\bar{x}^{(\mathbf{j})}) \triangleq \frac{\partial}{\partial u_i^{(j_i)}}, \quad i = 1, \dots, m, \end{aligned} \tag{3.1}$$

with the convention that  $\sum_{k=0}^{j_i-1} u_i^{(k+1)} \frac{\partial}{\partial u_i^{(k)}} \triangleq 0$  if  $j_i = 0$ .

They are naturally associated to the adjunction of  $\mathbf{j}$  pure integrators to  $u = u^{(0)}$  in (2.16) (with the same usual abuse of notations as in (2.18)):

$$\dot{\bar{x}}^{(\mathbf{j})} = g_0^{(\mathbf{j})}(\bar{x}^{(\mathbf{j})}) + \sum_{i=1}^m u_i^{(j_i+1)} g_i^{(j_i)}(\bar{x}^{(\mathbf{j})}) \tag{3.2}$$

or

$$\dot{x} = f(x, u), \quad \dot{u}_i^{(k)} = u_i^{(k+1)}, \quad k = 0, \dots, j_i, \quad i = 1, \dots, m,$$

$u^{(\mathbf{j}+1)} = (u_1^{(j_1+1)}, \dots, u_m^{(j_m+1)})$  being the new control vector of this *purely prolonged system*, whose state is  $(x, \bar{u}^{(\mathbf{j})}) = \bar{x}^{(\mathbf{j})}$ .

**Remark 3.1.** Note that the state of the  $\mathbf{j}$ -th prolonged system,  $\bar{x}^{(\mathbf{j})}$ , coincides with the image of  $\bar{x}$  by the projection  $p_{\mathbf{j}} : \bar{x} \in X \times \mathbb{R}_{\infty}^m \mapsto p_{\mathbf{j}}(\bar{x}) = \bar{x}^{(\mathbf{j})} \in X^{(\mathbf{j})}$  for all  $\mathbf{j}$ . In addition, the family of projections  $p_{\mathbf{i}, \mathbf{j}} : \bar{x}^{(\mathbf{i})} \in X^{(\mathbf{i})} \mapsto \bar{x}^{(\mathbf{j})} = p_{\mathbf{i}, \mathbf{j}}(\bar{x}^{(\mathbf{i})}) \in X^{(\mathbf{j})}$  for all  $\mathbf{i}, \mathbf{j}$  such that  $i_k \geq j_k$  for all  $k = 1, \dots, m$ , that we note  $\mathbf{i} \succeq \mathbf{j}$ , indeed satisfies  $p_{\mathbf{i}, \mathbf{j}} \circ p_{\mathbf{j}, \mathbf{k}} = p_{\mathbf{i}, \mathbf{k}}$  for all  $\mathbf{i} \succeq \mathbf{j} \succeq \mathbf{k}$  and thus allows us to identify the manifold  $X \times \mathbb{R}_{\infty}^m$  with the *projective limit* of the family  $(X^{(\mathbf{i})}, p_{\mathbf{i}, \mathbf{j}})$  for all  $\mathbf{i}$  and all  $\mathbf{j}$  such that  $\mathbf{i} \succeq \mathbf{j}$ , i.e.  $X \times \mathbb{R}_{\infty}^m \simeq \varprojlim X^{(\mathbf{i})}$  (see e.g. [3, Chap. I, §10]). A similar identification trivially holds for the associated tangent bundles, i.e.  $\text{TX} \times \text{T}\mathbb{R}_{\infty}^m \simeq \varprojlim \text{TX}^{(\mathbf{i})}$  relatively to the family  $\text{T}p_{\mathbf{i}, \mathbf{j}}$  of tangent projections, hence the identification of the Cartan field  $C_f$ , defined by (2.4), with  $\varprojlim g_0^{(\mathbf{j})}$ , the projective limit of the vector fields  $g_0^{(\mathbf{j})}$ . Nevertheless, this property does not hold for the control vector fields  $g_i^{(\mathbf{j})}$  since  $\text{T}p_{\mathbf{j}, \mathbf{k}}(g_i^{(\mathbf{j})})$  is not equal to  $g_i^{(\mathbf{k})}$ , for  $i = 1, \dots, m$  and  $\mathbf{j} \succeq \mathbf{k}$ . Moreover, the Lie bracket of vector fields is not preserved by this family of projections. This is one of the reasons why prolongations may enlarge the range of the system transformations.

**Remark 3.2.** Given an arbitrary point  $\bar{x}_0 \triangleq (x_0, \bar{u}_0)$  around which system (2.16) is defined, it is convenient to consider the shift

$$\theta : (x, \bar{u}) \in X \times \mathbb{R}_{\infty}^m \mapsto \theta(x, \bar{u}) = (x - x_0, \overline{(u - u_0)}) \triangleq (z, \bar{v}) \in X \times \mathbb{R}_{\infty}^m$$

such that  $\bar{x}_0$  is mapped to the origin of  $\text{TX} \times \text{T}\mathbb{R}_{\infty}^m$ , denoted by  $\bar{0}$ , thus inducing the shift of vector fields:

$$\theta_{\star}(g_i^{(\mathbf{j})})(z, \bar{v}) \triangleq g_i^{(\mathbf{j})}(z + x_0, \overline{(v + u_0)}), \quad i = 0, \dots, m, \quad (3.3)$$

now defined in a neighborhood of  $\bar{0}$ .

For the sake of simplicity, we will only consider such shifted vector fields in the sequel while keeping the same notation  $g_i^{(\mathbf{j})}$  as before, though abusive, but yet unambiguous.

We now introduce the following filtration of  $\mathbf{j}$ -th order purely prolonged distributions of  $\text{TX}^{(\mathbf{j})}$ :

$$G_0^{(\mathbf{j})} \triangleq \left\{ \frac{\partial}{\partial u_1^{(j_1)}}, \dots, \frac{\partial}{\partial u_m^{(j_m)}} \right\}, \quad G_{k+1}^{(\mathbf{j})} \triangleq G_k^{(\mathbf{j})} + \text{ad}_{g_0^{(\mathbf{j})}} G_k^{(\mathbf{j})}, \quad \forall k \geq 0 \quad (3.4)$$

Indeed, for  $\mathbf{j} = \mathbf{0}$ , i.e.  $j_1 = \dots = j_m = 0$ , this filtration coincides with the  $\mathbf{0}$ th order one given by (2.19). Similarly to the  $\mathbf{0}$ th order case,  $G_0^{(\mathbf{j})}$  is involutive with constant dimension, equal to  $m$ , by construction.

Moreover, since every  $G_k^{(\mathbf{j})} \subset \text{TX}^{(\mathbf{j})}$ , with  $\dim \text{TX}^{(\mathbf{j})} = n + m + |\mathbf{j}|$ , we have

**Proposition 3.3.** *There exists a finite integer  $k_{\star}^{(\mathbf{j})}$  such that  $G_k^{(\mathbf{j})} = G_{k_{\star}^{(\mathbf{j})}}^{(\mathbf{j})}$  for all  $k \geq k_{\star}^{(\mathbf{j})}$  and*

$$k_{\star}^{(\mathbf{j})} \leq n + |\mathbf{j}|. \quad (3.5)$$

*Proof.* Since

$$n + m + |\mathbf{j}| \geq \dim G_{k_\star}^{(\mathbf{j})} = \sum_{k=1}^{k_\star} \dim G_k^{(\mathbf{j})} / G_{k-1}^{(\mathbf{j})} + \dim G_0^{(\mathbf{j})} \geq k_\star^{(\mathbf{j})} + m,$$

we get (3.5).  $\square$

**Remark 3.4.** In full generality,  $k_\star^{(\mathbf{j})}$  depends on the point where it is evaluated. However, if  $\dim G_k^{(\mathbf{j})}$  is constant in an open dense subset of  $X^{(\mathbf{j})}$  for all large enough  $k$ , so is  $k_\star^{(\mathbf{j})}$ .

Let us inductively define the  $n$ -dimensional vector functions  $\gamma_{k,i}^{(\mathbf{j})}$ , for  $k \geq 1$ ,  $i = 1, \dots, m$ , and arbitrary  $\mathbf{j} = (j_1, \dots, j_m)$  as follows:

$$\gamma_{k+1,i}^{(\mathbf{j})} \triangleq L_{g_0^{(\mathbf{j})}} \gamma_{k,i}^{(\mathbf{j})} - \gamma_{k,i}^{(\mathbf{j})} \frac{\partial f}{\partial x} = L_{g_0^{(\mathbf{j})}} \gamma_{k,i}^{(\mathbf{j})} + \sum_{p=1}^m \sum_{l=0}^{j_p-1} u_p^{(l+1)} \frac{\partial \gamma_{k,i}^{(\mathbf{j})}}{\partial u_p^{(l)}} - \gamma_{k,i}^{(\mathbf{j})} \frac{\partial f}{\partial x} \quad (3.6)$$

with

$$\gamma_{1,i}^{(\mathbf{j})} = (-1)^{(j_i+1)} \frac{\partial f}{\partial u_i^{(0)}}. \quad (3.7)$$

For an arbitrary  $\mathbf{j}$  and given  $i = 1, \dots, m$ , thanks to (3.7), it is readily seen that  $\gamma_{1,i}^{(\mathbf{j})}$  depends at most of  $\bar{x}^{(0)}$  and thus, if  $k \leq j_i - 1$ , thanks to (3.6),  $\gamma_{k+1,i}^{(\mathbf{j})}$  depends at most of  $\bar{x}^{(\mathbf{j} \vee k)}$ .

**3.2. Vertical Distributions of Purely Prolonged Ones.** Before stating the next lemma, we need to recall the definition of *vertical bundle*. Given an arbitrary  $\mathbf{r} \in \mathbb{N}^m$  and the fiber bundle  $\pi_{\mathbf{r}} : X^{(\mathbf{r})} \rightarrow \mathbb{R}^{m+|\mathbf{r}|}$ , with  $\pi_{\mathbf{r}}(\bar{x}^{(\mathbf{r})}) = \bar{u}^{(\mathbf{r})}$ , its vertical space at  $\bar{x}^{(\mathbf{r})}$ , denoted by  $V_{\bar{x}^{(\mathbf{r})}} X^{(\mathbf{r})}$ , is the tangent space  $T_x X$ . Its vertical bundle, denoted by  $VX^{(\mathbf{r})}$ , is the vector bundle made of the vertical spaces at each  $\bar{x}^{(\mathbf{r})}$ , i.e. the set of linear combinations  $\sum_{i=1}^n \alpha_i(\bar{x}^{(\mathbf{r})}) \frac{\partial}{\partial x_i}$  whose coefficients  $\alpha_i$  are smooth functions that depend at most on  $\bar{x}^{(\mathbf{r})}$  and where  $(x_1, \dots, x_n)$  are local coordinates of  $X$ .

The same definition indeed holds for the vertical bundle  $V(X \times \mathbb{R}_\infty^m)$  associated to the fiber bundle  $\pi : X \times \mathbb{R}_\infty^m \rightarrow \mathbb{R}_\infty^m$ , i.e. the set of linear combinations of  $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$  whose coefficients are smooth functions of  $\bar{x}$ .

We now establish some comparison *formulae* between Lie brackets of the vector fields of the purely prolonged system and those of the original (non prolonged) one. More complete *formulae* may be found in lemma A.1 of the Appendix A.

**Lemma 3.5.** For all  $\mathbf{j} = (j_1, \dots, j_m) \in \mathbb{N}^m$  satisfying  $0 \leq j_1 \leq \dots \leq j_m$ , for all  $k \leq j_i$  and  $i = 1, \dots, m$ , we have:

$$\text{ad}_{g_0^{(\mathbf{j})}}^k \frac{\partial}{\partial u_i^{(j_i)}} = (-1)^k \frac{\partial}{\partial u_i^{(j_i-k)}} \quad (3.8)$$

and for all  $k \geq 1$ :

$$\text{ad}_{g_0^{(\mathbf{j})}}^{j_i+k} \frac{\partial}{\partial u_i^{(j_i)}} = (-1)^{j_i} \text{ad}_{g_0^{(\mathbf{j})}}^k \frac{\partial}{\partial u_i^{(0)}} = \gamma_{k,i}^{(\mathbf{j})} \frac{\partial}{\partial x} \in VX^{(\mathbf{j} \vee (k-1))}, \quad (3.9)$$

Moreover, we have

$$\left[ \frac{\partial}{\partial u_p^{(j_p-k)}}, \text{ad}_{g_0^{(j)}}^{l-j_q} \frac{\partial}{\partial u_q^{(0)}} \right] = 0, \quad \forall k < j_p, \quad \forall l \geq j_q \text{ s.t. } k+l < j_p + j_q + 1. \quad (3.10)$$

*Proof.* It is immediately seen that

$$\text{ad}_{g_0^{(j)}} \frac{\partial}{\partial u_i^{(j_i)}} = \left[ f \frac{\partial}{\partial x} + \sum_{k=1}^m \sum_{l=0}^{j_k-1} u_k^{(l+1)} \frac{\partial}{\partial u_k^{(l)}}, \frac{\partial}{\partial u_i^{(j_i)}} \right] = - \frac{\partial}{\partial u_i^{(j_{i-1})}}.$$

Iterating this computation up to  $k = j_i$  yields (3.8):

$$\text{ad}_{g_0^{(j)}}^{j_i} \frac{\partial}{\partial u_i^{(j_i)}} = (-1)^{j_i} \frac{\partial}{\partial u_i^{(0)}}.$$

Then, for  $k = j_i + 1$ , using the fact that  $[\frac{\partial}{\partial u_k^{(l)}}, \frac{\partial}{\partial u_i^{(0)}}] = 0$  for all  $i, k$  and  $l \geq 0$ , we have:

$$\begin{aligned} \text{ad}_{g_0^{(j)}}^{j_i+1} \frac{\partial}{\partial u_i^{(j_i)}} &= (-1)^{j_i} \left[ f \frac{\partial}{\partial x} + \sum_{k=1}^m \sum_{l=0}^{j_k-1} u_k^{(l+1)} \frac{\partial}{\partial u_k^{(l)}}, \frac{\partial}{\partial u_i^{(0)}} \right] \\ &= (-1)^{j_i} \text{ad}_{g_0^{(0)}} \frac{\partial}{\partial u_i^{(0)}} = (-1)^{(j_i+1)} \frac{\partial f}{\partial u_i^{(0)}}(\bar{x}^{(0)}) \frac{\partial}{\partial x} = \gamma_{1,i}^{(j)} \frac{\partial}{\partial x}, \end{aligned} \quad (3.11)$$

which proves that  $\text{ad}_{g_0^{(j)}}^{j_i+1} \frac{\partial}{\partial u_i^{(j_i)}} \in \text{VX}^{(0)}$  and that (3.9) holds at the order  $k = 1$ .

Assuming that (3.9) holds up to  $k = v$ , with  $\gamma_{v,i}^{(j)}$  depending at most on  $\bar{x}^{(j \vee (v-1))}$ , we have

$$\text{ad}_{g_0^{(j)}}^{j_i+v+1} \frac{\partial}{\partial u_i^{(j_i)}} = (-1)^{j_i} \left[ g_0^{(j)}, \text{ad}_{g_0^{(j)}}^v \frac{\partial}{\partial u_i^{(0)}} \right] = \left( L_{g_0^{(j)}} \gamma_{v,i}^{(j)} - \gamma_{v,i}^{(j)} \frac{\partial f}{\partial x} \right) \frac{\partial}{\partial x} = \gamma_{v+1,i}^{(j)} \frac{\partial}{\partial x},$$

and, according to (3.6)-(3.7), for all  $j_p + 1 \leq v < j_{p+1}$ ,  $p = 1, \dots, m-1$ , since we differentiate  $\gamma_{v,i}^{(j)}$  with respect to  $\sum_{k=1}^m \sum_{l=0}^{j_k-1} u_k^{(l+1)} \frac{\partial}{\partial u_k^{(l)}}$ , it is immediate to verify that  $\gamma_{v+1,i}^{(j)}$  depends at most on

$$\bar{x}^{(j \vee v)} = \left( x, u_1^{(0)}, \dots, u_1^{(j_1)}, \dots, u_p^{(0)}, \dots, u_p^{(j_p)}, u_{p+1}^{(0)}, \dots, u_{p+1}^{(v)}, \dots, u_m^{(0)}, \dots, u_m^{(v)} \right)$$

and on  $\bar{x}^{(j)} = \bar{x}^{(j \vee v)}$  if  $v \geq j_m$ , hence (3.9).

Concerning (3.10), since  $\text{ad}_{g_0^{(j)}}^{l-j_q} \frac{\partial}{\partial u_q^{(0)}} = \pm \gamma_{l-j_q,q}^{(j)} \frac{\partial}{\partial x}$  depends at most on  $\bar{x}^{(j \vee (l-j_q-1))}$ , the derivative of  $\gamma_{l-j_q,q}^{(j)}$  with respect to  $\frac{\partial}{\partial u_p^{(j_p-k)}}$  is indeed null if  $j_p - k > l - j_q - 1$ , hence the result.  $\square$

**3.3. Decomposition of Purely Prolonged Distributions.** Let us assume, without loss of generality, that the control components have been reordered in such a way that  $j_1 \leq j_2 \leq \dots \leq j_m$ . Moreover, we may suppose that  $j_1 = 0$ , as shown to be sufficient in the next section.

We now introduce two new filtrations of  $\text{TX}^{(j)}$ , noted  $\Gamma_k^{(j)}$  and  $\Delta_k^{(j)}$ , for  $k \geq 0$ , as follows

$$\Gamma_k^{(j)} \triangleq \bigoplus_{p=2}^m \left\{ \frac{\partial}{\partial u_p^{(j_p-l)}} \mid l = 0, \dots, k \vee (j_p - 1) \right\}, \quad (3.12)$$

$$\Delta_k^{(\mathbf{j})} \triangleq \sum_{p=1}^m \left\{ \text{ad}_{g_0^{(\mathbf{j})}}^{l-j_p} \frac{\partial}{\partial u_p^{(0)}} \mid l = j_p, \dots, k \right\}. \quad (3.13)$$

with the convention that  $\text{ad}_{g_0^{(\mathbf{j})}}^{k-j_p} \frac{\partial}{\partial u_p^{(0)}} = 0$  if  $k < j_p$ ,  $p = 1, \dots, m$ .

We indeed have  $\Gamma_k^{(\mathbf{j})} = \Gamma_{j_m-1}^{(\mathbf{j})} = \bigoplus_{p=2}^m \left\{ \frac{\partial}{\partial u_p^{(j_p)}}, \dots, \frac{\partial}{\partial u_p^{(1)}} \right\}$  for all  $k \geq j_m - 1$ . Thus,  $\dim \Gamma_k^{(\mathbf{j})} \leq |\mathbf{j}|$  for all  $k$ .

The definitions (3.12)–(3.13) and lemma 3.5 readily yield, for all  $k, l$  and  $\mathbf{j}$ ,

$$\begin{aligned} \overline{\Gamma_k^{(\mathbf{j})}} &= \Gamma_k^{(\mathbf{j})}, & [\Gamma_k^{(\mathbf{j})}, \Gamma_l^{(\mathbf{j})}] &\subset \Gamma_{k \wedge l}^{(\mathbf{j})}, \\ \Delta_l^{(\mathbf{j})} &\subset \mathbf{V}(X \times \mathbb{R}_\infty^m), & [\Gamma_k^{(\mathbf{j})}, \Delta_l^{(\mathbf{j})}] &\subset \mathbf{V}(X \times \mathbb{R}_\infty^m), & [\Gamma_k^{(\mathbf{j})}, \Delta_l^{(\mathbf{j})}] \cap \Gamma_k^{(\mathbf{j})} &= \{0\}. \end{aligned} \quad (3.14)$$

**Remark 3.6.** Note that, unlike the filtration  $\{G_k^{(\mathbf{j})}\}_{0 \leq k \leq k_\star^{(\mathbf{j})}}$  that is increasing for  $k \leq k_\star^{(\mathbf{j})}$ , the mapping  $k \mapsto \dim \Delta_k^{(\mathbf{j})}$  is only non-decreasing in general.

**Remark 3.7.** In our definition of  $\Delta_k^{(\mathbf{j})}$ , we consider Lie brackets of the form  $\text{ad}_{g_0^{(\mathbf{j})}}^k \frac{\partial}{\partial u_p^{(0)}}$ , as opposed to [2] where Lie brackets of the form  $\text{ad}_{g_0^{(0)}}^k \frac{\partial}{\partial u_p^{(0)}}$  are used.

**Example 3.8.** Let us consider the chained system (see e.g. [27]):

$$\begin{aligned} \dot{x}_1^{(3)} &= u_1 \\ \ddot{x}_2 &= u_2 \\ \dot{x}_3 &= u_1 u_2 \end{aligned} \quad (3.15)$$

We denote the state coordinates by

$$(x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_3, u_1^{(0)}, u_2^{(0)}),$$

( $n = 6$  and  $m = 2$ ), with  $x_{1,j} \triangleq x_1^{(j)}$ ,  $j = 1, 2, 3$ ,  $x_{2,j} \triangleq x_2^{(j)}$ ,  $j = 1, 2$  and  $u_i^{(0)} = u_i$ ,  $i = 1, 2$ .

The two input variables are  $u_i^{(1)} \triangleq \dot{u}_i$ ,  $i = 1, 2$ .

The (non prolonged) system vector fields are

$$\begin{aligned} g_0^{(0)} &\triangleq x_{1,2} \frac{\partial}{\partial x_{1,1}} + x_{1,3} \frac{\partial}{\partial x_{1,2}} + u_1^{(0)} \frac{\partial}{\partial x_{1,3}} + x_{2,2} \frac{\partial}{\partial x_{2,1}} + u_2^{(0)} \frac{\partial}{\partial x_{2,2}} + u_1^{(0)} u_2^{(0)} \frac{\partial}{\partial x_3} \\ g_1^{(0)} &\triangleq \frac{\partial}{\partial u_1^{(0)}}, & g_2^{(0)} &\triangleq \frac{\partial}{\partial u_2^{(0)}} \end{aligned} \quad (3.16)$$

The reader may easily verify that

$$\begin{aligned} G_0^{(0)} &= \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}} \right\}, & G_1^{(0)} &= G_0^{(0)} + \left\{ -\frac{\partial}{\partial x_{1,3}} - u_2^{(0)} \frac{\partial}{\partial x_3}, -\frac{\partial}{\partial x_{2,2}} - u_1^{(0)} \frac{\partial}{\partial x_3} \right\}, \\ G_2^{(0)} &= G_1^{(0)} + \left\{ \frac{\partial}{\partial x_{1,2}}, \frac{\partial}{\partial x_{2,1}} \right\}, & G_3^{(0)} &= G_2^{(0)} + \left\{ \frac{\partial}{\partial x_{1,1}} \right\} \end{aligned}$$

and  $G_k^{(0)} = G_3^{(0)}$  for all  $k \geq 3$ .

If we prolong the first input  $u_1^{(1)}$   $j_1$  times, with  $j_1 \geq 2$ , the new inputs are now  $u_1^{(j_1+1)}$  and  $u_2^{(1)}$ , and the prolonged vector fields are

$$\begin{aligned} g_0^{(\mathbf{j})} &\triangleq g_0^{(\mathbf{0})} + \sum_{l=0}^{j_1-1} u_1^{(l+1)} \frac{\partial}{\partial u_1^{(l)}} \\ g_1^{(\mathbf{j})} &\triangleq \frac{\partial}{\partial u_1^{(j_1)}}, \quad g_2^{(\mathbf{j})} \triangleq \frac{\partial}{\partial u_2^{(0)}} = g_2^{(\mathbf{0})} \end{aligned} \quad (3.17)$$

with  $\mathbf{j} \triangleq (j_1, 0)$ . We thus obtain:

$$\begin{aligned} G_0^{(\mathbf{j})} &= \left\{ \frac{\partial}{\partial u_1^{(j_1)}}, \frac{\partial}{\partial u_2^{(0)}} \right\}, \quad G_1^{(\mathbf{j})} = G_0^{(\mathbf{j})} + \left\{ -\frac{\partial}{\partial u_1^{(j_1-1)}}, -\frac{\partial}{\partial x_{2,2}} - u_1^{(0)} \frac{\partial}{\partial x_3} \right\}, \\ G_2^{(\mathbf{j})} &= G_1^{(\mathbf{j})} + \left\{ \frac{\partial}{\partial u_1^{(j_1-2)}}, -u_1^{(1)} \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_{2,1}} \right\} \end{aligned}$$

and, after some tedious calculations,

$$G_k^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_1^{(j_1)}}, \dots, \frac{\partial}{\partial u_1^{(1)}} \right\} \times \mathrm{TX}^{(\mathbf{0})} = \Gamma_k^{(\mathbf{j})} \oplus \Delta_k^{(\mathbf{j})} \quad \forall k \geq j_1 + 3$$

according to (3.12)-(3.13).

This example thus motivates the next proposition:

**Proposition 3.9.** *For all  $\mathbf{j} = (j_1, \dots, j_i, j_{i+1}, \dots, j_m)$  such that  $0 = j_1 \leq \dots \leq j_{m-1} \leq j_m$ , with  $j_m$  finite, denoting by  $\mathbf{j}' = (j_1, \dots, j_i, j'_{i+1}, \dots, j'_m)$  for some  $(j'_{i+1}, \dots, j'_m)$ , we have  $\Delta_k^{(\mathbf{j})} = \Delta_k^{(\mathbf{j}')}$  for all  $k = 0, \dots, (j_{i+1} \vee j'_{i+1}) - 1$ .*

Moreover, for all  $k \geq 0$ ,

$$\dim \Gamma_k^{(\mathbf{j})} = \begin{cases} \sum_{p=1}^i j_p + (k+1)(m-i) & \text{if } j_i \leq k < j_{i+1}, i = 1, \dots, m-1 \\ |\mathbf{j}| & \text{if } k \geq j_m \end{cases} \quad (3.18)$$

$$\dim \Delta_k^{(\mathbf{j})} \leq \begin{cases} ((k+1)i - \sum_{p=1}^i j_p) \vee (n+i) & \text{if } j_i \leq k < j_{i+1}, i = 1, \dots, m-1 \\ ((k+1)m - |\mathbf{j}|) \vee (n+m) & \text{if } k \geq j_m \end{cases} \quad (3.19)$$

and we have

$$G_k^{(\mathbf{j})} = \Gamma_k^{(\mathbf{j})} \oplus \Delta_k^{(\mathbf{j})}, \quad \forall k \geq 0. \quad (3.20)$$

Furthermore, the finite integer  $k_\star^{(\mathbf{j})}$ , satisfying (3.5), is such that  $\Delta_k^{(\mathbf{j})} = \Delta_{k_\star^{(\mathbf{j})}}^{(\mathbf{j})}$  and  $\Gamma_k^{(\mathbf{j})} = \Gamma_{k_\star^{(\mathbf{j})}}^{(\mathbf{j})}$  for all  $k \geq k_\star^{(\mathbf{j})}$ .

If, in addition,  $\dim \Delta_{k_\star^{(\mathbf{j})}}^{(\mathbf{j})} = m+n$ , then

$$n + |\mathbf{j}| \geq k_\star^{(\mathbf{j})} \geq j_m \wedge \frac{n + |\mathbf{j}|}{m}. \quad (3.21)$$

*Proof.* By definition, the generators of  $\Gamma_k^{(\mathbf{j})}$  are independent for all  $k$  and thus their number is equal to  $\dim \Gamma_k^{(\mathbf{j})}$ , hence (3.18). The dimension of  $\Delta_k^{(\mathbf{j})}$ , in turn, is lesser than, or equal to, the number of its generators, in number  $(k+1) + \dots + (k-j_i+1) = (k+1)i - \sum_{p=1}^i j_p$ , if  $j_i \leq k < j_{i+1}$  (respectively  $(k+1) + \dots + (k-j_m+1) = (k+1)m - |\mathbf{j}|$ , if  $k \geq j_m$ ), and, since, according to (3.9) of lemma 3.5,  $\Delta_k^{(\mathbf{j})}$  is contained in  $\{\frac{\partial}{\partial u_1^{(0)}}, \dots, \frac{\partial}{\partial u_m^{(0)}}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$  if  $j_i \leq k < j_{i+1}$ ,  $i = 1, \dots, m-1$  (respectively in  $\{\frac{\partial}{\partial u_1^{(0)}}, \dots, \frac{\partial}{\partial u_m^{(0)}}, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$  if  $k \geq j_m$ ), its dimension is bounded above by  $i+n$  (resp.  $m+n$ ), hence (3.19)

The proof of (3.20) is by induction. For  $k=0$ , by (3.12)-(3.13), we indeed have  $G_0^{(\mathbf{j})} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(j_2)}}, \dots, \frac{\partial}{\partial u_m^{(j_m)}} \right\} = \left\{ \frac{\partial}{\partial u_1^{(0)}} \right\} \oplus \left\{ \frac{\partial}{\partial u_2^{(j_2)}}, \dots, \frac{\partial}{\partial u_m^{(j_m)}} \right\} = \Delta_0^{(\mathbf{j})} \oplus \Gamma_0^{(\mathbf{j})}$ . Thus, (3.20) is valid at the order 0.

Assume now that (3.20) holds true up to the order  $\nu > 0$  with  $j_r \leq \nu < j_{r+1}$  for some  $r \in \{1, \dots, m\}$ , assuming that  $j_r < j_{r+1}$ . If  $j_r = j_{r+1}$ , the reader may immediately go to the case  $\nu+1 = j_{r+1}$  below.

At the order  $\nu+1$ , two cases are possible: either  $j_r \leq \nu+1 < j_{r+1}$  or  $\nu+1 = j_{r+1}$ . In the first case, using lemma 3.5, we get:

$$\begin{aligned} G_{\nu+1}^{(\mathbf{j})} &= G_\nu^{(\mathbf{j})} + ad_{g_0^{(\mathbf{j})}} G_\nu^{(\mathbf{j})} = \Gamma_\nu^{(\mathbf{j})} \oplus \Delta_\nu^{(\mathbf{j})} + ad_{g_0^{(\mathbf{j})}} \Gamma_\nu^{(\mathbf{j})} + ad_{g_0^{(\mathbf{j})}} \Delta_\nu^{(\mathbf{j})} \\ &= \Gamma_\nu^{(\mathbf{j})} \oplus \Delta_\nu^{(\mathbf{j})} + \left\{ ad_{g_0^{(\mathbf{j})}}^{\nu+1} \frac{\partial}{\partial u_1^{(0)}}, \dots, ad_{g_0^{(\mathbf{j})}}^{\nu+1-j_r} \frac{\partial}{\partial u_r^{(0)}}, \frac{\partial}{\partial u_{r+1}^{(j_{r+1}-\nu-1)}}, \dots, \frac{\partial}{\partial u_m^{(j_m-\nu-1)}} \right\} \\ &= \Gamma_{\nu+1}^{(\mathbf{j})} \oplus \Delta_{\nu+1}^{(\mathbf{j})}. \end{aligned}$$

In the second case, namely if  $\nu+1 = j_{r+1}$ ,

$$\begin{aligned} G_{\nu+1}^{(\mathbf{j})} &= \Gamma_\nu^{(\mathbf{j})} \oplus \Delta_\nu^{(\mathbf{j})} \\ &+ \left\{ ad_{g_0^{(\mathbf{j})}}^{\nu+1} \frac{\partial}{\partial u_1^{(0)}}, \dots, ad_{g_0^{(\mathbf{j})}}^{\nu+1-j_r} \frac{\partial}{\partial u_r^{(0)}}, \frac{\partial}{\partial u_{r+1}^{(0)}}, \frac{\partial}{\partial u_{r+2}^{(j_{r+2}-\nu-1)}}, \dots, \frac{\partial}{\partial u_m^{(j_m-\nu-1)}} \right\} \\ &= \Gamma_{\nu+1}^{(\mathbf{j})} \oplus \Delta_{\nu+1}^{(\mathbf{j})}. \end{aligned}$$

The case  $j_m \leq \nu+1$  follows the same lines:

$$G_{\nu+1}^{(\mathbf{j})} = \Gamma_\nu^{(\mathbf{j})} \oplus \Delta_\nu^{(\mathbf{j})} + \left\{ ad_{g_0^{(\mathbf{j})}}^{\nu+1} \frac{\partial}{\partial u_1^{(0)}}, \dots, ad_{g_0^{(\mathbf{j})}}^{\nu+1-j_m} \frac{\partial}{\partial u_m^{(0)}} \right\} = \Gamma_{\nu+1}^{(\mathbf{j})} \oplus \Delta_{\nu+1}^{(\mathbf{j})}$$

hence (3.20) is proven and the property of the number of iterations  $k_\star^{(\mathbf{j})}$  to simultaneously saturate the dimensions of  $\Gamma_k^{(\mathbf{j})}$  and  $\Delta_k^{(\mathbf{j})}$  immediately follows.

Moreover, if  $\dim \Delta_{k_\star^{(\mathbf{j})}}^{(\mathbf{j})} = m+n$ , we must have  $j_m \leq k_\star^{(\mathbf{j})}$  since otherwise, using definition (3.13) for  $k_\star^{(\mathbf{j})} < j_m$ ,  $\Delta_{k_\star^{(\mathbf{j})}}^{(\mathbf{j})}$  would not contain  $\{\frac{\partial}{\partial u_m^{(0)}}\}$  and its dimension would not exceed  $m-1+n$ . Consequently, applying once more (3.19) with  $\dim \Delta_{k_\star^{(\mathbf{j})}}^{(\mathbf{j})} = m+n$ , we get  $(k_\star^{(\mathbf{j})} + 1)m - |\mathbf{j}| \geq m+n$ , which, combined with (3.5), immediately yields (3.21).  $\square$

**Remark 3.10.** The inequality (3.21) reads  $k_\star^{(\mathbf{j})} - \frac{|\mathbf{j}|}{m} \geq \frac{n}{m}$  and may thus be interpreted as an estimate of the gap between  $k_\star^{(\mathbf{j})}$  and the average value  $\frac{|\mathbf{j}|}{m}$  of the prolongation lengths  $j_1, j_2, \dots, j_m$ , provided that the prolonged system satisfies the strong accessibility rank condition  $\dim \Delta_{k_\star^{(\mathbf{j})}}^{(\mathbf{j})} = m + n$ .

#### 4. FLATNESS BY PURE PROLONGATION

**4.1. Equivalence by Pure Prolongation.** Consider the two systems (2.5) with  $\dim x = n$ ,  $\dim y = n'$  and  $\dim u = \dim v = m$ .

Given arbitrary  $\mathbf{j}$  and  $\mathbf{l}$ , we recall that the associated prolonged vector fields are

$$g^{(\mathbf{j})} \triangleq g \frac{\partial}{\partial x} + \sum_{i=1}^m \sum_{p=0}^{j_i} u_i^{(p+1)} \frac{\partial}{\partial u_i^{(p)}}, \quad \gamma^{(\mathbf{l})} \triangleq \gamma \frac{\partial}{\partial y} + \sum_{i=1}^m \sum_{p=0}^{l_i} v_i^{(p+1)} \frac{\partial}{\partial v_i^{(p)}}.$$

The prolonged states are, respectively,  $\bar{x}^{(\mathbf{j})}$  and  $\bar{y}^{(\mathbf{l})}$ , and the control inputs  $u^{(\mathbf{j}+1)}$  and  $v^{(\mathbf{l}+1)}$ .

**Definition 4.1.** The systems (2.5) are equivalent by pure prolongation (in short  $P^2$ -equivalent) at a point  $\bar{x}_0 \in X \times \mathbb{R}_\infty^m$  if, and only if, there exist finite  $\mathbf{j}$  and  $\mathbf{l}$  such that the prolonged systems of order  $\mathbf{j}$  and  $\mathbf{l}$  respectively are equivalent by diffeomorphism and feedback, *i.e.* if, and only if, there exists a local diffeomorphism  $\varphi$  and a feedback  $W$ :

$$\bar{y}^{(\mathbf{l})} = \varphi(\bar{x}^{(\mathbf{j})}) \quad v^{(\mathbf{l}+1)} = W(\bar{x}^{(\mathbf{j})}, u^{(\mathbf{j}+1)})$$

with  $W$  invertible with respect to  $u^{(\mathbf{j}+1)}$  for all  $\bar{x}^{(\mathbf{j})}$  in a suitable neighborhood of  $\bar{x}_0$ , such that

$$\gamma^{(\mathbf{l})} \circ (\varphi \times W) = \varphi_\star g^{(\mathbf{j})}.$$

This equivalence indeed implies that  $n + |\mathbf{j}| = n' + |\mathbf{l}|$ .

**Remark 4.2.** The equivalence relation by *diffeomorphism and feedback* is easily seen to be strictly finer than the  $P^2$ -equivalence (take  $\mathbf{j} = \mathbf{l} = \mathbf{0}$ ), which in turn is strictly finer than the *Lie-Bäcklund equivalence* (see *e.g.* Example 5.4).

**Definition 4.3.** We say that system (2.16) is *flat by pure prolongation* (in short  $P^2$ -flat) at a point  $\bar{x}_0 \in X \times \mathbb{R}_\infty^m$  if, and only if, it is  $P^2$ -equivalent to  $\dot{y}_i = v_i$ ,  $i = 1, \dots, m$  and  $y$  is called  $P^2$ -flat output.

It is therefore immediate to remark that a system is  $P^2$ -flat if, and only if, there exists a pure prolongation of finite order  $\mathbf{j}$  such that the prolonged system is feedback linearizable at  $\bar{x}_0$ , thus recovering the definition of linearization by prolongation already introduced in [2, 11, 12].

Moreover, a  $P^2$ -flat output being obviously a flat output and pure prolongations being particular cases of Lie-Bäcklund isomorphisms, the class of  $P^2$ -flat systems is indeed contained in the class of Lie-Bäcklund equivalence to 0 (modulo the trivial field  $\tau$ ), *i.e.* constitutes a subclass of differentially flat systems.

The next lemma extends a well-know result (see *e.g.* [6, 35]) to our context (see also [12, 21]).

**Lemma 4.4.** *We consider system (3.2), denoted by  $\Sigma^{(\mathbf{j})}$ , with  $\mathbf{j} = (j_1, \dots, j_m)$ , assuming, without loss of generality, that  $0 \leq j_1 \leq \dots \leq j_m$ , possibly up to input renumbering. We denote by  $\mathbf{j}' = \mathbf{j} - \mathbf{j}_1 = (0, j_2 - j_1, \dots, j_m - j_1)$ , and by  $\Sigma^{(\mathbf{j}')}$  the corresponding system. Then  $\Sigma^{(\mathbf{j})}$  is  $P^2$ -flat at a given point  $(x_0, \bar{u}_0)$  if, and only if,  $\Sigma^{(\mathbf{j}')}$  is also  $P^2$ -flat at this point. Moreover, every  $P^2$ -flat output of  $\Sigma^{(\mathbf{j})}$  at  $(x_0, \bar{u}_0)$  is a  $P^2$ -flat output of  $\Sigma^{(\mathbf{j}')}$  at the same point, and conversely.*

*Proof.* Since  $\Sigma^{(\mathbf{j})}$  is  $P^2$ -flat, there exists  $\mathbf{j}_0$  and  $\mathbf{l}_0$  such that  $\Sigma^{(\mathbf{j}+\mathbf{j}_0)}$  is feedback equivalent to the linear system  $y^{(\mathbf{l}_0+1)} = w$ . Therefore, since, by assumption,  $\Sigma^{(\mathbf{j}+\mathbf{j}_0)} = \Sigma^{(\mathbf{j}'+\mathbf{j}_1+\mathbf{j}_0)}$ ,  $\Sigma^{(\mathbf{j}')}$  is also  $P^2$ -flat. The converse is trivial as well as the fact that  $\Sigma^{(\mathbf{j}+\mathbf{j}_0)}$  and  $\Sigma^{(\mathbf{j}'+\mathbf{j}_1+\mathbf{j}_0)}$  have the same  $P^2$ -flat output.  $\square$

**Remark 4.5.** In the single input case ( $m = 1$ ), this lemma shows that a  $P^2$ -flat system is equivalent, without prolongation, to a linear system. We thus trivially recover the fact that  $P^2$ -flatness and feedback linearizability are equivalent in this case (see [6] and remark 2.11)

**4.2. Necessary and Sufficient Conditions.** The following Proposition is a straightforward adaptation of theorem 2.9 for an arbitrary order  $\mathbf{j}$ . Note that, at this stage, nothing is said about a possible choice of  $\mathbf{j}$ , a question that will be dealt with in subsection 4.3, theorem 4.14.

**Proposition 4.6.** *The prolonged system of order  $\mathbf{j}$  is feedback linearizable at  $\bar{0}$  if, and only if,  $G_k^{(\mathbf{j})}$  is involutive with locally constant dimension for all  $k$  and such that  $G_{k_*}^{(\mathbf{j})} = \text{TX}^{(\mathbf{j})}$ .*

Again, flat outputs can be computed via Frobenius theorem, once established the list of Brunovsky's controllability indices of order  $\mathbf{j}$ , as follows:

**Definition 4.7.** Consider the sequence of integers

$$\rho_k^{(\mathbf{j})} \triangleq \dim G_k^{(\mathbf{j})} / G_{k-1}^{(\mathbf{j})} \quad \forall k \geq 1, \quad \rho_0^{(\mathbf{j})} \triangleq \dim G_0^{(\mathbf{j})} = m.$$

The Brunovsky controllability indices of order  $\mathbf{j}$  are defined by

$$\kappa_k^{(\mathbf{j})} \triangleq \#\{l \mid \rho_l^{(\mathbf{j})} \geq k\}, \quad k = 1, \dots, m,$$

As in the 0th order case, if the prolonged system of order  $\mathbf{j}$  is feedback linearizable at  $\bar{0}$ , we have:

- $\rho_k^{(\mathbf{j})}$ 's and  $\kappa_k^{(\mathbf{j})}$ 's are non increasing sequences,
- $\rho_k^{(\mathbf{j})} \leq m$  for all  $k$  and  $\rho_k^{(\mathbf{j})} = 0$  for all  $k \geq k_*^{(\mathbf{j})} + 1$ ,
- $\kappa_1^{(\mathbf{j})} = k_*^{(\mathbf{j})} + 1$ ,  $\kappa_m^{(\mathbf{j})} \geq 1$ ,
- $\sum_{k=0}^{k_*^{(\mathbf{j})}} \rho_k^{(\mathbf{j})} = \sum_{k=1}^m \kappa_k^{(\mathbf{j})} = \dim G_{k_*}^{(\mathbf{j})} = n + m + |\mathbf{j}|$ .

The list  $\kappa_1^{(\mathbf{j})}, \dots, \kappa_m^{(\mathbf{j})}$  is uniquely defined up to input permutation, invariant by prolonged state feedback and prolonged state diffeomorphism, and is thus equal to the list of controllability indices of the associated linear system (2.15) with  $r_i = \kappa_i^{(\mathbf{j})}$ ,  $i = 1, \dots, m$ .

Moreover, for all  $k$  and all  $i = 1, \dots, m$ , and possibly up to a suitable input reordering, we have

$$G_k^{(\mathbf{j})} = \bigoplus_{p=1}^m \left\{ \text{ad}_{s_0^{(\mathbf{j})}}^l \frac{\partial}{\partial u_p^{(\mathbf{j}p)}} \mid l = 0, \dots, k \vee (\kappa_p^{(\mathbf{j})} - 1) \right\}, \quad G_{\kappa_1^{(\mathbf{j})}-1}^{(\mathbf{j})} = G_{k_*^{(\mathbf{j})}}^{(\mathbf{j})} = \text{TX}^{(\mathbf{j})}$$

and flat outputs  $(y_1, \dots, y_m)$  are locally non trivial solutions of the system of PDE's

$$\left\langle G_k^{(\mathbf{j})}, dy_i \right\rangle = 0, \quad k = 0, \dots, \kappa_i^{(\mathbf{j})} - 2, \quad \text{with} \quad \left\langle G_{\kappa_i^{(\mathbf{j})}-1}^{(\mathbf{j})}, dy_i \right\rangle \neq 0, \quad i = 1, \dots, m. \quad (4.1)$$

Finally, the mapping

$$\bar{x}^{(\mathbf{j})} \mapsto (y_1, \dots, y_1^{(\kappa_1^{(\mathbf{j})}-1)}, \dots, y_m, \dots, y_m^{(\kappa_m^{(\mathbf{j})}-1)})$$

is a local diffeomorphism.

In virtue of lemma 4.4, it suffices to restrict our analysis to prolongations of order  $\mathbf{j} = (j_1, \dots, j_m)$  such that  $0 = j_1 \leq \dots \leq j_m$ .

We are now ready to state our main result.

**Theorem 4.8.** *A necessary and sufficient condition for  $P^2$ -flatness at  $\bar{0}$  is that there exists  $\mathbf{j} = (j_1, \dots, j_m) \in \mathbb{N}^m$ ,  $0 = j_1 \leq \dots \leq j_m$ , such that*

- (i)  $\overline{\Delta_k^{(\mathbf{j})}} = \Delta_k^{(\mathbf{j})}$  with  $\dim \Delta_k^{(\mathbf{j})}$  locally constant for all  $k \geq 0$ ,
- (ii)  $[\Gamma_k^{(\mathbf{j})}, \Delta_k^{(\mathbf{j})}] \subset \Delta_k^{(\mathbf{j})}$  for all  $k \geq 0$ ,
- (iii)  $k_*^{(\mathbf{j})}$  is such that  $\Delta_k^{(\mathbf{j})} = \text{TX} \times \text{TR}^m$  and  $\Gamma_k^{(\mathbf{j})} = \text{TR}^{|\mathbf{j}|}$  for all  $k \geq k_*^{(\mathbf{j})}$ .

*Proof.* By (3.20) of Proposition 3.9, we have  $G_k^{(\mathbf{j})} = \Gamma_k^{(\mathbf{j})} \oplus \Delta_k^{(\mathbf{j})}$  for all  $k \geq 0$ . Then,  $G_k^{(\mathbf{j})} = \overline{G_k^{(\mathbf{j})}}$  implies that  $[\Gamma_k^{(\mathbf{j})} \oplus \Delta_k^{(\mathbf{j})}, \Gamma_k^{(\mathbf{j})} \oplus \Delta_k^{(\mathbf{j})}] \subset \Gamma_k^{(\mathbf{j})} \oplus \Delta_k^{(\mathbf{j})}$ . Since  $\Gamma_k^{(\mathbf{j})} = \overline{\Gamma_k^{(\mathbf{j})}}$  for all  $k$ , and since  $[\Gamma_k^{(\mathbf{j})}, \Delta_k^{(\mathbf{j})}] \cap \Gamma_k^{(\mathbf{j})} = \{0\}$  by lemma 3.5 and (3.14), we deduce that  $[\Gamma_k^{(\mathbf{j})}, \Delta_k^{(\mathbf{j})}] + [\Delta_k^{(\mathbf{j})}, \Delta_k^{(\mathbf{j})}] \subset \Delta_k^{(\mathbf{j})}$ , hence  $[\Gamma_k^{(\mathbf{j})}, \Delta_k^{(\mathbf{j})}] \subset \Delta_k^{(\mathbf{j})}$  and  $[\Delta_k^{(\mathbf{j})}, \Delta_k^{(\mathbf{j})}] \subset \Delta_k^{(\mathbf{j})}$ , i.e.  $\overline{\Delta_k^{(\mathbf{j})}} = \Delta_k^{(\mathbf{j})}$ , for all  $k \geq 0$ .

Conversely,  $[\Gamma_k^{(\mathbf{j})}, \Delta_k^{(\mathbf{j})}] \subset \Delta_k^{(\mathbf{j})}$  and  $\Delta_k^{(\mathbf{j})} = \overline{\Delta_k^{(\mathbf{j})}}$  for all  $k \geq 0$  trivially implies that  $G_k^{(\mathbf{j})} = \overline{G_k^{(\mathbf{j})}}$  for all  $k \geq 0$ .

Moreover, since  $\dim \Gamma_k^{(\mathbf{j})}$  is constant by construction, the fact that  $G_k^{(\mathbf{j})}$  has locally constant dimension is equivalent to the fact that  $\Delta_k^{(\mathbf{j})}$  has locally constant dimension too for all  $k \geq 0$ , hence (i).

Finally, (iii) is an immediate consequence of the condition that  $G_k^{(\mathbf{j})} = \text{TX} \times \text{TR}^{m+|\mathbf{j}|}$  for all  $k \geq k_*^{(\mathbf{j})}$ , and the theorem is proven.  $\square$

**Remark 4.9.** Expressed in words, the above necessary and sufficient conditions read (i) involutivity with locally constant dimension of the  $\Delta_k^{(\mathbf{j})}$ 's, (ii) invariance of  $\Delta_k^{(\mathbf{j})}$  by  $\Gamma_k^{(\mathbf{j})}$  and (iii) strong controllability rank condition (see e.g. [37, 13]).

**4.3. The Pure Prolongation Algorithm.** From now on, for every sequence  $\mathbf{l} = (l_1, \dots, l_m) \in \mathbb{N}^m$ , we systematically re-order the indices  $\{1, \dots, m\}$  by a suitable permutation  $\alpha: \{1, \dots, m\} \rightarrow \{1, \dots, m\}$  such that  $0 = l_{\alpha(1)} \leq \dots \leq l_{\alpha(m)}$ . Moreover, for simplicity's sake, the permutation  $\alpha$  will be omitted. We will thus abusively replace  $l_{\alpha(i)}$  by  $l_i$ , i.e.  $0 = l_1 \leq \dots \leq l_m$ .

In the following algorithm we assume that the computations are done in a suitable open dense neighborhood of  $\bar{0}$  where all the distributions involved have constant dimension. Note that, in general, the dimension of some of these distributions may drop on some submanifolds that constitute *apparent or intrinsic flatness singularities*, as defined in [18, 19]. However, investigating such singularities is far beyond the scope of this paper.

4.3.1. *Initialization.* Consider the filtration  $\{G_k^{(0)} \mid k \geq 0\}$  defined by (2.19). If every  $G_k^{(0)}$  satisfies the conditions of theorem 2.9, the system is feedback linearizable and no prolongation is needed. In particular, for  $m = 1$ , a case where flatness and feedback linearizability are equivalent, the results of this section are pointless.

Otherwise, we have the following alternative:

- either there must exist  $k_0 \leq n$  such that  $G_{k_0}^{(0)}$  is not involutive while every  $G_k^{(0)}$  is involutive for  $k < k_0$  (note that  $k_0 \geq 1$  since  $G_0^{(0)}$  is always involutive),
- or the  $G_k^{(0)}$ 's are all involutive but with  $\max_{k \geq 0} \dim G_k^{(0)} = \dim G_{k_*}^{(0)} < n + m$ .

First Case. There exists  $k_0 \in \{1, \dots, n\}$ , first index for which  $G_{k_0}^{(0)}$  is not involutive.

Let  $H_1^{(0)} \neq \{0\}$  be any involutive distribution included in  $G_1^{(0)}$ , of the form

$$H_1^{(0)} \triangleq \left\{ \frac{\partial}{\partial u_1^{(0)}}, \dots, \text{ad}_{g_0^{(0)}} \frac{\partial}{\partial u_1^{(0)}}, \dots, \frac{\partial}{\partial u_{p_0}^{(0)}}, \dots, \text{ad}_{g_0^{(0)}} \frac{\partial}{\partial u_{p_0}^{(0)}} \right\} \quad (4.2)$$

for some  $p_0 \leq m$ .

By (A.3),  $\text{ad}_{g_0^{(0)}} \frac{\partial}{\partial u_p^{(0)}} = \text{ad}_{g_0^{(l_q-k)}} \frac{\partial}{\partial u_p^{(0)}}$  for all  $\mathbf{l}$  and, thanks to (3.10), we have  $[\frac{\partial}{\partial u_q^{(l_q-k)}}, \text{ad}_{g_0^{(0)}} \frac{\partial}{\partial u_p^{(0)}}] = 0$  for all  $l_q - k > 1$  with  $q > p_0$  and all  $p = 1, \dots, p_0$ . Therefore we can choose

$$\Delta_1^{(\mathbf{l})} \triangleq \left\{ \frac{\partial}{\partial u_1^{(0)}}, \dots, \text{ad}_{g_0^{(0)}} \frac{\partial}{\partial u_1^{(0)}}, \dots, \frac{\partial}{\partial u_{p_0}^{(0)}}, \dots, \text{ad}_{g_0^{(0)}} \frac{\partial}{\partial u_{p_0}^{(0)}} \right\} = H_1^{(0)} \quad (4.3)$$

and

$$\Gamma_1^{(\mathbf{l})} \triangleq \left\{ \frac{\partial}{\partial u_{p_0+1}^{(l_{p_0+1})}}, \frac{\partial}{\partial u_{p_0+1}^{(l_{p_0+1}-1)}}, \dots, \frac{\partial}{\partial u_m^{(l_m)}}, \frac{\partial}{\partial u_m^{(l_m-1)}} \right\} \quad (4.4)$$

for any

$$\mathbf{l} = \left( \underbrace{0, \dots, 0}_{p_0}, l_{p_0+1}, \dots, l_m \right), \quad 2 = l_{p_0+1} \leq \dots \leq l_m \quad (4.5)$$

since they naturally satisfy  $\overline{\Delta_1^{(\mathbf{l})}} = \overline{H_1^{(0)}} = H_1^{(0)} = \Delta_1^{(\mathbf{l})}$  and  $[\Gamma_1^{(\mathbf{l})}, \Delta_1^{(\mathbf{l})}] \subset \Delta_1^{(\mathbf{l})}$  for all  $\mathbf{l}$  given by (4.5), i.e. with  $l_{p_0+1} = 2$ .

However if, for  $l_{p_0+1} = 1$ ,

$$\Delta_1^{(\mathbf{l})} \triangleq \left\{ \frac{\partial}{\partial u_1^{(0)}}, \dots, \text{ad}_{g_0^{(0)}} \frac{\partial}{\partial u_1^{(0)}}, \dots, \frac{\partial}{\partial u_{p_0}^{(0)}}, \dots, \text{ad}_{g_0^{(0)}} \frac{\partial}{\partial u_{p_0}^{(0)}}, \frac{\partial}{\partial u_{p_0+1}^{(0)}} \right\} \quad (4.6)$$

and

$$\Gamma_1^{(\mathbf{l})} \triangleq \left\{ \frac{\partial}{\partial u_{p_0+1}^{(1)}}, \frac{\partial}{\partial u_{p_0+2}^{(l_2)}}, \frac{\partial}{\partial u_{p_0+2}^{(l_2-1)}}, \dots, \frac{\partial}{\partial u_m^{(l_m)}}, \frac{\partial}{\partial u_m^{(l_m-1)}} \right\} \quad (4.7)$$

also satisfy  $\overline{\Delta_1^{(\mathbf{l})}} = \Delta_1^{(\mathbf{l})}$  and  $[\Gamma_1^{(\mathbf{l})}, \Delta_1^{(\mathbf{l})}] \subset \Delta_1^{(\mathbf{l})}$ , the previous choice (4.3)-(4.4)-(4.5) maybe replaced by (4.6)-(4.7) with  $\mathbf{l} = l_{p_0+1} \leq \dots \leq l_m$ .

These choices, namely (4.3)-(4.4)-(4.5), or (4.6)-(4.7) with  $l_{p_0+1} = 1$ , for any  $H_1^{(0)}$  given by (4.2), constitutes the initialization of the algorithm.

Note that there are at most  $\sum_{p=1}^{m-1} C_m^p = \sum_{p=1}^{m-1} \frac{m!}{p!(m-p)!}$  possibilities of such initialization.

Second Case.

**Lemma 4.10.** *Assume that the  $G_k^{(0)}$ 's are all involutive with  $\max_{k \geq 0} \dim G_k^{(0)} < n + m$ . Then  $G_k^{(\mathbf{j})} = \overline{G_k^{(\mathbf{j})}} = G_k^{(0)}$  for all  $k$  and  $\mathbf{j}$  and  $\max_{k \geq 0} \dim G_k^{(\mathbf{j})} < m + n$  for all  $\mathbf{j}$ .*

*Proof.* According to (A.3) of lemma A.1, every  $\text{ad}_{g_0^{(j)}}^k \frac{\partial}{\partial u_p^{(j_p)}}$  is equal to  $\pm \text{ad}_{g_0^{(0)}}^{k-j_p} \frac{\partial}{\partial u_p^{(0)}}$  + terms in  $\overline{G_{k-1}^{(0)}} = G_{k-1}^{(0)}$ , thanks to the involutivity of the  $G_k^{(0)}$ 's. Therefore,  $G_k^{(\mathbf{j})} \subset G_k^{(0)}$ , the converse inclusion being immediate using (A.3). Thus  $\dim G_k^{(0)} = \dim G_k^{(\mathbf{j})}$  for all  $k$  and  $\mathbf{j}$ , Q.E.D.  $\square$

We immediately conclude that, in this case, the strong controllability rank condition does not hold, which contradicts (iii) of theorem 2.9, hence the non-flatness by pure prolongation. Therefore, this case must be discarded.

4.3.2. *Recursion.* For all  $k \geq 1$  and all  $\mathbf{l}$  given by:

$$\mathbf{l} = \left( \underbrace{0, \dots, 0}_{p_0}, l_{p_0+1}, \dots, l_m \right), \quad 1 \leq l_{p_0+1} \leq \dots \leq l_m, \quad (4.8)$$

and denoting by  $C\text{-}\min_{\mathbf{l} \in L} (l_{p_0+1}, \dots, l_m)$  (resp.  $C\text{-}\max_{\mathbf{l} \in L} (l_{p_0+1}, \dots, l_m)$ ) the componentwise minimum (resp. maximum) of a collection of  $(m - p_0)$ -tuples  $(l_{p_0+1}, \dots, l_m)$  in a set  $L$ , with  $1 \leq l_{p_0+1} \leq \dots \leq l_m$ , let us introduce the following numbers

$$\begin{aligned} \sigma_{\Delta}(k) &\triangleq (\sigma_{p_0+1, \Delta}(k), \dots, \sigma_{m, \Delta}(k)) \\ &\triangleq C\text{-}\min\{1 \leq l_{p_0+1} \leq \dots \leq l_m \mid \overline{\Delta_k^{(\mathbf{l})}} = \Delta_k^{(\mathbf{l})}\}, \end{aligned} \quad (4.9)$$

$$\begin{aligned} \sigma_{\Gamma, \Delta}(k) &\triangleq (\sigma_{p_0+1, \Gamma, \Delta}(k), \dots, \sigma_{m, \Gamma, \Delta}(k)) \\ &\triangleq C\text{-}\min\{1 \leq l_{p_0+1} \leq \dots \leq l_m \mid [\Gamma_k^{(\mathbf{l})}, \Delta_k^{(\mathbf{l})}] \subset \Delta_k^{(\mathbf{l})}\} \end{aligned} \quad (4.10)$$

and

$$\begin{aligned} &(j_{p_0+1}, \dots, j_m) \\ &\triangleq C\text{-}\max_{k \geq 1} (\sigma_{p_0+1, \Delta}(k) \wedge \sigma_{p_0+1, \Gamma, \Delta}(k), \dots, \sigma_{m, \Delta}(k) \wedge \sigma_{m, \Gamma, \Delta}(k)). \end{aligned} \quad (4.11)$$

**Remark 4.11.** For  $k = 1$ , thanks to (4.5)-(4.3)-(4.4), we get

$$\sigma_{p_0+1, \Delta}(1) \wedge \sigma_{p_0+1, \Gamma, \Delta}(1) \leq 2,$$

and if  $\overline{\Delta_1^{(\mathbf{l})}} = \Delta_1^{(\mathbf{l})}$  and  $[\Gamma_1^{(\mathbf{l})}, \Delta_1^{(\mathbf{l})}] \subset \Delta_1^{(\mathbf{l})}$ , with (4.6)-(4.7) and  $l_{p_0+1}, \dots, l_m \geq 1$ , we get

$$\sigma_{p_0+1, \Delta}(1) \wedge \sigma_{p_0+1, \Gamma, \Delta}(1) = 1.$$

**Remark 4.12.** In view of the definition (3.13) of  $\Delta_k^{(\mathbf{l})}$ , if  $l_q \leq k < l_{q+1}$  for some  $q \in \{1, \dots, m\}$  and  $k \geq 1$ ,  $\Delta_k^{(\mathbf{l})}$  does not depend on  $l_{q+1}, \dots, l_m$ . Thus

$$\sigma_{\Delta}(k) = \left( \sigma_{1, \Delta}(k), \dots, \sigma_{q, \Delta}(k), \underbrace{\sigma_{q, \Delta}(k), \dots, \sigma_{q, \Delta}(k)}_{m-q} \right).$$

**Proposition 4.13.** *We have*

$$[\Gamma_k^{(1)}, \Delta_k^{(1)}] = \{0\}, \quad \text{and} \quad \sigma_{p_0+1, \Gamma, \Delta}(k) \leq 2k \quad \forall l_{p_0+1} > 2k-1, \forall k \geq 0. \quad (4.12)$$

Moreover,  $\Delta_k^{(1)}$  is independent of  $l_{p_0+1}$  if  $l_{p_0+1} > k$ . Therefore, if  $\Delta_k^{(1)}$  is involutive for all  $k$ , we get

$$\sigma_{p_0+1, \Delta}(k) \wedge \sigma_{p_0+1, \Gamma, \Delta}(k) \leq 2k \quad (4.13)$$

and  $j_{p_0+1} \leq \dots \leq j_m < +\infty$ . The minimal prolongation orders are thus obtained by minimizing (4.11) over all initializations built on (4.2).

*Proof.* (4.13) is an immediate consequence of (3.10) and of the definitions of  $\Delta_k^{(1)}$  and  $\Gamma_k^{(1)}$ . Moreover, since, according to remark 3.6,  $k \mapsto \dim \Delta_k^{(1)}$  is non decreasing and bounded by  $n+m$ , then there exists a finite  $K^*$  such that  $\Delta_{K^*}^{(1)} = \Delta_k^{(1)}$  for all  $k \geq K^*$  and thus the maximum with respect to  $k$  in (4.11) is achieved for  $k = K^*$ . The C-minimal prolongation orders are thus obtained by varying the initializations built on (4.2).  $\square$

We thus have proven:

**Theorem 4.14.** *We have the following alternative:*

(1) *If, for every choice of initialization built on (4.2), there exists a  $k$  for which*

$$\max_{p \in \{p_0+1, \dots, m\}} \sigma_{p, \Delta}(k) = +\infty \quad \text{or} \quad \max_k \dim \Delta_k^{(\mathbf{j})} < n+m \quad \text{for all } \mathbf{j},$$

*then the system is not flat by pure prolongation.*

(2) *Otherwise,  $\mathbf{j}$  is finite and given by (4.11) and the minimal prolongation orders are obtained by C-minimizing (4.11) over all initializations.*

*Proof.* Straightforward from what precedes.  $\square$

We immediately deduce the following algorithm:

**Algorithm 1.** *Flatness by Pure Prolongation*

**Input:** the vector fields  $g_0^{(0)}, g_1^{(0)} = \frac{\partial}{\partial u_1^{(0)}}, \dots, g_m^{(0)} = \frac{\partial}{\partial u_m^{(0)}}$  (see (2.17))

**output:** the minimal lengths  $0 = j_1 \leq \dots \leq j_m$  and  $k_\star^{(\mathbf{j})}$  or FAIL if the system is not flat by pure prolongation.

**Initialization:** Construct  $G_k^{(0)}$  for all  $k \leq n$ . If  $\exists k_0 \leq n$  such that  $G_{k_0}^{(0)}$  is not involutive, choose an involutive subdistribution  $H_1^{(0)} \subset G_{k_0}^{(0)}$ , given by (4.2), with  $p_0 \leq m-1$  and compute  $\Delta_1^{(1)}$  and  $\Gamma_1^{(1)}$  by (4.3)-(4.4)-(4.5), or (4.6)-(4.7) with  $l_{p_0+1} = 1$ .

**Step  $k \geq 1$ :** Compute  $\Delta_k^{(1)}, \Gamma_k^{(1)}, \sigma_\Delta(k)$  and  $\sigma_{\Gamma, \Delta}(k)$  by (4.9) and (4.10). Continue up to the first  $k_1$  such that  $\sigma_{\Gamma, \Delta}(k_1) \wedge \sigma_\Delta(k_1)$  is C-maximal. Then  $\mathbf{j}$  is given by (4.11).

If  $\Delta_k^{(1)}$  is non involutive for some  $k \leq k_1$ , change the initialization and restart and if  $\Delta_k^{(1)}$  is non involutive for all initializations, then FAIL.

**Step  $k_\star^{(\mathbf{j})}$ :** Determine  $k_\star^{(\mathbf{j})}$ . If  $\dim \Delta_{k_\star^{(\mathbf{j})}}^{(\mathbf{j})} = n+m$ , STOP. Otherwise, FAIL.

## 5. EXAMPLES

All the examples of this paper concern two input systems, *i.e.*  $m = 2$ , except example 5.5, with 3 inputs, resuming the example 2 of [19].

In the two input examples, the prolongation index  $\mathbf{j} = (j_1, j_2)$  is supposed to be such that  $0 = j_1 \leq j_2$ , up to a suitable input permutation. However, for the sake of readability, we will keep the original input numbering unchanged and thus consider that  $\mathbf{j} = (0, j_2)$  or  $(j_1, 0)$  depending on the context. At the exception of this modification, we strictly apply algorithm 1 in all the examples.

The first example gives a detailed presentation of the application of this algorithm, in particular the role played by  $\sigma_{\Gamma, \Delta}(k)$  and  $\sigma_{\Delta}(k)$ . The second one shows the importance of the sole number  $\sigma_{1, \Delta}(k)$  to determine the prolongation length, and the third one, borrowed from [7], and carried on again in [2, Section 3.1], is reported here to compare our approach with the one of [2]. Then, the pendulum example is presented to show that non flat systems by pure prolongation can be detected in a finite number of steps. To conclude this section, the last example, with 3 inputs, compares the constuction proposed in [19, Example 2] with algorithm 1, leading to a minimal prolongation, strictly smaller than the one of [19].

**5.1. Chained System (continued).** Recall from example 3.8 that the system equations are given by (3.15):

$$\begin{aligned} x_1^{(3)} &= u_1 \\ \ddot{x}_2 &= u_2 \\ \dot{x}_3 &= u_1 u_2 \end{aligned}$$

This system has been proven to be flat in [27, section 3.1.1] with the following flat output

$$y_1 = x_3 - \ddot{x}_1 u_2 + \dot{x}_1 \dot{u}_2 - x_1 \ddot{u}_2, \quad y_2 = x_2. \quad (5.1)$$

**5.1.1. Associated Non Prolonged Distributions.** Let us first show that system (3.15) is not static feedback linearizable. Recall again from example 3.8 that we have

$$G_0^{(0)} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}} \right\} = \overline{G_0^{(0)}},$$

$$G_1^{(0)} = G_0^{(0)} + \left\{ -\frac{\partial}{\partial x_{1,3}} - u_2^{(0)} \frac{\partial}{\partial x_3}, -\frac{\partial}{\partial x_{2,2}} - u_1^{(0)} \frac{\partial}{\partial x_3} \right\} \neq \overline{G_1^{(0)}},$$

since, e.g.,  $[g_2^{(0)}, \text{ad}_{g_0^{(0)}} g_1^{(0)}] = -\frac{\partial}{\partial x_3} \notin G_1^{(0)}$ , and  $\dim G_1^{(0)} = 4$  whereas  $\dim \overline{G_1^{(0)}} = 5$ ,

$$G_2^{(0)} = G_1^{(0)} + \left\{ \frac{\partial}{\partial x_{1,2}}, \frac{\partial}{\partial x_{2,1}} \right\} \neq \overline{G_2^{(0)}}, \quad \dim \overline{G_2^{(0)}} = 7,$$

$$G_3^{(0)} = G_2^{(0)} + \left\{ \frac{\partial}{\partial x_{1,1}} \right\} \neq \overline{G_3^{(0)}}$$

and  $G_k^{(0)} = G_3^{(0)}$  for all  $k \geq 3$ . Moreover,  $\dim G_3^{(0)} = 7 < \dim \overline{G_3^{(0)}} = n + m = 8$ . We conclude that the system is not feedback linearizable.

Since  $G_1^{(0)}$  is not involutive, there are two possible initializations:

$$H_{1,1}^{(0)} = \{g_1^{(0)}, \text{ad}_{g_0^{(0)}} g_1^{(0)}\} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, -\frac{\partial}{\partial x_{1,3}} - u_2^{(0)} \frac{\partial}{\partial x_3} \right\} = \overline{H_{1,1}^{(0)}} \quad (5.2)$$

or

$$H_{1,2}^{(0)} = \{g_2^{(0)}, \text{ad}_{g_0^{(0)}} g_2^{(0)}\} = \left\{ \frac{\partial}{\partial u_2^{(0)}}, -\frac{\partial}{\partial x_{2,2}} - u_1^{(0)} \frac{\partial}{\partial x_3} \right\} = \overline{H_{1,2}^{(0)}}. \quad (5.3)$$

We use the second possibility, which amounts to prolonging the first input.

**5.1.2. Flatness by Pure Prolongation of the First Input.** Let us now apply theorem 4.8 and algorithm 1 with  $j_2 = 0$ , i.e.  $g_2^{(j_2)} \triangleq g_2^{(0)} = \frac{\partial}{\partial u_2^{(0)}}$  to determine if this system is flat by pure prolongation and compute  $j_1 \geq 1$ . Recall that we have set  $g_0^{(l_1,0)} = g_0^{(0)} + \sum_{p=0}^{l_1-1} u_1^{(p+1)} \frac{\partial}{\partial u_1^{(p)}}$ .

• **k = 0.** We have  $\Gamma_0^{(l_1,0)} = \left\{ \frac{\partial}{\partial u_1^{(l_1)}} \right\}$ ,  $\Delta_0^{(l_1,0)} = \left\{ \frac{\partial}{\partial u_2^{(0)}} \right\} = \overline{\Delta_0^{(l_1,0)}}$  and  $[\Gamma_0^{(l_1,0)}, \Delta_0^{(l_1,0)}] \subset \Delta_0^{(l_1,0)}$  for all  $l_1 \geq 1$ .

• **k = 1.** If  $l_1 \geq 2$ ,

$$\Gamma_1^{(l_1,0)} = \left\{ \frac{\partial}{\partial u_1^{(l_1)}}, \frac{\partial}{\partial u_1^{(l_1-1)}} \right\}$$

and

$$\Delta_1^{(l_1,0)} = \left\{ \frac{\partial}{\partial u_2^{(0)}}, \text{ad}_{g_0^{(l_1,0)}} \frac{\partial}{\partial u_2^{(0)}} \right\} = \left\{ \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial x_{2,2}} + u_1^{(0)} \frac{\partial}{\partial x_3} \right\} = \overline{\Delta_1^{(l_1,0)}}.$$

Moreover,  $[\Gamma_1^{(l_1,0)}, \Delta_1^{(l_1,0)}] = \{0\} \subset \Delta_1^{(l_1,0)}$ .

Now, for  $l_1 = 1$ , we have  $\Gamma_1^{(1,0)} = \left\{ \frac{\partial}{\partial u_1^{(1)}} \right\}$  and

$$\Delta_1^{(1,0)} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \text{ad}_{g_0^{(1,0)}} \frac{\partial}{\partial u_2^{(0)}} \right\} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial x_{2,2}} + u_1^{(0)} \frac{\partial}{\partial x_3} \right\}$$

which is not involutive.

Thus  $\sigma_{1,\Delta}(1) = 2$  and  $\sigma_{1,\Gamma,\Delta}(1) = 0$  which implies that

$$j_1 = \max_{k \geq 0} \sigma_{1,\Delta}(k) \wedge \sigma_{1,\Gamma,\Delta}(k) \geq \sigma_{1,\Delta}(1) \wedge \sigma_{1,\Gamma,\Delta}(1) = 2.$$

• **k = 2.** If  $l_1 \geq 3$ , we have

$$\Gamma_2^{(l_1,0)} = \left\{ \frac{\partial}{\partial u_1^{(l_1)}}, \frac{\partial}{\partial u_1^{(l_1-1)}}, \frac{\partial}{\partial u_1^{(l_1-2)}} \right\}$$

and

$$\begin{aligned} \Delta_2^{(l_1,0)} &= \left\{ \frac{\partial}{\partial u_2^{(0)}}, \text{ad}_{g_0^{(l_1,0)}} \frac{\partial}{\partial u_2^{(0)}}, \text{ad}_{g_0^{(l_1,0)}}^2 \frac{\partial}{\partial u_2^{(0)}} \right\} \\ &= \left\{ \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial x_{2,2}} + u_1^{(0)} \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_{2,1}} - u_1^{(1)} \frac{\partial}{\partial x_3} \right\} = \overline{\Delta_2^{(l_1,0)}}. \end{aligned}$$

Moreover, it is readily verified that  $[\Gamma_2^{(l_1,0)}, \Delta_2^{(l_1,0)}] \subset \Delta_2^{(l_1,0)}$  only if  $l_1 \geq 4$ , condition (ii) of theorem 4.8 being violated if  $l_1 = 3$  and we have  $\sigma_{1,\Delta}(2) = 0$  and  $\sigma_{1,\Gamma,\Delta}(2) = 4$  which implies that  $j_1 = \max_{k \geq 0} \sigma_{1,\Delta}(k) \wedge \sigma_{1,\Gamma,\Delta}(k) \geq \max_{r=1,2} \sigma_{1,\Delta}(r) \wedge \sigma_{1,\Gamma,\Delta}(r) \geq 4$ .

• **k = 3.** Again, if  $l_1 \geq 4$ , we have:

$$\Gamma_3^{(l_1,0)} = \left\{ \frac{\partial}{\partial u_1^{(l_1)}}, \frac{\partial}{\partial u_1^{(l_1-1)}}, \frac{\partial}{\partial u_1^{(l_1-2)}}, \frac{\partial}{\partial u_1^{(l_1-3)}} \right\}$$

and

$$\begin{aligned} \Delta_3^{(l_1,0)} &= \left\{ \frac{\partial}{\partial u_2^{(0)}}, \text{ad}_{g_0^{(l_1,0)}} \frac{\partial}{\partial u_2^{(0)}}, \text{ad}_{g_0^{(l_1,0)}}^2 \frac{\partial}{\partial u_2^{(0)}}, \text{ad}_{g_0^{(l_1,0)}}^3 \frac{\partial}{\partial u_2^{(0)}} \right\} \\ &= \left\{ \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial x_{2,2}} + u_1^{(0)} \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_{2,1}} - u_1^{(1)} \frac{\partial}{\partial x_3}, u_1^{(2)} \frac{\partial}{\partial x_3} \right\} \\ &= \left\{ \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial x_{2,2}}, \frac{\partial}{\partial x_{2,1}}, \frac{\partial}{\partial x_3} \right\} = \overline{\Delta_3^{(l_1,0)}} \end{aligned}$$

provided that  $u_1^{(2)} \neq 0$ . We also indeed have  $[\Gamma_3^{(l_1,0)}, \Delta_3^{(l_1,0)}] \subset \Delta_3^{(l_1,0)}$  for  $l_1 \geq 4$  hence  $\sigma_{1,\Gamma,\Delta}(3) = 4$  and  $j_1 \geq \max_{r=1,2,3} \sigma_{1,\Delta}(r) \wedge \sigma_{1,\Gamma,\Delta}(r) \geq 4$ .

• **k = 4.** If  $l_1 \geq 5$ , we have  $\Gamma_4^{(l_1,0)} = \left\{ \frac{\partial}{\partial u_1^{(l_1)}}, \dots, \frac{\partial}{\partial u_1^{(l_1-4)}} \right\}$  and  $\Delta_4^{(l_1,0)} = \Delta_3^{(l_1,0)}$  since  $\text{ad}_{g_0^{(l_1,0)}}^4 \frac{\partial}{\partial u_2^{(0)}} = -u_1^{(3)} \frac{\partial}{\partial x_3} \in \Delta_3^{(l_1,0)}$ .

If now  $l_1 = 4$ ,  $\Gamma_4^{(4,0)} = \left\{ \frac{\partial}{\partial u_1^{(4)}}, \dots, \frac{\partial}{\partial u_1^{(1)}} \right\}$  and

$$\begin{aligned} \Delta_4^{(4,0)} &= \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \text{ad}_{g_0^{(4,0)}} \frac{\partial}{\partial u_2^{(0)}}, \dots, \text{ad}_{g_0^{(4,0)}}^4 \frac{\partial}{\partial u_2^{(0)}} \right\} \\ &= \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial x_{2,2}}, \frac{\partial}{\partial x_{2,1}}, \frac{\partial}{\partial x_3} \right\} = \overline{\Delta_4^{(4,0)}}. \end{aligned}$$

We thus immediately get  $[\Gamma_4^{(l_1,0)}, \Delta_4^{(l_1,0)}] \subset \Delta_4^{(l_1,0)}$  for all  $l_1 \geq 4$ .

- $k \geq 5$ . Finally, the reader may easily check that  $\Gamma_k^{(4,0)} = \Gamma_4^{(4,0)}$  for all  $k \geq 5$  and

$$\begin{aligned}\Delta_5^{(4,0)} &= \Delta_4^{(4,0)} + \left\{ \text{ad}_{g_0^{(4,0)}} \frac{\partial}{\partial u_1^{(0)}} \right\} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial x_{1,3}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial x_{2,2}}, \frac{\partial}{\partial x_{2,1}}, \frac{\partial}{\partial x_3} \right\} \\ &= \overline{\Delta_5^{(4,0)}} \\ \Delta_6^{(4,0)} &= \Delta_5^{(4,0)} + \left\{ \text{ad}_{g_0^{(4,0)}}^2 \frac{\partial}{\partial u_1^{(0)}} \right\} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial x_{1,3}}, \frac{\partial}{\partial x_{1,2}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial x_{2,2}}, \frac{\partial}{\partial x_{2,1}}, \frac{\partial}{\partial x_3} \right\} \\ &= \overline{\Delta_6^{(4,0)}} \\ \Delta_7^{(4,0)} &= \Delta_6^{(4,0)} + \left\{ \text{ad}_{g_0^{(4,0)}}^3 \frac{\partial}{\partial u_1^{(0)}} \right\} = \mathbb{T}\mathbb{R}^8.\end{aligned}$$

We also indeed have  $[\Gamma_k^{(4,0)}, \Delta_k^{(4,0)}] \subset \Delta_k^{(4,0)}$  for all  $k \geq 5$ .

Hence, for all  $k \geq 0$ , the minimal  $\mathbf{j}$  is equal to  $(4, 0)$  and we conclude that system (3.15), with the first input channel prolonged up to  $j_1 = 4$ , *i.e.* controlled by  $u_1^{(5)}$ , is feedback linearizable.

Note that the bounds (3.5) and (3.21) are indeed satisfied. They read  $n + j_1 = 10 \geq k_\star^{(4,0)} = 7 \geq \frac{n+j_1}{2} \wedge j_1 = 5$ , but they are not tight.

Accordingly, evaluating the bound on the number of integrators needed to linearize the system proposed in [21, 36] for  $m = 2$ , we find  $2n - 3 = 9$ , and the one proposed by [12] gives  $2n - \frac{1}{6}(8 + 24 - 14) = 2n - 3 = 9$  as well.

**5.1.3. Flat output computation.** The prolonged system is now expressed in the state coordinates

$$\bar{x}^{(4,0)} \triangleq (x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_3, u_1^{(0)}, u_1^{(1)}, u_1^{(2)}, u_1^{(3)}, u_1^{(4)}, u_2^{(0)})$$

still with  $x_{1,j} \triangleq x_1^{(j-1)}$ ,  $j = 1, 2, 3$ ,  $x_{2,j} \triangleq x_2^{(j-1)}$ ,  $j = 1, 2$  and  $u_i^{(0)} = u_i$ ,  $i = 1, 2$ . We indeed still have  $n = 6$  but the prolonged state dimension is now equal to  $n + m + |j| = 12$  with the two input variables  $u_1^{(5)}$  and  $u_2^{(1)}$ .

The prolonged system vector fields are

$$g_0^{(4,0)} \triangleq g_0^{(0)} + \sum_{j=0}^3 u_1^{(j+1)} \frac{\partial}{\partial u_1^{(j)}}, \quad g_1^{(4)} \triangleq \frac{\partial}{\partial u_1^{(4)}}, \quad g_2^{(0)} \triangleq \frac{\partial}{\partial u_2^{(0)}}$$

and the corresponding distributions are, according to (3.20),

$$\begin{aligned}
G_0^{(4,0)} &= \left\{ \frac{\partial}{\partial u_1^{(4)}}, \frac{\partial}{\partial u_2^{(0)}} \right\} = \overline{G_0^{(4,0)}}, \\
G_1^{(4,0)} &= G_0^{(4,0)} + \left\{ \frac{\partial}{\partial u_1^{(3)}}, -\frac{\partial}{\partial x_{2,2}} - u_1^{(0)} \frac{\partial}{\partial x_3} \right\} = \overline{G_1^{(4,0)}}, \\
G_2^{(4,0)} &= G_1^{(4,0)} + \left\{ \frac{\partial}{\partial u_1^{(2)}}, \frac{\partial}{\partial x_{2,1}} - u_1^{(1)} \frac{\partial}{\partial x_3} \right\} = \overline{G_2^{(4,0)}}, \\
G_3^{(4,0)} &= G_2^{(4,0)} + \left\{ \frac{\partial}{\partial u_1^{(1)}}, -u_1^{(2)} \frac{\partial}{\partial x_3} \right\} = \overline{G_3^{(4,0)}}, \\
G_4^{(4,0)} &= G_3^{(4,0)} + \left\{ \frac{\partial}{\partial u_1^{(0)}} \right\} = \overline{G_4^{(4,0)}}, \\
G_5^{(4,0)} &= G_4^{(4,0)} + \left\{ \frac{\partial}{\partial x_{1,3}} \right\} = \overline{G_5^{(4,0)}}, \\
G_6^{(4,0)} &= G_5^{(4,0)} + \left\{ \frac{\partial}{\partial x_{1,2}} \right\} = \overline{G_6^{(4,0)}}, \\
G_7^{(4,0)} &= G_6^{(4,0)} + \left\{ \frac{\partial}{\partial x_{1,1}} \right\} = \overline{G_7^{(4,0)}} = \text{TR}^{12}.
\end{aligned}$$

This confirms that the system (3.15) is flat by pure prolongation in any neighborhood excluding  $u_1^{(2)} = 0$ , with  $k_*^{(4,0)} = 7$ . The reader may easily verify that  $\rho_k^{(4,0)} = 2$  for  $k = 0, \dots, 3$  and  $\rho_k^{(4,0)} = 1$  for  $k = 4, \dots, 7$ , which yields  $\kappa_1^{(4,0)} = 8$  and  $\kappa_2^{(4,0)} = 4$ .

The corresponding flat outputs are thus obtained by solving the set of P.D.E.'s

$$\begin{aligned}
\left\langle G_k^{(4,0)}, dy_1 \right\rangle &= 0, \quad k = 0, \dots, 6, & \left\langle G_7^{(4,0)}, dy_1 \right\rangle &\neq 0 \\
\left\langle G_k^{(4,0)}, dy_2 \right\rangle &= 0, \quad k = 0, \dots, 2, & \left\langle G_3^{(4,0)}, dy_2 \right\rangle &\neq 0
\end{aligned} \tag{5.4}$$

whose solution is

$$y_1 = x_1, \quad y_2 = x_3 - x_{2,2}u_1^{(0)} + x_{2,1}u_1^{(1)} = x_3 - \dot{x}_2u_1 + x_2\dot{u}_1. \tag{5.5}$$

**Remark 5.1.** Using the first initialization (5.2), which amounts to prolonging the second input, we leave to the reader the verification that one obtains a prolongation of order 6, thus non minimal, with flat outputs given by (5.1).

**Remark 5.2.** In [27, section 3.1.1], the authors consider a dual notion of minimality, called  $r$ -flatness, where  $r$  is the minimal number over all possible flat outputs of the maximal number of derivatives of the inputs that appear in the flat outputs, *i.e.*

$$r = \min_{Y: \text{ flat output}} \max_{i=1, \dots, m} \{s_i \mid \mathbf{s} = (s_1, \dots, s_m), y = Y(x, \bar{u}^{(\mathbf{s})})\}.$$

They conjectured that  $r$  should be equal to 1 in the present case (with their notations,  $\alpha_1 = 3$ ,  $\alpha_2 = 2$  and  $\min(\alpha_1, \alpha_2) - 1 = 1$ ). As the reader may easily verify, it is indeed minimal since the minimal  $\mathbf{j}$  is  $(4, 0)$  and moreover since, by the equations of the first line of (5.4),  $y_1$  neither

can depend on  $\bar{u}_1^{(4)}$  nor on  $u_2^{(0)}$ , and, by the second line,  $y_2$  cannot depend on  $u_1^{(4)}, u_1^{(3)}, u_1^{(2)}$  but explicitly depends on  $u_1^{(1)}$  by the definition of  $G_2^{(4,0)}$ .

## 5.2. 4-dimensional Driftless Bilinear System [27, 28, 29].

$$\begin{aligned}\dot{x}_1 &= u_1 \\ \dot{x}_2 &= x_3 u_1 \\ \dot{x}_3 &= x_4 u_1 \\ \dot{x}_4 &= u_2\end{aligned}\tag{5.6}$$

It is immediate to verify that this system is flat with flat output

$$y_1 = x_1, \quad y_2 = x_2\tag{5.7}$$

(see [28, 29] and [27, theorems 4 and 5]) but not static feedback linearizable.

According to our formalism, we consider the state  $(x_1, x_2, x_3, x_4, u_1^{(0)}, u_2^{(0)})$  of dimension 6, with  $n = 4$  and  $m = 2$ , and the new inputs  $(u_1^{(1)}, u_2^{(1)})$ . The associated vector fields are

$$g_0^{(0)} = u_1^{(0)} \left( \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} \right) + u_2^{(0)} \frac{\partial}{\partial x_4}, \quad g_1^{(0)} = \frac{\partial}{\partial u_1^{(0)}}, \quad g_2^{(0)} = \frac{\partial}{\partial u_2^{(0)}}.$$

The reader may easily verify that  $G_1^{(0)}$  is not involutive and that there are two possible initializations:

$$H_{1,1}^{(0)} = \{g_1^{(0)}, \text{ad}_{g_0^{(0)}} g_1^{(0)}\} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} \right\} = \overline{H_{1,1}^{(0)}}$$

or

$$H_{1,2}^{(0)} = \{g_2^{(0)}, \text{ad}_{g_0^{(0)}} g_2^{(0)}\} = \left\{ \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial x_4} \right\} = \overline{H_{1,2}^{(0)}}.$$

We use the second one, which amounts to prolonging the first input  $u_1^{(1)}$ .

5.2.1. *the distributions  $\Gamma_k^{(1)}$  and  $\Delta_k^{(1)}$ .* We set  $\mathbf{l} = (l_1, 0)$ ,  $l_1 \geq 1$ . We thus have

$$g_0^{(l_1,0)} = g_0^{(0)} + \sum_{p=0}^{l_1-1} u_1^{(p+1)} \frac{\partial}{\partial u_1^{(p)}}, \quad g_1^{(l_1)} = \frac{\partial}{\partial u_1^{(l_1)}}, \quad g_2^{(0)} = \frac{\partial}{\partial u_2^{(0)}}$$

and

$$\Gamma_0^{(l_1,0)} = \left\{ \frac{\partial}{\partial u_1^{(l_1)}} \right\}, \quad \Delta_0^{(l_1,0)} = \left\{ \frac{\partial}{\partial u_2^{(0)}} \right\}$$

for all  $l_1 \geq 1$ .

•  $\mathbf{k} = \mathbf{1}$ .. If  $l_1 \geq 2$ , we have

$$\Gamma_1^{(l_1,0)} = \left\{ \frac{\partial}{\partial u_1^{(l_1)}}, \frac{\partial}{\partial u_1^{(l_1-1)}} \right\}, \quad \Delta_1^{(l_1,0)} = \Delta_0^{(l_1,0)} + \left\{ \text{ad}_{g_0^{(l_1,0)}} \frac{\partial}{\partial u_2^{(0)}} \right\} = \left\{ \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial x_4} \right\}$$

and if  $l_1 = 1$ ,

$$\Gamma_1^{(1,0)} = \left\{ \frac{\partial}{\partial u_1^{(1)}} \right\}, \quad \Delta_1^{(1,0)} = \Delta_0^{(1,0)} + \left\{ \frac{\partial}{\partial u_1^{(0)}}, \text{ad}_{g_0^{(1,0)}} \frac{\partial}{\partial u_2^{(0)}} \right\} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial x_4} \right\}.$$

We thus have  $\Delta_1^{(l_1,0)} = \overline{\Delta_1^{(l_1,0)}}$  and  $[\Gamma_1^{(l_1,0)}, \Delta_1^{(l_1,0)}] = \{0\} \subset \Delta_1^{(l_1,0)}$  for all  $l_1 \geq 1$ , hence  $\sigma_{1,\Gamma,\Delta}(1) = \sigma_{1,\Delta}(1) = 0$ .

• **k = 2..** If  $l_1 \geq 3$ , we have

$$\Gamma_2^{(l_1,0)} = \left\{ \frac{\partial}{\partial u_1^{(l_1)}}, \frac{\partial}{\partial u_1^{(l_1-1)}}, \frac{\partial}{\partial u_1^{(l_1-2)}} \right\}, \quad \Delta_2^{(l_1,0)} = \left\{ \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_3} \right\},$$

$$[\Gamma_2^{(l_1,0)}, \Delta_2^{(l_1,0)}] = \{0\}$$

and if  $l_1 = 2$ ,

$$\Gamma_2^{(2,0)} = \left\{ \frac{\partial}{\partial u_1^{(2)}}, \frac{\partial}{\partial u_1^{(1)}} \right\}, \quad \Delta_2^{(2,0)} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_3} \right\},$$

$$[\Gamma_2^{(2,0)}, \Delta_2^{(2,0)}] = \{0\}.$$

Finally, if  $l_1 = 1$ ,

$$\Gamma_2^{(1,0)} = \left\{ \frac{\partial}{\partial u_1^{(1)}} \right\}, \quad \Delta_2^{(1,0)} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, -\frac{\partial}{\partial x_1} - x_3 \frac{\partial}{\partial x_2} - x_4 \frac{\partial}{\partial x_3}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_3} \right\}$$

$$\neq \overline{\Delta_2^{(1,0)}}.$$

We thus have  $\Delta_2^{(l_1,0)} = \overline{\Delta_2^{(l_1,0)}}$  and  $[\Gamma_2^{(l_1,0)}, \Delta_2^{(l_1,0)}] \subset \Delta_2^{(l_1,0)}$  for all  $l_1 \geq 2$  but, for  $l_1 = 1$ ,  $\Delta_2^{(1,0)}$  is not involutive. Therefore,  $\sigma_{1,\Delta}(2) = 2$  and our search may be restricted to  $l_1 \geq 2$ .

• **k = 3..** If  $l_1 \geq 4$ , we have

$$\Gamma_3^{(l_1,0)} = \left\{ \frac{\partial}{\partial u_1^{(l_1)}}, \frac{\partial}{\partial u_1^{(l_1-1)}}, \frac{\partial}{\partial u_1^{(l_1-2)}}, \frac{\partial}{\partial u_1^{(l_1-3)}} \right\}, \quad \Delta_3^{(l_1,0)} = \left\{ \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2} \right\}.$$

If  $l_1 = 3$ ,

$$\Gamma_3^{(3,0)} = \left\{ \frac{\partial}{\partial u_1^{(3)}}, \frac{\partial}{\partial u_1^{(2)}}, \frac{\partial}{\partial u_1^{(1)}} \right\}, \quad \Delta_3^{(3,0)} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2} \right\}.$$

If  $l_1 = 2$ ,

$$\Gamma_3^{(2,0)} = \left\{ \frac{\partial}{\partial u_1^{(2)}}, \frac{\partial}{\partial u_1^{(1)}} \right\}, \quad \Delta_3^{(2,0)} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1} \right\} = \text{TR}^6$$

Therefore,  $\Delta_k^{(2,0)} = \overline{\Delta_k^{(2,0)}}$  and  $[\Gamma_k^{(2,0)}, \Delta_k^{(2,0)}] \subset \Delta_k^{(2,0)}$  for all  $k \geq 0$  and  $\Delta_3^{(2,0)} = \text{TR}^6$ .

We conclude that the conditions of theorem 4.8 hold true whenever  $j_1 \geq 2$ , which proves that system (5.6) with the pure prolongation of order  $\mathbf{j} = (2, 0)$  is feedback linearizable.

The reader may easily check that, using the first initialization  $H_{1,1}^{(0)}$ , since  $\Delta_2^{(0,l_2)}$  is not involutive for all  $l_2 \geq 3$ , no pure prolongation of the second input channel,  $u_2^{(1)}$ , can lead to the linearizability conditions. Therefore the minimal prolongation is  $\mathbf{j} = (2, 0)$ .

5.2.2. *Flat output computation.* The prolonged state is now  $\bar{x}^{(2,0)} \triangleq (x_1, x_2, x_3, x_4, u_1^{(0)}, u_1^{(1)}, u_1^{(2)}, u_2^{(0)})$  of dimension 8, and the new inputs are  $(u_1^{(3)}, u_2^{(1)})$ .

$$\begin{aligned} G_0^{(2,0)} &= \left\{ \frac{\partial}{\partial u_1^{(2)}}, \frac{\partial}{\partial u_2^{(0)}} \right\} = \overline{G_0^{(2,0)}}, \\ G_1^{(2,0)} &= \left\{ \frac{\partial}{\partial u_1^{(2)}}, \frac{\partial}{\partial u_1^{(1)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial x_4} \right\} = \overline{G_1^{(2,0)}}, \\ G_2^{(2,0)} &= \left\{ \frac{\partial}{\partial u_1^{(2)}}, \frac{\partial}{\partial u_1^{(1)}}, \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_3} \right\} = \overline{G_2^{(2,0)}}, \\ G_3^{(2,0)} &= \left\{ \frac{\partial}{\partial u_1^{(2)}}, \frac{\partial}{\partial u_1^{(1)}}, \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2} \right\} \\ &= \overline{G_3^{(2,0)}} = \mathbb{T}\mathbb{R}^8 \end{aligned}$$

hence the feedback linearizability of the purely prolonged system with  $\rho_0^{(2,0)} = \rho_1^{(2,0)} = \rho_2^{(2,0)} = \rho_3^{(2,0)} = 2$  and  $\kappa_1^{(2,0)} = \kappa_2^{(2,0)} = 4$ .

Flat outputs  $(y_1, y_2)$  are locally non trivial solutions of the system (5.8), i.e. :

$$\left\langle G_k^{(2,0)}, dy_i \right\rangle = 0, \quad k = 0, 1, 2, \quad \text{with} \quad \left\langle G_3^{(2,0)}, dy_i \right\rangle \neq 0, \quad i = 1, 2. \quad (5.8)$$

It is immediate to verify that

$$y_1 = x_1, \quad y_2 = x_2$$

is a solution of (5.8) and that the mapping

$$\bar{x}^{(2,0)} \mapsto \left( y_1, \dots, y_1^{(3)}, y_2, \dots, y_2^{(3)} \right)$$

is a local diffeomorphism.

5.3. **An Example from [7].** In our formalism, considering the inputs  $(u_1, u_2) \triangleq (u_1^{(0)}, u_2^{(0)})$  as part of the state, with  $n = 4$  and  $m = 2$ , this example from [7, Example 2] reads:

$$\begin{aligned} \dot{x}_1 &= x_2 + x_3 u_2^{(0)} \\ \dot{x}_2 &= x_3 + x_1 u_2^{(0)} \\ \dot{x}_3 &= u_1^{(0)} + x_2 u_2^{(0)} \\ \dot{x}_4 &= u_2^{(0)} \\ \dot{u}_1^{(0)} &= u_1^{(1)} \\ \dot{u}_2^{(0)} &= u_2^{(1)}. \end{aligned} \quad (5.9)$$

It is shown in [7] that this system does not satisfy the sufficient, but otherwise not necessary, condition for dynamic linearization of theorem 4.2 of this paper. Nevertheless, it satisfies the algorithm 1 of [2, section 3.1], that constitutes a sufficient condition for flatness by pure prolongation, without proof of minimality of the obtained prolongation. We show here that it is

linearizable by pure prolongation by application of algorithm 1, thus providing the minimal prolongation.

The non prolonged vector fields are

$$\begin{aligned} g_0^{(0)} &= (x_2 + x_3 u_2^{(0)}) \frac{\partial}{\partial x_1} + (x_3 + x_1 u_2^{(0)}) \frac{\partial}{\partial x_2} + (u_1^{(0)} + x_2 u_2^{(0)}) \frac{\partial}{\partial x_3} + u_2^{(0)} \frac{\partial}{\partial x_4} \\ g_1^{(0)} &= \frac{\partial}{\partial u_1^{(0)}}, \quad g_2^{(0)} = \frac{\partial}{\partial u_2^{(0)}}. \end{aligned}$$

and it is easily seen that

$$G_1^{(0)} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial x_3}, -x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4} \right\} \neq \overline{G_1^{(0)}}$$

and that the subdistributions

$$H_{1,1}^0 = \{g_1^{(0)}, \text{ad}_{g_0^{(0)}} g_1^{(0)}\} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial x_3} \right\} = \overline{H_{1,1}^0}$$

or

$$H_{1,2}^0 = \{g_2^{(0)}, \text{ad}_{g_0^{(0)}} g_2^{(0)}\} = \left\{ \frac{\partial}{\partial u_2^{(0)}}, -x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4} \right\} = \overline{H_{1,2}^0}$$

can be taken as possible initializations.

Choosing  $H_{1,1}^0$  amounts to prolonging the second input at an arbitrary order  $l_2 \geq 1$  and set  $\mathbf{l} = (0, l_2)$ . For  $l_2 \geq 1$ , we denote, as before,

$$\begin{aligned} g_0^{(0,l_2)} &= (x_2 + x_3 u_2^{(0)}) \frac{\partial}{\partial x_1} + (x_3 + x_1 u_2^{(0)}) \frac{\partial}{\partial x_2} + (u_1^{(0)} + x_2 u_2^{(0)}) \frac{\partial}{\partial x_3} \\ &\quad + u_2^{(0)} \frac{\partial}{\partial x_4} + \sum_{p=0}^{l_2-1} u_2^{(p+1)} \frac{\partial}{\partial u_2^{(p)}} \\ g_1^{(0)} &= \frac{\partial}{\partial u_1^{(0)}}, \quad g_2^{(l_2)} = \frac{\partial}{\partial u_2^{(l_2)}} \end{aligned}$$

and we indeed have

$$\Gamma_0^{(0,l_2)} = \left\{ \frac{\partial}{\partial u_2^{(l_2)}} \right\}, \quad \Delta_0^{(0,l_2)} = \left\{ \frac{\partial}{\partial u_1^{(0)}} \right\} = \overline{\Delta_0^{(0,l_2)}}, \quad [\Gamma_0^{(0,l_2)}, \Delta_0^{(0,l_2)}] \subset \Delta_0^{(0,l_2)}, \quad \forall l_2 \geq 1.$$

•  $\mathbf{k} = \mathbf{1}$ . For all  $l_2 \geq 2$ ,  $\text{ad}_{g_0^{(0,l_2)}} \frac{\partial}{\partial u_1^{(0)}} = -\frac{\partial}{\partial x_3}$  and

$$\begin{aligned} \Gamma_1^{(0,l_2)} &= \left\{ \frac{\partial}{\partial u_2^{(l_2)}}, \frac{\partial}{\partial u_2^{(l_2-1)}} \right\}, \quad \Delta_1^{(0,l_2)} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, -\frac{\partial}{\partial x_3} \right\} = \overline{\Delta_1^{(0,l_2)}}, \\ &[\Gamma_1^{(0,l_2)}, \Delta_1^{(0,l_2)}] \subset \Delta_1^{(0,l_2)}. \end{aligned}$$

For  $l_2 = 1$ :

$$\Gamma_1^{(0,1)} = \left\{ \frac{\partial}{\partial u_2^{(1)}} \right\}, \quad \Delta_1^{(0,1)} = \left\{ \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_1^{(0)}}, -\frac{\partial}{\partial x_3} \right\} = \overline{\Delta_1^{(0,1)}},$$

$$[\Gamma_1^{(0,1)}, \Delta_1^{(0,1)}] \subset \Delta_1^{(0,1)}.$$

• **k = 2.** For all  $l_2 \geq 3$ ,  $\text{ad}_{g_0}^2 \frac{\partial}{\partial u_1^{(0)}} = u_2^{(0)} \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$  and

$$\Gamma_2^{(0,l_2)} = \left\{ \frac{\partial}{\partial u_2^{(l_2)}}, \frac{\partial}{\partial u_2^{(l_2-1)}}, \frac{\partial}{\partial u_2^{(l_2-2)}} \right\},$$

$$\Delta_2^{(0,l_2)} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, -\frac{\partial}{\partial x_3}, u_2^{(0)} \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right\} = \overline{\Delta_2^{(0,l_2)}},$$

$$[\Gamma_2^{(0,l_2)}, \Delta_2^{(0,l_2)}] \subset \Delta_2^{(0,l_2)}.$$

But for  $l_2 = 2$ :

$$\Gamma_2^{(0,2)} = \left\{ \frac{\partial}{\partial u_2^{(2)}}, \frac{\partial}{\partial u_2^{(1)}} \right\}, \quad \Delta_2^{(0,2)} = \left\{ \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_1^{(0)}}, -\frac{\partial}{\partial x_3}, u_2^{(0)} \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right\} \neq \overline{\Delta_2^{(0,2)}}$$

therefore,  $\sigma_{2,\Delta}(2) = 2$  and we must exclude  $j_2 = 2$ .

• **k = 3.** For all  $l_2 \geq 4$ ,  $\text{ad}_{g_0}^3 \frac{\partial}{\partial u_1^{(0)}} = (u_2^{(1)} - 1) \frac{\partial}{\partial x_1} - (u_2^{(0)})^2 \frac{\partial}{\partial x_2} - u_2^{(0)} \frac{\partial}{\partial x_3}$  and, if we exclude the points where  $u_2^{(0)} = 0$  and  $u_2^{(1)} = 1$ ,

$$\Gamma_3^{(0,l_2)} = \left\{ \frac{\partial}{\partial u_2^{(l_2)}}, \frac{\partial}{\partial u_2^{(l_2-1)}}, \frac{\partial}{\partial u_2^{(l_2-2)}}, \frac{\partial}{\partial u_2^{(l_2-3)}} \right\},$$

$$\Delta_3^{(0,l_2)} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, -\frac{\partial}{\partial x_3}, u_2^{(0)} \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, (u_2^{(1)} - 1) \frac{\partial}{\partial x_1} + (u_2^{(0)})^2 \frac{\partial}{\partial x_2} - u_2^{(0)} \frac{\partial}{\partial x_3} \right\}$$

$$= \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1} \right\} = \overline{\Delta_3^{(0,l_2)}},$$

$$[\Gamma_3^{(0,l_2)}, \Delta_3^{(0,l_2)}] \subset \Delta_3^{(0,l_2)}.$$

The reader may then easily check that the same holds for  $l_2 = 3$ :

$$\Gamma_3^{(0,3)} = \left\{ \frac{\partial}{\partial u_2^{(3)}}, \frac{\partial}{\partial u_2^{(2)}}, \frac{\partial}{\partial u_2^{(1)}} \right\},$$

$$\Delta_3^{(0,3)} = \left\{ \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1} \right\} = \overline{\Delta_3^{(0,3)}}.$$

• **For all  $k \geq 4$ .** Since  $\text{ad}_{g_0}^{k(0,l_2)} \frac{\partial}{\partial u_1^{(0)}}$  and  $\text{ad}_{g_0}^{k-3(0,l_2)} \frac{\partial}{\partial u_2^{(0)}}$  are linear combinations of  $\frac{\partial}{\partial x_1}$ ,  $\frac{\partial}{\partial x_2}$  and  $\frac{\partial}{\partial x_3}$  only, we have, for all  $l_2 \geq 4$ :

$$\Delta_k^{(0,l_2)} = \overline{\Delta_k^{(0,l_2)}}, \quad \left[ \Gamma_k^{(0,l_2)}, \Delta_k^{(0,l_2)} \right] \subset \Delta_k^{(0,l_2)},$$

and, for  $l_2 = 3$ , using the fact that  $\text{ad}_{g_0}^{(0,3)} \frac{\partial}{\partial u_2^{(0)}} = -x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}$ , we have

$$\Gamma_4^{(0,3)} = \left\{ \frac{\partial}{\partial u_2^{(3)}}, \frac{\partial}{\partial u_2^{(2)}}, \frac{\partial}{\partial u_2^{(1)}} \right\},$$

$$\begin{aligned} \Delta_4^{(0,3)} &= \left\{ \frac{\partial}{\partial u_2^{(0)}}, -x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}, \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1} \right\} \\ &= \left\{ \frac{\partial}{\partial u_2^{(0)}}, \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1} \right\} = \overline{\Delta_4^{(0,3)}} = \text{TRR}^6 \end{aligned}$$

and

$$\Delta_k^{(0,3)} = \text{TRR}^6 \quad \forall k \geq 4,$$

hence  $j_2 = 3$  and  $k_*^{(0,3)} = 4$ .

We conclude that the conditions of theorem 4.8 are satisfied for all  $k$  provided that  $j_2 = 3$ .

On the contrary, initializing the algorithm by  $H_{1,2}^0$ , which amounts to prolonging the first input  $u_1^{(1)}$ , we may easily check that, for all  $l_1 \geq 3$ ,

$$\Delta_2^{(l_1,0)} = \left\{ \frac{\partial}{\partial u_2^{(0)}}, -x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}, (x_1 - u_1^{(0)}) \frac{\partial}{\partial x_1} - x_3 \frac{\partial}{\partial x_3} \right\}$$

is not involutive, thus contradicting condition (i) of theorem 4.8, which proves that the minimal prolongation of the second input is equal to 3.

Let us finally give the construction of the flat output and prolonged state diffeomorphism. The prolonged state is  $(x_1, x_2, x_3, x_4, u_1^{(0)}, u_2^{(0)}, u_2^{(1)}, u_2^{(2)}, u_2^{(3)})$  of dimension  $9 = n + m + |\mathbf{j}|$ .

We get

$$\begin{aligned} G_0^{(0,3)} &= \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(3)}} \right\} = \overline{G_0^{(0,3)}}, \quad G_1^{(0,3)} = \left\{ \frac{\partial}{\partial x_3}, \frac{\partial}{\partial u_2^{(2)}} \right\} \oplus G_0^{(0,3)} = \overline{G_1^{(0,3)}}, \\ G_2^{(0,3)} &= \left\{ u_2^{(0)} \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, \frac{\partial}{\partial u_2^{(1)}} \right\} \oplus G_1^{(0,3)} = \overline{G_2^{(0,3)}}, \\ G_3^{(0,3)} &= \left\{ (u_2^{(1)} - 1) \frac{\partial}{\partial x_1} - u_2^{(0)} \frac{\partial}{\partial x_3}, \frac{\partial}{\partial u_2^{(0)}} \right\} \oplus G_2^{(0,3)} = \overline{G_3^{(0,3)}}, \\ G_4^{(0,3)} &= \left\{ -x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4} \right\} \oplus G_3^{(0,3)} = \text{TRR}^9 \end{aligned}$$

The Brunovsky's controllability indices are  $\kappa_1^{(0,3)} = 5$  and  $\kappa_2^{(0,3)} = 4$  and the system of PDE's that the flat outputs must satisfy is :

$$\begin{aligned} \langle G_k^{(0,3)}, dy_1 \rangle &= 0, \quad k = 0, \dots, 3, & \langle G_4^{(0,3)}, dy_1 \rangle &\neq 0 \\ \langle G_k^{(0,3)}, dy_2 \rangle &= 0, \quad k = 0, \dots, 2, & \langle G_3^{(0,3)}, dy_2 \rangle &\neq 0. \end{aligned}$$

Its solution is given by  $y_1 = x_4, y_2 = x_1 - u_2^{(0)} x_2$ .

**5.4. The Pendulum Example** [10, section II. C]. This model of pendulum in the vertical plane has been studied in [10, section II. C], [23, section 6.2.3],[24, section 5.3] where it is shown to be flat. We prove here that it is not flat by pure prolongation.

Though naturally control-affine, it is presented here in its prolonged form (2.16):

$$\begin{aligned} \dot{x}_1 &= x_2 & \dot{x}_2 &= u_1^{(0)} \\ \dot{y}_1 &= y_2 & \dot{y}_2 &= u_2^{(0)} \\ \dot{\theta}_1 &= \theta_2 & \dot{\theta}_2 &= -\frac{u_1^{(0)}}{\varepsilon} \cos \theta_1 + \frac{u_2^{(0)} + 1}{\varepsilon} \sin \theta_1 \\ \dot{u}_1^{(0)} &= u_1^{(1)} & \dot{u}_2^{(0)} &= u_2^{(1)}. \end{aligned} \tag{5.10}$$

The state is  $(x_1, x_2, y_1, y_2, \theta_1, \theta_2, u_1^{(0)}, u_2^{(0)})$ , of dimension  $n + m = 6 + 2 = 8$ . The associated non prolonged vector fields are

$$\begin{aligned} g_0^{(0)} &= x_2 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial y_1} + \theta_2 \frac{\partial}{\partial \theta_1} + \frac{1}{\varepsilon} \sin \theta_1 \frac{\partial}{\partial \theta_2} \\ &+ u_1^{(0)} \left( \frac{\partial}{\partial x_2} - \frac{1}{\varepsilon} \cos \theta_1 \frac{\partial}{\partial \theta_2} \right) + u_2^{(0)} \left( \frac{\partial}{\partial y_2} + \frac{1}{\varepsilon} \sin \theta_1 \frac{\partial}{\partial \theta_2} \right) \\ g_1^{(0)} &= \frac{\partial}{\partial u_1^{(0)}} & g_2^{(0)} &= \frac{\partial}{\partial u_2^{(0)}} \end{aligned} \tag{5.11}$$

We have

$$G_0^{(0)} = \{g_1^{(0)}, g_2^{(0)}\} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}} \right\}$$

and

$$\begin{aligned} G_1^{(0)} &= \{g_1^{(0)}, g_2^{(0)}, \text{ad}_{g_0^{(0)}} g_1^{(0)}, \text{ad}_{g_0^{(0)}} g_2^{(0)}\} \\ &= \left\{ \frac{\partial}{\partial u_1^{(0)}}, \frac{\partial}{\partial u_2^{(0)}}, -\frac{\partial}{\partial x_2} + \frac{1}{\varepsilon} \cos \theta_1 \frac{\partial}{\partial \theta_2}, -\frac{\partial}{\partial y_2} - \frac{1}{\varepsilon} \sin \theta_1 \frac{\partial}{\partial \theta_2} \right\} = \overline{G_1^{(0)}} \end{aligned}$$

but

$$\begin{aligned} G_2^{(0)} &= G_1^{(0)} + \left\{ \frac{\partial}{\partial x_1} - \frac{1}{\varepsilon} \cos \theta_1 \frac{\partial}{\partial \theta_1} - \frac{\theta_2}{\varepsilon} \sin \theta_1 \frac{\partial}{\partial \theta_2}, \frac{\partial}{\partial y_1} + \frac{1}{\varepsilon} \sin \theta_1 \frac{\partial}{\partial \theta_1} - \frac{\theta_2}{\varepsilon} \cos \theta_1 \frac{\partial}{\partial \theta_2} \right\} \\ &\neq \overline{G_2^{(0)}}. \end{aligned}$$

As before, we may initialize the pure prolongation algorithm by choosing

$$H_{1,1}^0 = \{g_1^{(0)}, \text{ad}_{g_0^{(0)}} g_1^{(0)}\} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, -\frac{\partial}{\partial x_2} + \frac{1}{\varepsilon} \cos \theta_1 \frac{\partial}{\partial \theta_2} \right\} = \overline{H_{1,1}^0}$$

which amounts to prolonging  $u_2^{(1)}$ , or

$$H_{1,2}^0 = \{g_2^{(0)}, \text{ad}_{g_0^{(0)}} g_2^0\} = \left\{ \frac{\partial}{\partial u_2^{(0)}}, -\frac{\partial}{\partial y_2} - \frac{1}{\varepsilon} \sin \theta_1 \frac{\partial}{\partial \theta_2} \right\} = \overline{H_{1,2}^0}$$

which amounts to prolonging  $u_1^{(1)}$ .

It is noticeable that the inputs  $u_1^{(1)}$  and  $u_2^{(1)}$  play a symmetric role. Thus, one may choose indifferently one of them as the non prolonged input. Let us choose  $u_1^{(1)}$  as non prolonged input, with the initialization  $H_{1,1}^0$ . Thus, the vector fields associated to a prolongation of length  $l_2$  on the second input read:

$$\begin{aligned} g_0^{(0,l_2)} &= x_2 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial y_1} + \theta_2 \frac{\partial}{\partial \theta_1} + \frac{1}{\varepsilon} \sin \theta_1 \frac{\partial}{\partial \theta_2} + u_1^{(0)} \left( \frac{\partial}{\partial x_2} - \frac{1}{\varepsilon} \cos \theta_1 \frac{\partial}{\partial \theta_2} \right) \\ &\quad + u_2^{(0)} \left( \frac{\partial}{\partial y_2} + \frac{1}{\varepsilon} \sin \theta_1 \frac{\partial}{\partial \theta_2} \right) + \sum_{p=0}^{l_2-1} u_2^{(p+1)} \frac{\partial}{\partial u_2^{(p)}} \end{aligned} \quad (5.12)$$

$$g_1^{(0)} = \frac{\partial}{\partial u_1^{(0)}} \quad g_2^{(l_2)} = \frac{\partial}{\partial u_2^{(l_2)}}.$$

• **k = 1.** For all  $l_2 \geq 2$ ,

$$\Gamma_1^{(0,l_2)} = \left\{ \frac{\partial}{\partial u_2^{(l_2)}}, \frac{\partial}{\partial u_2^{(l_2-1)}} \right\}, \quad \Delta_1^{(0,l_2)} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, -\frac{\partial}{\partial x_2} + \frac{1}{\varepsilon} \cos \theta_1 \frac{\partial}{\partial \theta_2} \right\},$$

and if  $l_2 = 1$ ,

$$\Gamma_1^{(0,1)} = \left\{ \frac{\partial}{\partial u_2^{(1)}} \right\}, \quad \Delta_1^{(0,1)} = \left\{ \frac{\partial}{\partial u_1^{(0)}}, -\frac{\partial}{\partial x_2} + \frac{1}{\varepsilon} \cos \theta_1 \frac{\partial}{\partial \theta_2}, \frac{\partial}{\partial u_2^{(0)}} \right\}.$$

Thus, we have  $\overline{\Delta_1^{(0,l_2)}} = \Delta_1^{(0,l_2)}$  and  $[\Gamma_1^{(0,l_2)}, \Delta_1^{(0,l_2)}] \subset \Delta_1^{(0,l_2)}$  for all  $l_2 \geq 1$ .

• **k = 2.** For all  $l_2 \geq 3$ ,

$$\begin{aligned} \Gamma_2^{(0,l_2)} &= \left\{ \frac{\partial}{\partial u_2^{(l_2)}}, \frac{\partial}{\partial u_2^{(l_2-1)}}, \frac{\partial}{\partial u_2^{(l_2-2)}} \right\}, \\ \Delta_2^{(0,l_2)} &= \left\{ \frac{\partial}{\partial u_1^{(0)}}, -\frac{\partial}{\partial x_2} + \frac{1}{\varepsilon} \cos \theta_1 \frac{\partial}{\partial \theta_2}, \frac{\partial}{\partial x_1} - \frac{1}{\varepsilon} \cos \theta_1 \frac{\partial}{\partial \theta_1} - \frac{1}{\varepsilon} \theta_2 \sin \theta_1 \frac{\partial}{\partial \theta_2} \right\}. \end{aligned}$$

If  $l_2 = 2$ ,

$$\begin{aligned} \Gamma_2^{(0,2)} &= \left\{ \frac{\partial}{\partial u_2^{(2)}}, \frac{\partial}{\partial u_2^{(1)}} \right\}, \\ \Delta_2^{(0,2)} &= \left\{ \frac{\partial}{\partial u_1^{(0)}}, -\frac{\partial}{\partial x_2} + \frac{1}{\varepsilon} \cos \theta_1 \frac{\partial}{\partial \theta_2}, \frac{\partial}{\partial x_1} - \frac{1}{\varepsilon} \cos \theta_1 \frac{\partial}{\partial \theta_1} - \frac{1}{\varepsilon} \theta_2 \sin \theta_1 \frac{\partial}{\partial \theta_2}, \frac{\partial}{\partial u_2^{(0)}} \right\}. \end{aligned}$$

If  $l_2 = 1$ ,

$$\begin{aligned}\Gamma_2^{(0,1)} &= \left\{ \frac{\partial}{\partial u_2^{(1)}} \right\}, \\ \Delta_2^{(0,1)} &= \left\{ \frac{\partial}{\partial u_1^{(0)}}, -\frac{\partial}{\partial x_2} + \frac{1}{\varepsilon} \cos \theta_1 \frac{\partial}{\partial \theta_2}, \frac{\partial}{\partial x_1} - \frac{1}{\varepsilon} \cos \theta_1 \frac{\partial}{\partial \theta_1} - \frac{1}{\varepsilon} \theta_2 \sin \theta_1 \frac{\partial}{\partial \theta_2}, \right. \\ &\quad \left. \frac{\partial}{\partial u_2^{(0)}}, -\frac{\partial}{\partial y_2} - \frac{1}{\varepsilon} \sin \theta_1 \frac{\partial}{\partial \theta_2} \right\}.\end{aligned}$$

Again, we have  $[\Gamma_2^{(0,l_2)}, \Delta_2^{(0,l_2)}] \subset \Delta_2^{(0,l_2)}$  for all  $l_2 \geq 1$ , but  $\overline{\Delta_2^{(0,l_2)}} \neq \Delta_2^{(0,l_2)}$  for all  $l_2 \geq 1$  since, e.g.

$$\left[ -\frac{\partial}{\partial x_2} + \frac{1}{\varepsilon} \cos \theta_1 \frac{\partial}{\partial \theta_2}, \frac{\partial}{\partial x_1} - \frac{1}{\varepsilon} \cos \theta_1 \frac{\partial}{\partial \theta_1} - \frac{1}{\varepsilon} \theta_2 \sin \theta_1 \frac{\partial}{\partial \theta_2} \right] = -\frac{1}{\varepsilon^2} \sin 2\theta_1 \frac{\partial}{\partial \theta_2} \notin \Delta_2^{(0,l_2)}$$

for all  $l_2 \geq 1$ . Changing the non prolonged input  $u_1^{(0)}$  in  $u_2^{(0)}$ , as previously announced, a similar calculation, left to the reader, shows that  $\Delta_2^{(l_1,0)}$  is not involutive for all  $l_1 \geq 1$ . Thus, according to the first item of theorem 4.14, system (5.10) is not flat by pure prolongation, though differentially flat, as shown in [10, section C].

### 5.5. An example with 3 inputs [19].

$$\begin{aligned}\dot{x}_1 &= u_1^{(0)} \\ \dot{x}_2 &= x_3 u_1^{(0)} \\ \dot{x}_3 &= x_4 u_1^{(0)} + x_1 u_3^{(0)} \\ \dot{x}_4 &= u_2^{(0)}.\end{aligned}\tag{5.13}$$

The state is  $(x_1, \dots, x_4, u_1^{(0)}, u_2^{(0)}, u_3^{(0)})$  (of dimension  $n + m = 4 + 3 = 7$ ) and the control inputs are  $(u_1^{(1)}, u_2^{(1)}, u_3^{(1)})$ . The reader may easily see that  $G_1^{(0)}$  is not involutive and thus that this system is not static feedback linearizable.

This example may be found in [19] in the context of control affine systems with  $n$  states and  $n - 1$  inputs with the property that there exists  $i = 1, 2, 3$  such that

$$\dim\{f_1, f_2, f_3, \text{ad}g f_i\} = 4,\tag{5.14}$$

with

$$f_1 = \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3}, \quad f_2 = \frac{\partial}{\partial x_4}, \quad f_3 = x_1 \frac{\partial}{\partial x_3}, \quad g = u_1^{(0)} f_1 + u_2^{(0)} f_2 + u_3^{(0)} f_3,$$

a property satisfied for  $i = 3$  since, in a suitable neighborhood excluding  $u_1^{(0)} = 0$ ,

$$\text{ad}g f_3 = u_1^{(0)} [f_1, f_3] = u_1^{(0)} \left( \frac{\partial}{\partial x_3} - x_1 \frac{\partial}{\partial x_2} \right) \notin \{f_1, f_2, f_3\}.$$

Therefore, in [19, theorem 3], flat outputs are exhibited as 3 independent first integrals of  $f_3$ , yielding the prolongation  $\dot{u}_1^{(1)} = u_1^{(2)}$ ,  $\dot{u}_2^{(1)} = u_2^{(2)}$ , with  $\mathbf{j} = (1, 1, 0)$  and thus  $|\mathbf{j}| = 2$ .

We show here that this prolongation is not minimal and we compute the minimal one.

We initialize the algorithm by remarking that  $\{f_2, f_3\}$  is the unique maximal involutive sub-distribution of  $\{f_1, f_2, f_3\}$ , thus indicating that  $u_2^{(1)}$  and  $u_3^{(1)}$  are suitable candidates of non prolonged inputs. We thus, consider the following vector fields, corresponding to prolongations of  $u_1^{(1)}$ , for all  $l \geq 1$ :

$$g_0^{(l,0,0)} \triangleq u_1^{(0)} \left( \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2} + x_4 \frac{\partial}{\partial x_3} \right) + u_2^{(0)} \frac{\partial}{\partial x_4} + u_3^{(0)} x_1 \frac{\partial}{\partial x_3} + \sum_{p=0}^{l-1} u_1^{(p+1)} \frac{\partial}{\partial u_1^{(p)}},$$

$$g_1^{(l)} = \frac{\partial}{\partial u_1^{(l)}}, \quad g_2^{(0)} = \frac{\partial}{\partial u_2^{(0)}}, \quad g_3^{(0)} = \frac{\partial}{\partial u_3^{(0)}}.$$

The reader may easily check that  $\Delta_k^{(l,0,0)} = \overline{\Delta_k^{(l,0,0)}}$  and  $[\Gamma_k^{(l,0,0)}, \Delta_k^{(l,0,0)}] \subset \Delta_k^{(l,0,0)}$  for all  $l \geq 1$  and  $k = 0, 1, 2$ , and that  $\dim \Delta_0^{(1,0,0)} = 2$ ,  $\dim \Delta_1^{(1,0,0)} = 5$ ,  $\dim \Delta_2^{(1,0,0)} = n + m = 7$ . Therefore, the prolongation  $\mathbf{j} = (1, 0, 0)$ , with  $|\mathbf{j}| = 1$ , is minimal and  $\rho_0^{(1,0,0)} = 3$ ,  $\rho_1^{(1,0,0)} = 3$  and  $\rho_2^{(1,0,0)} = 1$ . Thus  $\kappa_1^{(1,0,0)} = 3$ ,  $\kappa_2^{(1,0,0)} = 2$  and  $\kappa_3^{(1,0,0)} = 2$ . Consequently, the minimally purely prolonged system is equivalent by diffeomorphism and feedback to

$$y_1^{(3)} = u_1^{(2)}, \quad \ddot{y}_2 = u_2^{(1)} \quad y_3^{(3)} = W(y_1, \dots, y_1^{(3)}, y_2, \dot{y}_2, y_3, \dot{y}_3, \ddot{y}_3, u_3^{(1)})$$

with flat outputs

$$y_1 = x_1, \quad y_2 = x_4, \quad y_3 = x_2$$

and with

$$W(y_1, \dots, y_1^{(3)}, y_2, \dot{y}_2, y_3, \dot{y}_3, \ddot{y}_3, u_3^{(1)}) \triangleq y_2 \dot{y}_1 \left( \ddot{y}_1 - \frac{(\dot{y}_1)^2}{y_1} \right) + \dot{y}_2 (\dot{y}_1)^2 + \dot{y}_3 \frac{y_1^{(3)}}{\dot{y}_1} + \left( \ddot{y}_3 - \dot{y}_3 \frac{\ddot{y}_1}{\dot{y}_1} \right) \left( \frac{\dot{y}_1}{y_1} + 2 \frac{\ddot{y}_1}{\dot{y}_1} \right) + y_1 \dot{y}_1 u_3^{(1)},$$

$W$  being indeed invertible with respect to  $u_3^{(1)}$  as long as  $y_1 \dot{y}_1 \neq 0$ .

## 6. CONCLUDING REMARKS

We have established necessary and sufficient conditions for a system to be flat by pure prolongation, *i.e.* belonging to the equivalence class of 0 with respect to the *equivalence by pure prolongation relation*. These conditions extend preliminary results of [6, 7, 35, 36, 2, 12] thanks to a thorough study of purely prolonged vector fields. We then deduce a computationally tractable algorithm giving the minimal prolongation in a finite number of steps using only Lie brackets and linear algebra.

Possible extensions of this work towards general flatness necessary and sufficient conditions are under study.

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#### APPENDIX A. A COMPARISON FORMULA

We establish a comparison formula expressing the iterated Lie brackets  $\text{ad}_{g_0}^k g_i^{(j_i)}$  for all  $k \geq 0$ , in terms of a combination of Lie brackets of the vector fields  $g_0^{(0)} = f \frac{\partial}{\partial x}$ ,  $g_1^{(0)} = \frac{\partial}{\partial u_1^{(0)}}$ ,  $\dots$ ,  $g_m^{(0)} = \frac{\partial}{\partial u_m^{(0)}}$  of the *original system* (2.16).

For simplicity's sake, we introduce a fictitious input  $u_0 \triangleq u_0^{(0)} \triangleq t$  so that  $u_0^{(1)} = 1$ ,  $j_0 = 1$ , and  $u_0^{(k)} = 0$  for all  $k > 1$ . Thus,  $\sum_{l=0}^{j_0-1} u_0^{(l+1)} g_0^{(l)} = u_0^{(1)} g_0^{(0)} = g_0^{(0)}$ , and

$$g_0^{(j)} = g_0^{(0)} + \sum_{k=1}^m \sum_{l=0}^{j_k-1} u_k^{(l+1)} g_k^{(l)} = \sum_{k=0}^m \sum_{l=0}^{j_k-1} u_k^{(l+1)} g_k^{(l)}.$$

**Lemma A.1.** *For every  $i = 1, \dots, m$ , we have:*

$$\text{ad}_{g_0}^k g_i^{(j_i)} = (-1)^k \frac{\partial}{\partial u_i^{(j_i-k)}} = (-1)^k g_i^{(j_i-k)} \quad \text{if } k \leq j_i, \quad (\text{A.1})$$

and, for  $k = j_i + v$ , for all  $v \geq 1$ ,

$$\text{ad}_{g_0}^{j_i+v} g_i^{(j_i)} = (-1)^{j_i} \sum_{(r, \mathbf{l}, \mathbf{p}) \in \mathbb{J}_{i,v}^{(j_i)}} c_{v, \mathbf{l}}^{\mathbf{p}} u_{p_r}^{(l_r)} \cdots u_{p_1}^{(l_1)} [g_{p_r}^{(0)}, \dots, [g_{p_1}^{(0)}, g_i^{(0)}] \dots] \quad (\text{A.2})$$

$$= (-1)^{j_i} \text{ad}_{g_0}^v g_i^{(0)} + \sum_{(q, r, \mathbf{l}, \mathbf{p}) \in \mathbb{I}_{i,v}^{(j_i)}} c_{v, q, \mathbf{l}}^{\mathbf{p}} u_{p_r}^{(l_r)} \cdots u_{p_1}^{(l_1)} [g_{p_r}^{(0)}, \dots, [g_{p_1}^{(0)}, \text{ad}_{g_0}^q g_i^{(0)}] \dots] \quad (\text{A.3})$$

where  $\mathbb{J}_{i,v}^{(\mathbf{j})}$  (resp.  $\mathbb{I}_{i,v}$ ) is the set of multi-integers  $(r, \mathbf{l}, \mathbf{p}) \triangleq (r, l_1, \dots, l_r, p_1, \dots, p_r)$  (resp.  $(q, r, \mathbf{l}, \mathbf{p}) \triangleq (q, r, l_1, \dots, l_r, p_1, \dots, p_r)$ ) defined by

$$\mathbb{J}_{i,v}^{(\mathbf{j})} \triangleq \{(r, \mathbf{l}, \mathbf{p}) \mid 1 \leq r \leq v, \quad 0 \leq p_\alpha \leq m, \quad 1 \leq l_\alpha \leq j_\alpha, \quad \alpha = 1, \dots, r, \quad |\mathbf{l}| = v\}, \quad (\text{A.4})$$

(resp.

$$\mathbb{I}_{i,v}^{(\mathbf{j})} \triangleq \{q, r \geq 1, \quad 0 \leq p_\alpha \leq m, \quad 1 \leq l_\alpha \leq j_\alpha, \quad \alpha = 1, \dots, r, \quad p_r \neq 0, \quad r \leq |\mathbf{l}| = v - q\}) \quad (\text{A.5})$$

with  $|\mathbf{l}| \triangleq \sum_{\alpha=1}^r l_\alpha$ ,  $c_{v,\mathbf{l}}^{\mathbf{p}} \triangleq c_{v,l_1,\dots,l_r}^{p_1,\dots,p_r} \in \mathbb{Z}$  (resp.  $c_{v,q,\mathbf{l}}^{\mathbf{p}} \triangleq c_{v,q,l_1,\dots,l_r}^{p_1,\dots,p_r} \in \mathbb{Z}$ ) and with the convention that the summation of the right-hand side of (A.2) (resp. (A.3)) vanishes if  $\mathbb{J}_{i,v}^{(\mathbf{j})} = \emptyset$  (resp.  $\mathbb{I}_{i,v}^{(\mathbf{j})} = \emptyset$ ), each bracket  $[g_{p_r}^{(0)}, \dots, [g_{p_1}^{(0)}, g_i^{(0)}] \dots]$  in (A.2) (resp.  $[g_{p_r}^{(0)}, \dots, [g_{p_1}^{(0)}, \text{ad}_{g_0}^q g_i^{(0)}] \dots]$  in (A.3)) depending at most on  $\bar{x}^{(0)}$ .

*Proof.* By induction. It is immediately seen that

$$\text{ad}_{g_0}^{(\mathbf{j})} g_i^{(j_i)} = [f \frac{\partial}{\partial x} + \sum_{k=1}^m \sum_{l=0}^{j_k-1} u_k^{(l+1)} \frac{\partial}{\partial u_k^{(l)}}, \frac{\partial}{\partial u_i^{(j_i)}}] = -\frac{\partial}{\partial u_i^{(j_i-1)}} = -g_i^{(j_i-1)}$$

Iterating this computation up to  $k = j_i$  yields (A.1). In particular:

$$\text{ad}_{g_0}^{j_i} g_i^{(j_i)} = (-1)^{j_i} \frac{\partial}{\partial u_i^{(0)}} = (-1)^{j_i} g_i^{(0)}.$$

Then, for  $k = j_i + 1$ , using the fact that  $[\frac{\partial}{\partial u_k^{(l)}}, \frac{\partial}{\partial u_i^{(0)}}] = 0$  for all  $i, k$  and  $l \geq 0$ , we have:

$$\begin{aligned} \text{ad}_{g_0}^{j_i+1} g_i^{(j_i)} &= [g_0^{(\mathbf{j})}, \text{ad}_{g_0}^{j_i} g_i^{(j_i)}] = (-1)^{j_i} [g_0^{(\mathbf{j})}, g_i^{(0)}] \\ &= (-1)^{j_i} [f \frac{\partial}{\partial x} + \sum_{k=1}^m \sum_{l=0}^{j_k-1} u_k^{(l+1)} \frac{\partial}{\partial u_k^{(l)}}, \frac{\partial}{\partial u_i^{(0)}}] \\ &= (-1)^{j_i} [f \frac{\partial}{\partial x}, \frac{\partial}{\partial u_i^{(0)}}] = (-1)^{j_i} \text{ad}_{g_0} g_i^{(0)} \end{aligned} \quad (\text{A.6})$$

which proves that (A.2) and (A.3) hold at the order  $k = j_i + 1$ , i.e.  $v = 1$ , the summation of the right-hand side of (A.3) being equal to 0 since  $\mathbb{I}_{i,1}^{(\mathbf{j})} = \emptyset$ . Furthermore, a direct calculation shows that

$$(-1)^{j_i} \text{ad}_{g_0} g_i^{(0)} = (-1)^{(j_i+1)} \frac{\partial f}{\partial u_i^{(0)}} (\bar{x}^{(0)}) \frac{\partial}{\partial x}, \quad (\text{A.7})$$

which proves that  $\text{ad}_{g_0}^{j_i+1} g_i^{(j_i)} \in \text{VX}^{(0)}$ , where  $\text{VX}^{(0)}$  is the vertical bundle of  $X^{(0)}$ , i.e. the set of vector fields that are linear combinations of  $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ , and whose coefficients are smooth functions that depend at most on  $\bar{x}^{(0)}$  (but not of  $u^{(l)}$  for all  $l \geq 1$ ). It results that  $[\frac{\partial}{\partial u_p^{(l)}}, \text{ad}_{g_0} g_i^{(0)}] = [g_p^{(l)}, \text{ad}_{g_0} g_i^{(0)}] = 0$  for all  $l \geq 1$ , all  $p = 1, \dots, m$ , and all  $i = 1, \dots, m$ .

Assume now that (A.2) and (A.3) hold up to  $k = j_i + \nu$ , with  $\nu > 1$ , and that all the brackets  $[g_{p_r}^{(0)}, \dots, [g_{p_1}^{(0)}, g_i^{(0)}] \dots]$  and  $[g_{p_r}^{(0)}, [\dots, [g_{p_1}^{(0)}, \text{ad}_{g_0}^q g_i^{(0)}] \dots]]$  depend at most on  $\bar{x}^{(0)}$ . We immediately deduce that

$$\left[ \frac{\partial}{\partial u_{p_{r+1}}^{(l_{r+1})}}, [g_{p_r}^{(0)}, [\dots, [g_{p_1}^{(0)}, \text{ad}_{g_0}^q g_i^{(0)}] \dots]] \right] = [g_{p_{r+1}}^{(l_{r+1})}, [g_{p_1}^{(0)}, [\dots, [g_{p_r}^{(0)}, \text{ad}_{g_0}^q g_i^{(0)}] \dots]]] = 0$$

for all  $l_{r+1} \geq 1$ ,  $p_{r+1} \in \{1, \dots, m\}$ ,  $q \in \{0, \dots, \nu - 1\}$ ,  $r \geq 1$ ,  $p_1, \dots, p_r \in \{0, \dots, m\}$ , and  $i \in \{1, \dots, m\}$ . Thus:

$$\begin{aligned} \text{ad}_{g_0}^{j_i + \nu + 1} g_i^{(j_i)} &= [g_0^{(\mathbf{j})}, \text{ad}_{g_0}^{j_i + \nu} g_i^{(j_i)}] \\ &= (-1)^{j_i} \sum_{(r, \mathbf{l}, \mathbf{p}) \in \mathbb{J}_{i, \nu}^{(\mathbf{j})}} [g_0^{(\mathbf{j})}, c_{\nu, \mathbf{l}}^{\mathbf{p}} u_{p_r}^{(l_r)} \cdots u_{p_1}^{(l_1)} [g_{p_r}^{(0)}, \dots, [g_{p_1}^{(0)}, g_i^{(0)}] \dots]] \\ &= (-1)^{j_i} \sum_{(r, \mathbf{l}, \mathbf{p}) \in \mathbb{J}_{i, \nu}^{(\mathbf{j})}} \sum_{p_{r+1}=0}^m c_{\nu, \mathbf{l}}^{\mathbf{p}} u_{p_{r+1}}^{(1)} u_{p_r}^{(l_r)} \cdots u_{p_1}^{(l_1)} [g_{p_{r+1}}^{(0)}, [g_{p_r}^{(0)}, \dots, [g_{p_1}^{(0)}, g_i^{(0)}] \dots]] \\ &\quad + (-1)^{j_i} \sum_{(r, \mathbf{l}, \mathbf{p}) \in \mathbb{J}_{i, \nu}^{(\mathbf{j})}} \sum_{p_{r+1}=1}^m \sum_{l_{r+1}=0}^{j_{p_{r+1}}} c_{\nu, \mathbf{l}}^{\mathbf{p}} u_{p_{r+1}}^{(l_{r+1}+1)} \frac{\partial}{\partial u_{p_{r+1}}^{(l_{r+1})}} \left( u_{p_r}^{(l_r)} \cdots u_{p_1}^{(l_1)} \right) [g_{p_r}^{(0)}, \dots, [g_{p_1}^{(0)}, g_i^{(0)}] \dots] \\ &\triangleq \sum_{(r, \lambda, \mu) \in \mathbb{K}_{i, \nu+1}^{(\mathbf{j})}} c_{\nu+1, \lambda}^{\mu} u_{\mu_{r+1}}^{(\lambda_{r+1})} \cdots u_{\mu_1}^{(\lambda_1)} [g_{\mu_{r+1}}^{(0)}, \dots, [g_{\mu_1}^{(0)}, g_i^{(0)}] \dots]. \end{aligned}$$

Applying the Leibnitz rule to the penultimate line, using the fact that  $|\mathbf{l}| = \sum_{\alpha=1}^r l_\alpha = \nu$  by the induction assumption, we get  $1 + |\mathbf{l}| = \sum_{\alpha=1}^{r+1} \lambda_\alpha = \nu + 1$ , with  $\lambda_\alpha \geq 1$ ,  $\alpha = 1, \dots, r+1$ , and  $0 \leq \mu_\alpha \leq m$ , which proves that the latter formula holds for all  $(r, \lambda, \mu) \in \mathbb{K}_{i, \nu+1}^{(\mathbf{j})} \subset \mathbb{J}_{i, \nu+1}^{(\mathbf{j})}$ . We leave as an exercise to the reader the proof of the converse inclusion  $\mathbb{J}_{i, \nu+1}^{(\mathbf{j})} \subset \mathbb{K}_{i, \nu+1}^{(\mathbf{j})}$ . Therefore (A.2) is valid for all  $\nu$ . The proof of (A.3) follows exactly the same lines. The lemma is proven.  $\square$