Introduction to Mathematical Morphology

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Introduction to Mathematical Morphology

Lecture Notes

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Contents

Symbols and Notations

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1. Morphological operations

Dilation by a compact $K : A \rightarrow A \oplus \check{K}$

Erosion by a compact $K : A \rightarrow A \ominus \check{K} = \{x, K_x \subset A\}$

Opening by a compact $K : A_K = A \ominus \check{K} \oplus K$

Closing by a compact $K : A^K = A \oplus \check{K} \ominus K$

2. Random sets

A, B: random closed sets (RACS)

A^c: complementary set of A

$\check{A} = \{-x, x \in A\}$: transposed set of A

A_i: component of a multi-component random set

$B(r)$: closed ball with radius r

K: compact set

E: topological space

$\mathcal{F}$, $\mathcal{G}$ et $\mathcal{K}$: closed, open and compact sets of E

$\mathcal{F}^K = \{F \in \mathcal{F};\ F \cap K = \emptyset,\ K \in \mathcal{K}\}$

$\mathcal{F}_G = \{F \in \mathcal{F};\ F \cap G \neq \emptyset,\ G \in \mathcal{G}\}$

probability P

$G_K(s)$: generating function of the random variable $N(K)$

Choquet's capacity $T(K) = P\{K \cap A \neq \emptyset\} = P\{\mathcal{F}_K\} = 1 - P\{K \subset A^c\} = 1 - Q(K)$

$p = P\{x \in A\}$

$q = P\{x \in A^c\}$

Covariance $C(h) = P\{x \in A, x + h \in A\}$

Covariance $Q(h) = P\{x \in A^c, x + h \in A^c\}$

Covariance $C_{ij}(h) = P\{x \in A_i, x + h \in A_j\}$ for a multi-component random set

Three points Probability $Q(h_1, h_2) = P\{x \in A^c, x + h_1 \in A^c, x + h_2 \in A^c\}$

Segment l: $P(l) = P\{l \subset A\}$; $Q(l) = P\{l \subset A^c\}$

Hexagon $H(r)$: $P(H(r)) = P\{H(r) \subset A\}$; $Q(H(r) = P\{H(r) \subset A^c\}$

$R(x, A)$: distance between the point x and the set A

3. MEASUREMENTS

$\mu(A)$: measure of A

μ_n Lebesgue measure in R^n

Volume of A: $V(A)$

Integral of mean curvature of A: $M(A)$

Surface area of A: $S(A)$

Perimeter of A (in R^2): $L(A)$

Connectivity number in R^2: $N_A(A)$

Connectivity number in R^3: $N_V(A) - G_V(A)$

4. RANDOM FUNCTIONS

Random Function $Z(x)$

test function $g(x)$

$\Phi(E)$: set of functions from $E \to \overline{R}$

upper semi continuous random functions (usc RF)

$\Phi_f \subset \Phi$: set of usc functions from $E \to \overline{R}$

Choquet's capacity $T(g) = P\{x \in D_Z(g)\}; D_Z(g)^c = \{x, Z(x+y) < g(y), \forall y \in K\}$

lower semi continuous random functions (lsc RF)

$\Phi_g \subset \Phi$: set of lsc functions from $E \to \overline{R}$

functional $P(g) = P\{x \in H_Z(g)\}; H_Z(g) = \{x, Z(x+y) \geq g(y), \forall y \in K\}$

subgraph Γ^φ of the function φ: $\Gamma^\varphi = \{x, z\}$, $x \in E$, $z \in \overline{R}$, with $z \leq \varphi(x)$

overgraph Γ_φ: $\Gamma_\varphi = \{x, z\}$, $x \in E$, $z \in \overline{R}$, with $z \geq \varphi(x)$

$\vee$: supremum; $Z_\vee(K) = \vee_{x \in K}\{Z(x)\}$

$\wedge$: infimum; $Z_\wedge(K) = \wedge_{x \in K}\{Z(x)\}$

thresholding: $A_Z(z) = \{x, Z(x) \geq z\}$

spatial law: $F(x, z) = P\{Z(x_1) < z_1, ..., Z(x_m) < z_m\}$ with $x \in E^m$ and $z \in \overline{R}^m$

spatial law: $T(x, z) = P\{Z(x_1) \geq z_1, ..., Z(x_m) \geq z_m\}$ with $x \in E^m$ and $z \in \overline{R}^m$

$F(z)$, $G(z)$ distribution functions (with density, or pdf $f(z)$ and $g(z)$)

S: coefficient of variation of a distribution

$D^2[Z]$: variance of the random variable Z

Bivariate distribution $F_{ij}(h, z_1, z_2) = P\{Z_i(x) < z_1, Z_j(x+h) < z_2\}$

Bivariate distribution $T_{ij}(h, z_1, z_2) = P\{Z_i(x) \geq z_1, Z_j(x+h) \geq z_2\}$

Bivariate distribution $T_2(h, z_1, z_2) = P\{Z(x) \geq z_1, Z(x+h) \geq z_2\}$

Covariance $C(x, x+h)$ and second order central correlation function $\overline{W}_2(x, x+h)$

$\gamma_1(h)$, $\gamma_2(h)$: variograms of order 1 and 2

Central correlation function of order m $\overline{W}_m(x)$, with $x \in E^m$

5. MODELS

5.1. Random sets. Boolean model: RACS A with primary grain A'

Intensity θ

Primary grain A':

geometrical covariogram $K(h) = \overline{\mu}_n(A' \cap A'_{-h})$

normalized covariogram $r(h) = \frac{K(h)}{K(0)}$

$s(h_1, h_2) = \frac{\overline{\mu}_n(A' \cap A'_{-h_1} \cap A'_{-h_2})}{K(0)}$

IBRS: infinitesimal Boolean random set; time t; Intensity $\theta(t)$

Dead leaves tesselation: $N_i(t)$: specific number of intact grains; $\varphi_{A'}$, $\varphi_{A'_i}$: pdf of grains and of intact grains

Poisson varieties $V_k(\omega)$, with intensity $\theta(\omega)$

5.2. Random functions (RF). BRF: Boolean random function $Z(x)$

DLRF: Dead leaves random function

SARF: Sequential alternate random function

$Z'_t(x)$: Primary random function, with subgraph $\Gamma^{Z'_t} = A'(t)$ and sections $A_{Z'_t}(z)$

$\varphi(Z)$: transformation of the RF Z by the anamorphosis φ

Dilution RF: $\Phi(U, X)$ and $\phi_t(U, X)$: multivariate characteristic functions of the RF $Z(x)$ and $Z'_t(x)$

$Z * \check{p}(x)$: convolution of the RF $Z(x)$ by a weight function $p(x)$

Introduction to random structures

Introduction to Random Structures

Introduction

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Introduction

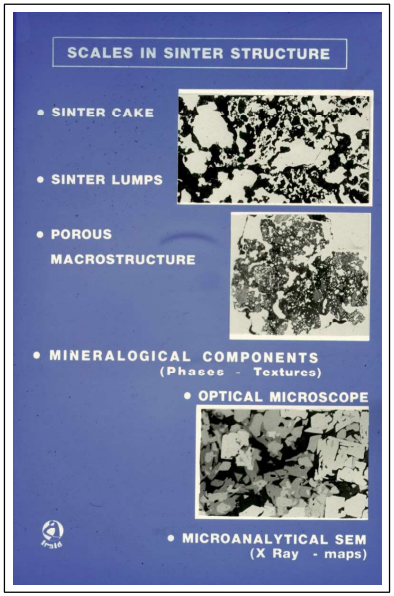
- Why use models of **Random Structures**?
- No specific role of **Randomness**, as opposed to **Determinism.**
- Choice of a **methodology** based on a **probabilistic approach.**

Random Structure Models

- **Aim:** to estimate, from spoiled or missing data, structural, morphological, or physical propertiesof heterogeneous media.
- **Method: Use of a probabilistic approach**
  - for practical reasons
  - to get efficient solutions to the studied problems

Heterogeneity

- **Microstructures, Textures:**
  - more or less important heterogeneity at different scales
  - structured aspects
- How to account for these two points?
- How to replace fluctuating data at a small scale by global characteristics?
- Going from **Microscopic to Macroscopic**
- Relationships **Physical Behavior-Textures**



Partial knowledge

- Regionalized data are known from a **sampling** procedure:
  - *Selection of objects in a population*
    - Grains of a powder
    - Biological cells
    - trees, plants
    - individuals
  - *Partial knowledge of objects*
    - Ore deposit explored by probes
    - 2D or 1D sclices in a material
    - Finite number of zones examined with the microscope,...

Representativity

How to **extrapolate** partial available information to

- a full single object?
- a population of objects?

Typical models:

- *Periodic medium*, known from a single period (satisfactory for ordered media like perfect crystals)
- *Probabilistic model* (concerns most practical situations for heterogeneous media)

Aims of Probabilistic Models

Two main aspects:

- Descriptive
- Predictive (probem solving)

Descriptive aspects

Simplified representation of a complex structure:

- **Summary of structural data** in few parameters (1, 2,...) for relation with the physical properties, or for classification of structures
- **Genetic models:** simplified construction to simulate the physical processes driving the formation of the structure
- **Realistic simulations** of random media

Predictive aspects

What can be obtained from partial data, using probabilistic models? Typical examples from Geostatistics:

- Give the **precision** of a global estimation, such as the integral

$$I = \int_R f(x)dx$$

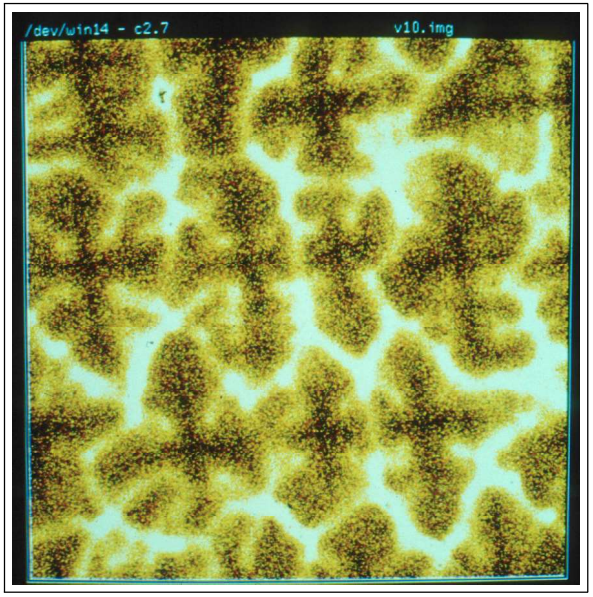
estimated from sampling points by

$$I^* = a \sum_{p=-\infty}^{p=+\infty} f(x_0 + pa)$$

- Statistical definition of a **RVE (Representative Volume Element)**

Predictive aspects

- **Interpolation of data** at points without information (orebody deposit exploration, missing data such as non available slices for a 3D image analysis of a microstructure,..) by **kriging**
- **Restore noisy data** by filtering (e.g. by **kriging**)



Microsegregation in steel (noisy image)

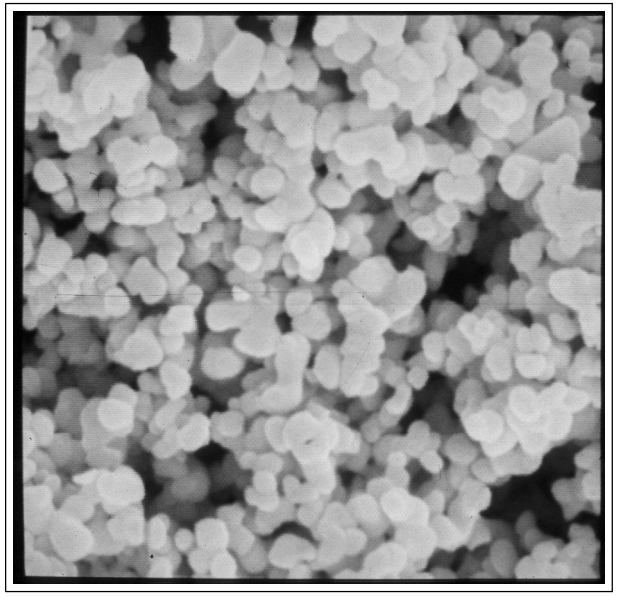
Predictive aspects

- Solution of **optimisation problems by simulations:**
  - Orebody deposits exploitation (**Conditional simulations** that respect data at points with information)
  - Oil reservoir production (**Conditional simulations)**
  - Reliability computation, **fracture statistics**
  - **Homogenization**: computation of the effective behaviour of random heterogeneous media (physical effective properties), from **probabilistic information** on the microstructure, or from **simulations of random media**
  - Materials conception and optimization,...

Predictive aspects

Further practical problems:

- **Stereology:** how to reach 3D morphological properties from 1D or 2D morphological properties (obtained on slices or on projections)
- **Counting and measuring:**
  - number of imbricated objects in a texture, without segmenting the image
  - estimation of a powder size distribution from images of overlapping objects



SEM Micrograph of Powder

Predictive aspects

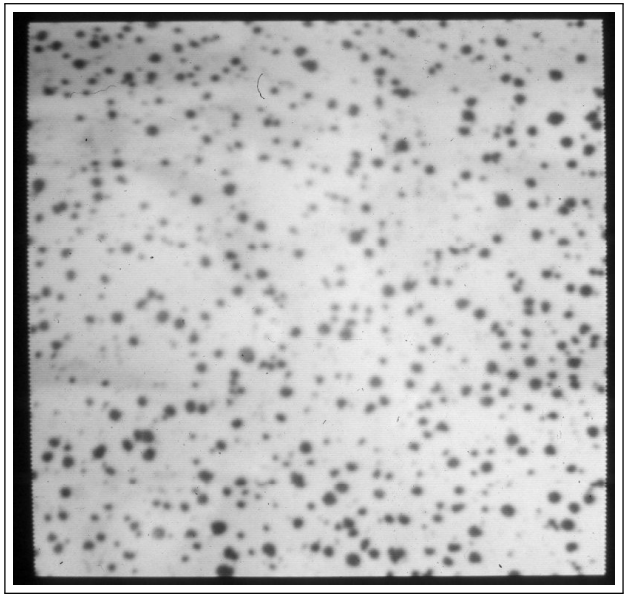
- **Change of support:** change of statistical properties of a medium with the size and shape of the support of data
- **Change of scale:** prediction of the macroscopic physical behavior of a medium from its microscopic behavior; for instance, in *Fracture Statistics of brittle materials*, prediction of the fracture probability of parts (or planes, buildings,...) from data on small scale samples

Possible construction of **algorithms** based on theoretical probabilistic models, to be used more generally in a heuristic way (filtering, counting, change of support, change of scale,...)

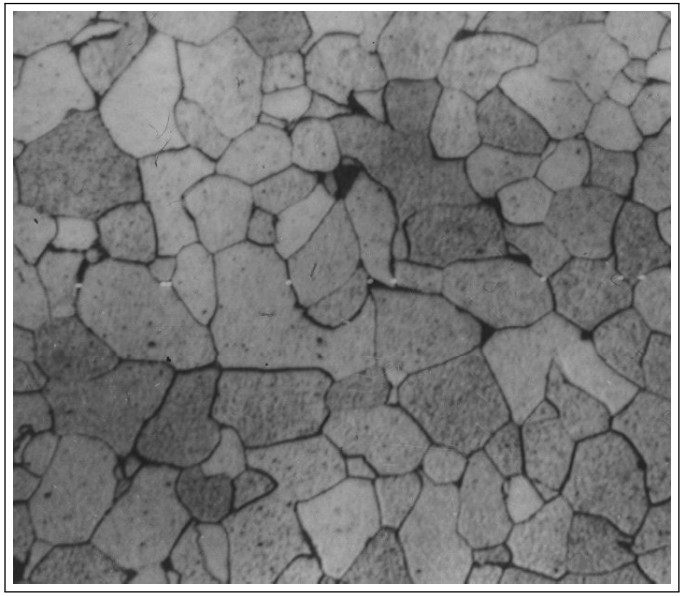
Main types of data and of models

Examples of regionalized data and corresponding types of models:

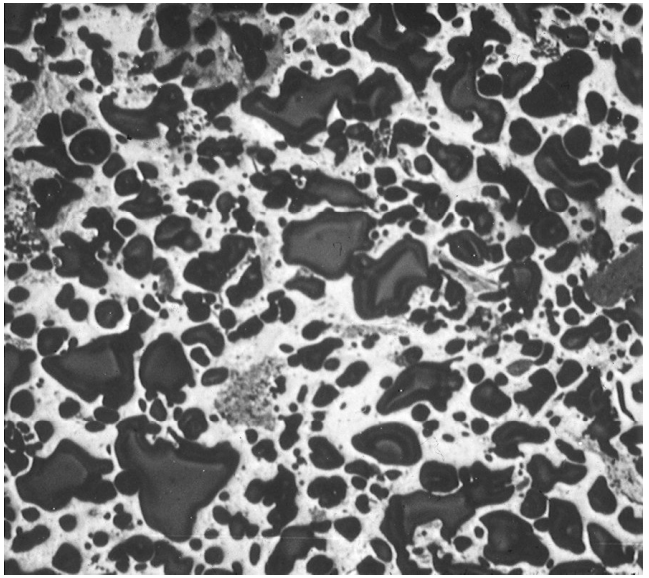
- Dispersions of small particles in a matrix (non metallic inclusions in steel,...), modelled by realizations of **stochastic point processes**
- Granular structures (polycrystals), assimilated to **random tessellations of space** (each class corresponding to a grain)
- Two phase (porous media) or multiphase structures (composite materials with several components) may be simulated by **random sets** (binary), or **multicomponent random sets**



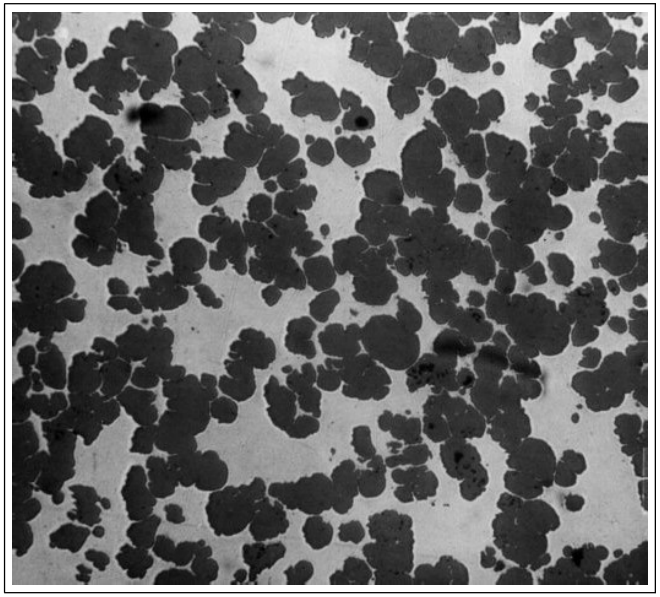
Nodular graphite in pig iron



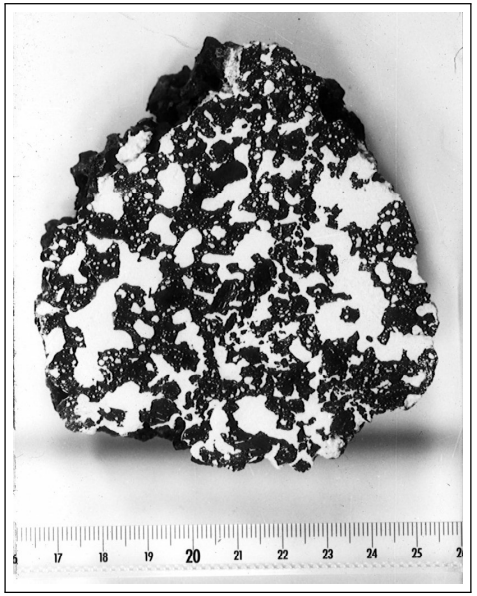
Polycristal in steel



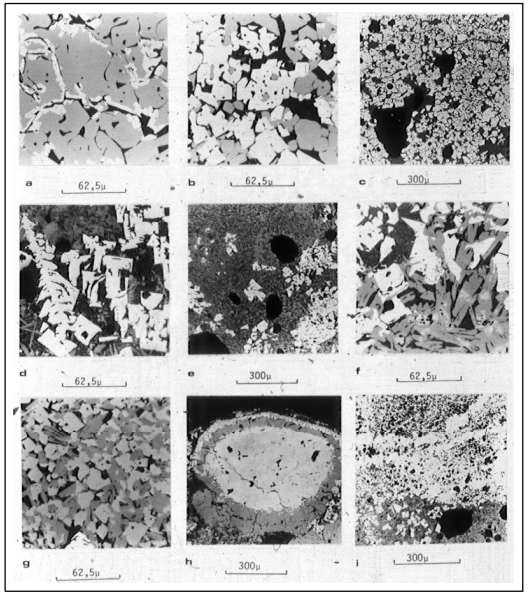
Metallurgical coke



Fe Ag composite



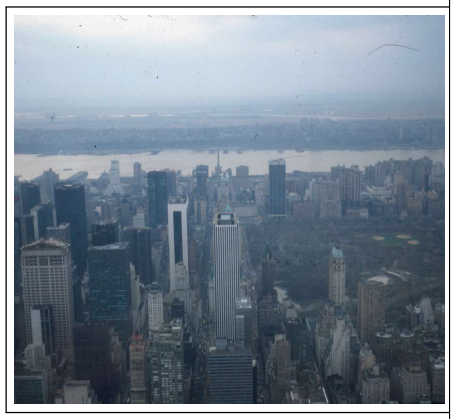
Iron ore sinter

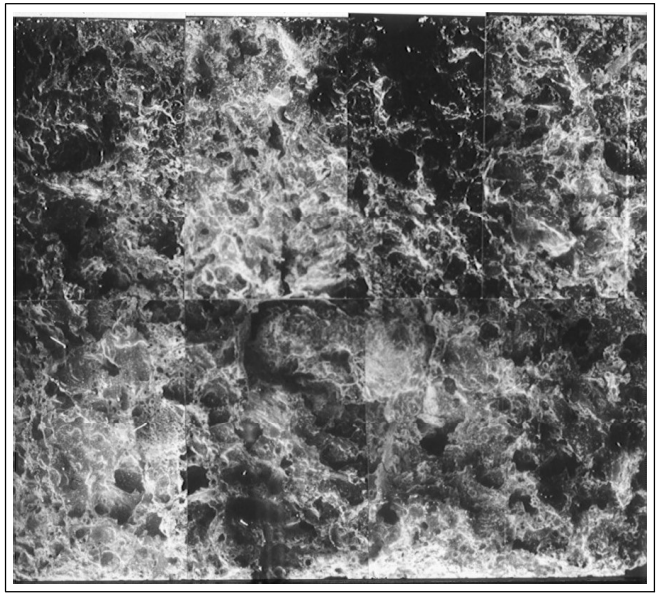


Mineral textures in iron ore sinters

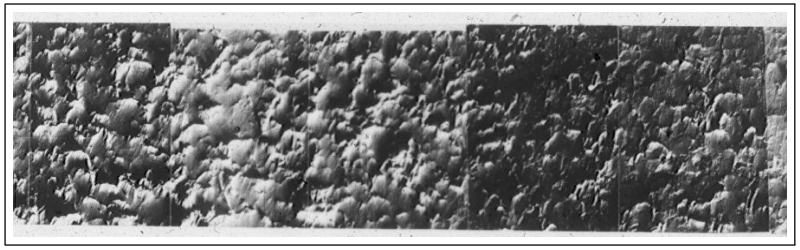
Main types of data and of models

- Rough surfaces (steel plate, fracture surface,...), chemical concentration mappings (X ray images obtained with an electron microprobe), and more generally grey level images (video images, secondary electron images in a scanning electron microscope,...) can be represented by **random functions**
- Multivariate data (multi species chemical mappings, components of a vector or of a tensor in every point x of space), modelled by **multivariate random function models**

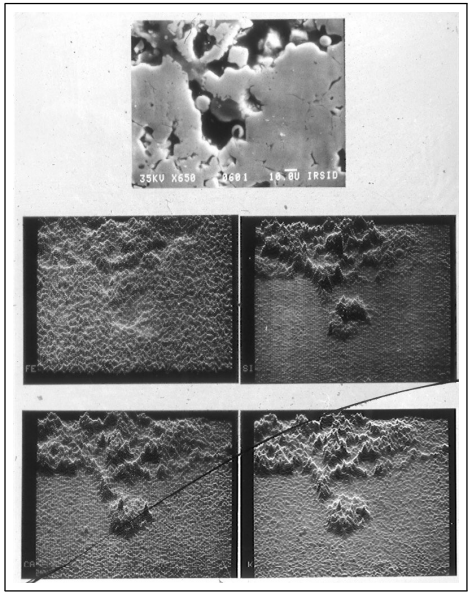




Microfractography of graphite



Roughness of a steel rolling mill



Chemical maps in an iron ore sinter

Main types of data and of models

- Sequential images (change of a microstructure during its formation, its deformations under loading, its degradation; successive grounds in a perspective view) associated to **sequential random sets or random functions**
- Data on a network connecting vertices (roads, porous medium, cracks,...) and their properties depending on connectivity, modelled by **random graphs**
- Arborescent data, modelled by **random trees**

Construction of random structures

To build **random structures** are required:

- a set of **events** and their combination (σ **algebra**)

e.g. open sets on a topological space (Borel σ algebra)

- a **probability**

Characterization of a random set model

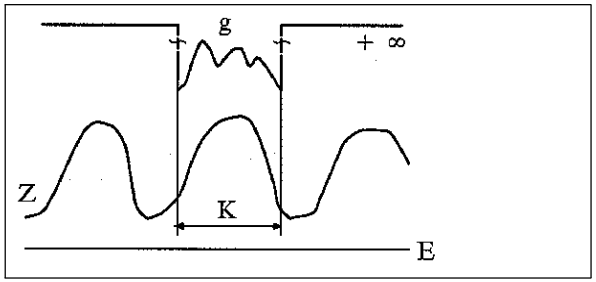
Models derived from the theory of **Random Sets** by **G. MATHERON.** For a **random closed set** A **(RACS)**, characterization by the **CHOQUET capacity** $T(K)$ defined on the compact sets K

$$T(K) = P\{K \cap A \neq \emptyset\} = 1 - P\{K \subset A^c\} = 1 - Q(K)$$

Characterization of a random function model

For an **upper semi continuous random functions (usc RF)** $Z(x)$ defined in R^n, characterization by the **CHOQUET capacity** $T(g)$ defined on the lower semi continuous functions (lsc) g with compact support K

$$T(g) = P\{x \in D_Z(g)\}; D_Z(g)^c = \{x, Z(x+y) < g(y), \forall y \in K\}$$

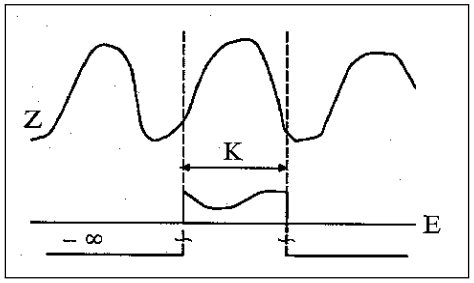


usc RF: $x \in D_Z(g)^c$

Characterization of a random function model

For a **lower semi continuous random functions (lsc RF)** , characterization by the **functional** $P(g)$ defined on the upper semi continuous functions (usc) g with compact support K

$$P(g) = P\{x \in H_Z(g)\}; H_Z(g) = \{x, Z(x+y) \geq g(y), \forall y \in K\}$$



Characterization of a random model

- The CHOQUET capacity of a RACS is equivalent to the distribution function of a random variable
- Two models (RACS, usc RF, lsc RF) with the same functionnal $T(K)$, $T(g)$, $P(g)$ cannot be distinguished (theoretically as well as experimentally)
- The functionnal $T(K)$, $T(g)$, $P(g)$ connects theory and experimentation; it is used to estimate the parameters of a model and to test its validity

Construction of Random Structure Models

Main steps:

- Choice of basic assumptions
- Computation of the functionnal $T(K)$

Construction of Random Structure Models

Choice of basic assumptions

- **Genetic models**: construction from elementary processes and geometrical modes of interaction
- Definition of a RF as a solution of Stochastic partial differential equations (e.g. **Reaction-Diffusion models)**

Construction of Random Structure Models
Choice of basic assumptions

- Prior general properties satisfied by a class of models:
  - stationarity
  - independent increments
  - infinite divisibility
  - Markov property (lack of memory)
  - stable distribution after transformation,...
- **Asymptotic limit** of families of models: Gaussian RF ("Central limit " theorem); Boolean RF obtained by the supremum $\vee$ of independant RF

Construction of Random Structure Models
Calculation of the CHOQUET capacity

Determination of the functional $T(K), T(g), P(g)$ as a function of

- the assumptions
- the parameters of the model
- the compact K or the function g

Construction of Random Structure Models
Calculation of the CHOQUET capacity

For a given model, the functionnal T is obtained:

- by theoretical calculation
- by estimation
  - on simulations
  - on real structures (possible estimation of the parameters from the "experimental" T , and tests of the validity of assumption)

Construction of Random Structure Models
Calculation of the CHOQUET capacity

- The functions $T(K)$ (K being variable) are **consistent** (which is not the case of any prior analytical model)
- After specification and validation of the model from available data, possible **predictive implementation** of its properties (such as $T(K)$ for compacts K not used during the identification step). Examples: 3D properties deduced from 2D observations (stereology); change of support by $\vee$ or $\wedge$ in the case of a change of scale in fracture statistics

General properties of the proposed models

- Most random structure models defined in the **Euclidean space** R^n :
  - more general than stochastic processes limited to the 1D space R, where the order relation is used;
  - different from discrete models defined on a grid, even if the discrete counterpart of the euclidean models is easily defined
- Models depending on a **few number of parameters**, not to ask too much from the available data, and for realistic experimental identification and test

General properties of the proposed models

- Mostly **stationary and ergodic** version, to allow a statistical inference from a single realization
- Good **stereological properties,** for access from slices or projections
- Facilities for **simulations** (and often for **conditional simulations)**
- Approach based on the Choquet capacity extended to convolutional neural networks (CNN) in the context of Deep Learning, for the discrimination of various point processes (Mohammadi et al.)

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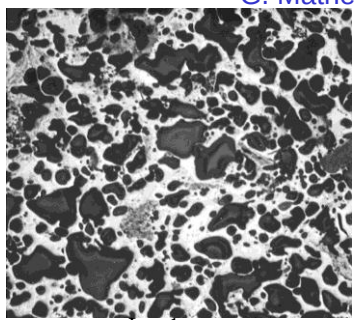
Introduction to mathematical Morphology and to morphological segmentation of images

Introduction to Mathematical Morphology
Outline

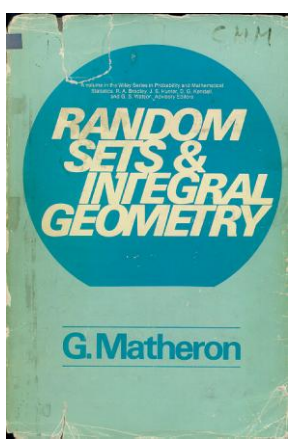
- Origin and purpose
- Mathematical Morphology for binary images
- Mathematical morphology on grey level images
- Morphological segmentation

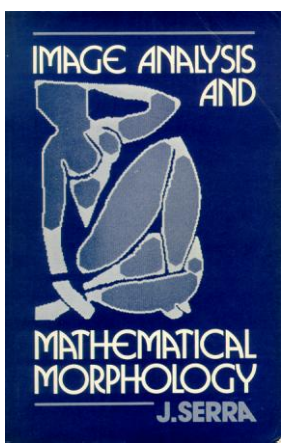
Mathematical Morphology
Origin – Motivations
G. Matheron, 1967



L = 1 mm

- Characterization of the morphology of a heterogeneous medium?
- Prediction of the macroscopic behaviour of a porous medium (composition of permeabilities)?
- Representation of a heterogeneous medium by a model?

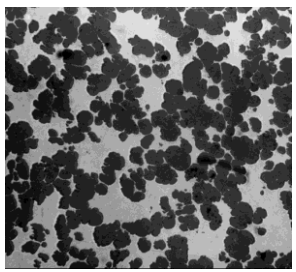




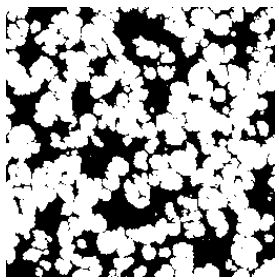
Mathematical morphology analysis

- Initial motivation: **how to quantify complex porous media?**
- Spatial arrangement from **Covariance**, **sizing by morphological opening** (G. Matheron, J. Serra)

Mathematical Morphology for binary images



Fe – Ag (0-255 grey level image)



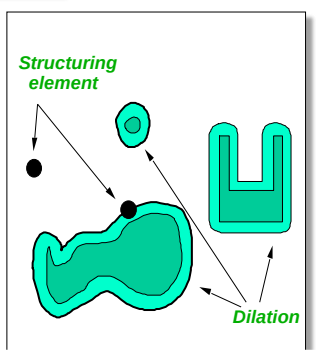
Fe (Threshold 0-100)

Basic operations of binary mathematical morphology

- Qualitative description of the morphology by means of comparison with a reference shape
- Binary image: set A and complementary set A^c
- **Structuring element** K: choice of an object moved by translation in the image K_x
- Answer to a question about the relative location of K_x and of A, recorded for each location x
- For a binary, answer, indicator function of the set $\Phi(A)$ obtained by transformation of A ($k(x) = 1$ when x in $\Phi(A)$)

Set Dilation

K_x hits A ?

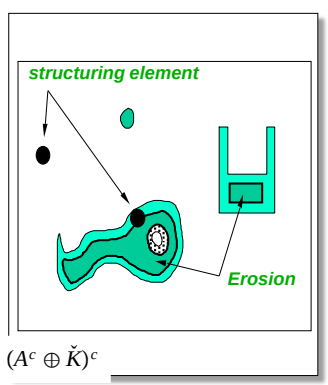


$$A \oplus \check{K} = \{x, K_x \cap A \neq \emptyset\} = \cup_{y \in K} A_{-y} = \cup_{x \in A, y \in K} \{x - y\}$$

Set Erosion

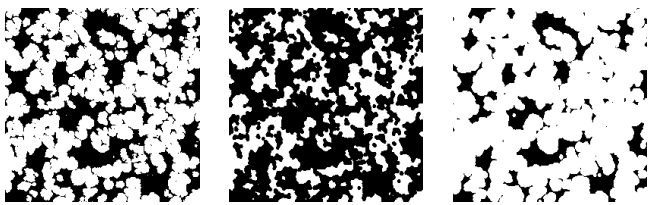
K_x included in A ?

- Erosion obtained by the complementary set of the result of the dilation of A^c by K
- Erosion and dilation **dual operations** with respect to complementation



$$A \ominus \check{K} = \{x, K_x \subset A\} = \cap_{y \in K} A_{-y} = (A^c \oplus \check{K})^c$$

Basic Operations of Mathematical Morphology: erosion and dilation



Erosion hexagonal (2) | Dilation hexagonal (2)

Basic Operations of Mathematical Morphology

$(A \ominus K_1) \ominus K_2 = A \ominus (K_1 \oplus K_2)$

$(A \oplus K_1) \oplus K_2 = A \oplus (K_1 \oplus K_2)$

$(A \cap B) \ominus K = (A \ominus K) \cap (B \ominus K)$

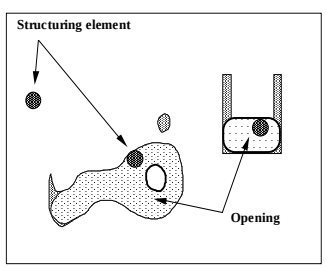
$A \oplus (K_1 \cup K_2) = (A \oplus K_1) \cup (A \oplus K_2)$

If A is a convex set, $A \ominus K$ is a convex set

If A and K are convex sets, $A \oplus K$ is a convex set

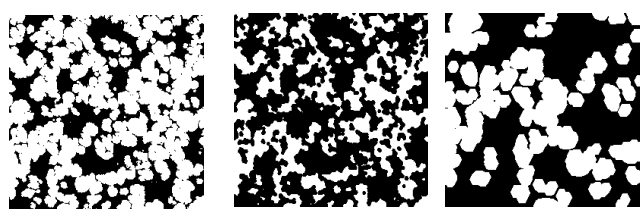
Basic Operations of Mathematical Morphology: Opening

- First erode A by K, then dilate by K, to obtain $\gamma_B(A)$
- The opened set $\gamma_B(A)$ is the union of the structuring elements K_x which are included in the set A
- The opening is an **idempotent operation**
- The opening operation can be used to **remove noise by filtering**, and to measure **size distribution** of the elements of A



When K is a disc, the opening amends the caps, removes the small islands and opens isthmuses.

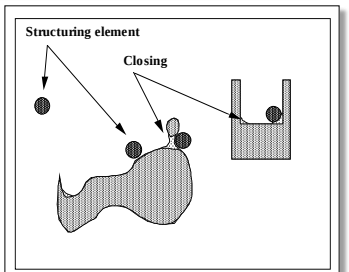
Basic Operations of Mathematical Morphology: opening



Erosion hexagonal (2) | hexagonal opening(6)

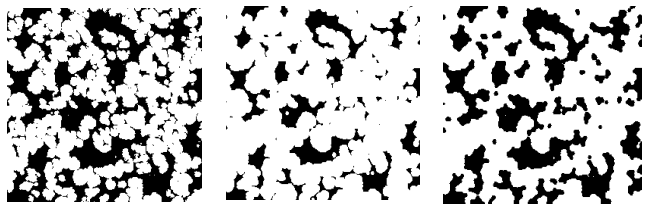
Basic Operations of Mathematical Morphology: Closing

- First dilate *A* by *K*, then erode by *K*, to obtain $\phi_B(A)$
- The closed set $\phi_B(A)$ is also obtained by the complementary set of the opening by *K* of the set A^c (opening and closing **dual operations** with respect to complementation)
- The closing is an **idempotent operation**
- The opening operation can be used to **remove noise by filtering**, and to measure **size distribution** of the elements of A^c



When K is a disc, the closing closes the channels, fills the small lakes, and partly the gulfs.

Basic Operations of Mathematical Morphology: closing



Dilation hexagonal (2)

Closing hexagonal (2)

Mathematical morphology on grey level images

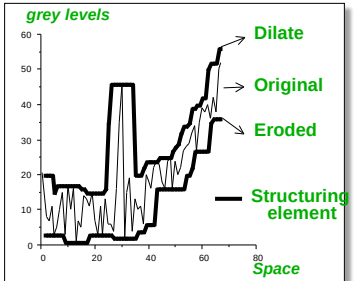
Dilation and Erosion by a flat structuring Element

Definition **:** The dilation (erosion) of a function Z(x) by a flat structuring element *K* is introduced as the dilation (erosion) of each set A_t (λ) by *K*.They are said to be **planar**

This definition leads to the following formulae *(J.Serra)* :

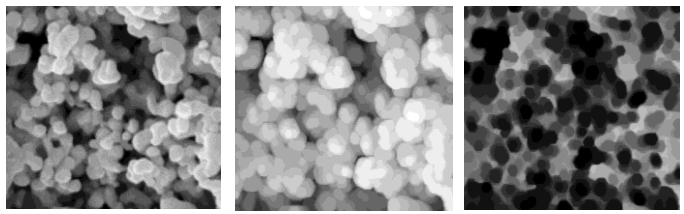
$$Z \oplus K(x) = Sup\{Z(x), x \in K\}$$

$$Z \ominus K(x) = Inf\{Z(x)x, x \in K\}$$



- Erosion shrinks peaks. Peaks thinner that the structuring element disappear. It expands the valleys and the sinks
- Dilation produces the dual effects

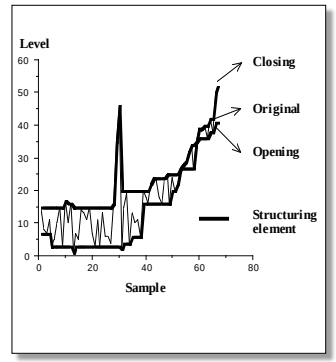
Dilation and Erosion by a flat structuring Element



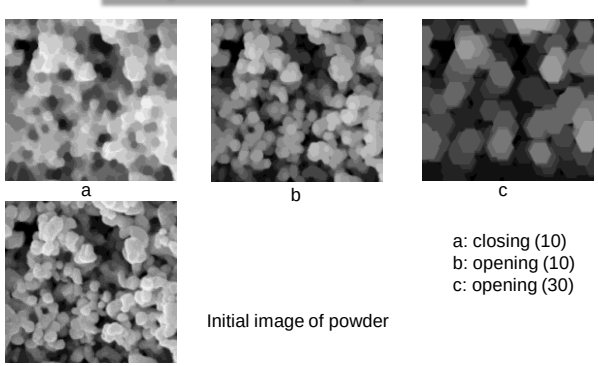
Powder (SEM) | Dilation (hexagon 10) | Erosion (hexagon 10)

Opening and closing of functions by flat structuring element

- Opening: erosion followed by a dilation by *K*
- Closing: dilation followed by an erosion by *K*
- Opening and erosion create a ***simpler function*** than the original. They smooth (**filter**) the image in a nonlinear way
- The ***opening*** (closing) removes ***positive*** (negative) ***peaks*** that are thinner than the structuring element.
- The opening (closing) remains below (above) the original function



Opening and closing of functions by flat structuring element



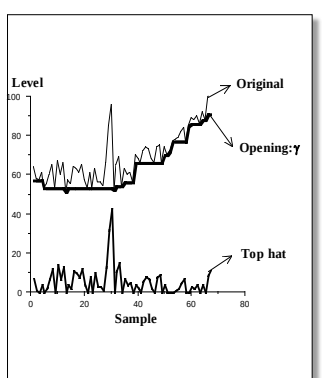
a | b | c

a: closing (10)
b: opening (10)
c: opening (30)

Initial image of powder

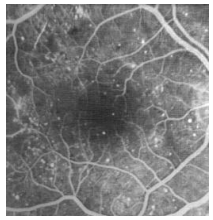
Top-hat transformation

- **Opening top hat**:
  Z(x) - Opening(Z(x))
- **Closing top hat**:
  Closing(Z(x)) - Z(x)
- Top-hat used to extract contrasted components with respect to the background. The opening top-hat extracts positive components and the closing top hat the negative ones
- Typically, top-hats remove the slow drifts, and thus performs a contrast enhancement

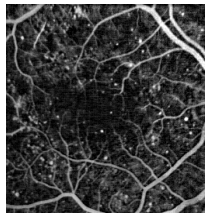


Example of Top-hat

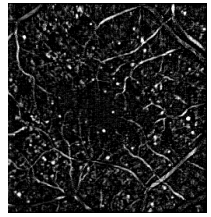
Aim :*extraction of the aneurisms in retian (small white spots)..*



Negative image of the retina.



Top hat by an hexagon opening of size 10.



Top hat by the sup of three segments openings of size 10.

Morphological filters

- Morphological filter: **increasing and idempotent** transformation
- Basic filters: opening and closing operations
- Serial combinations: **filters more progressive** than the openings γ_t and closing φ_t used in the combination

$$\varphi_t \gamma_t \text{ or } \gamma_t \varphi_t$$

- **Sequential alternate filters** Mn and Nn

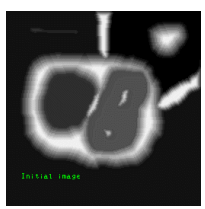
$$\gamma n < ... < \gamma 1 < \varphi 1 < ... < \varphi n$$
$$Mn = \gamma n \varphi n ... \gamma i \varphi i ... \gamma 1 \varphi 1$$
$$Nn = \varphi n \gamma n ... \varphi i \gamma i ... \varphi 1 \gamma 1$$

Morphological Segmentation

- **AIM**: extraction in images of domains with a similar morphology
- **PRINCIPLE**
  - extraction of pertinent markers
  - construction of **Watershed Lines** (2D), or of **separation surfaces** (3D), boundaries of domains

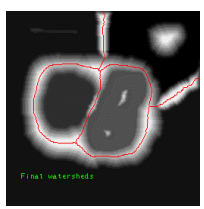
Beucher, S., & Lantuéjoul, C. (1979). Sur l'utilisation de la ligne de partage des eaux en détection de contours. *Rapport Interne N-598, Ecole des Mines de Paris, Centre de Morphologie Mathématique*, 21-46
Meyer, F., & Beucher, S. (1990). Morphological segmentation. *Journal of visual communication and image representation*, *1*(1), 21-46.

Watershed Line
S. Beucher, Ch. Lantéjoul, 1979



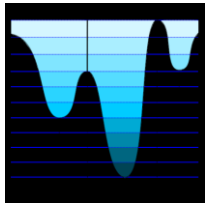
Grey level Image
Gradient image

Representation by a topographical surface



Flooding from minima: basins and watersheds

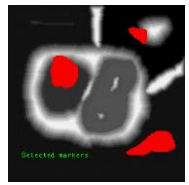
Construction of the Watershed by flooding from minima

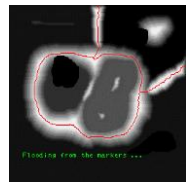


- Dam built to prevent the merging of two lakes
- Elevation of the water level until the complete flooding of the relief
- Boundaries : *watershed lines*
- In 3D, boundaries are watershed surfaces
- Advantage: closed boundaries are generated

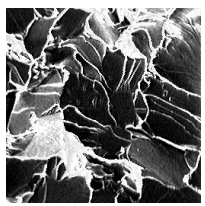
Watershed from markers

- Possible use of markers, instead of filling all basins
- Depending on applications, choice of different types of markers

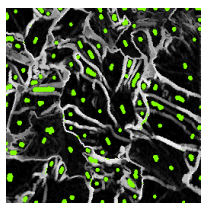




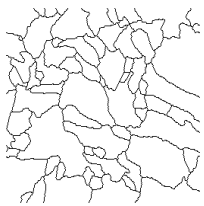
Segmentation: cleavage facets S. Beucher



Fractography of steel (IRSID)

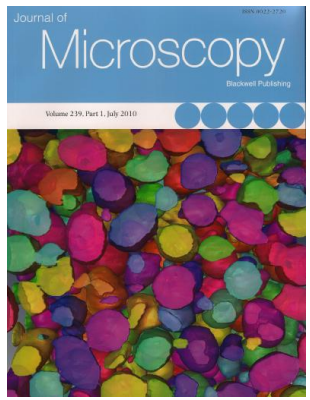


Markers



Contour of cleavage zones

Probabilistic segmentation



Segmentation of 3D microtomographic images of granular materials with the **Stochastic watershed**

(J. Angulo & D. Jeulin, M. Faessel)

Use of **random markers** and estimation of **contour probabilities**, to generate a hierarchy

Angulo, J., & Jeulin, D. (2007, October) Stochastic watershed segmentation. In ISMM (1) (pp. 265-276)

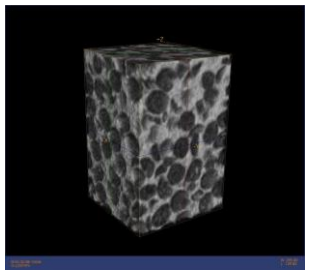
Faessel, M., & Jeulin, D. (2010). Segmentation of 3D microtomographic images of granular materials with the stochastic watershed. Journal of microscopy, 239(1), 17-31.

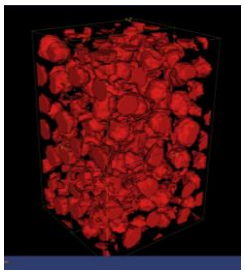
3D Segmentation of grains by the probabilistic watershed

(D. Jeulin & M. Faessel)

Two phase granular material (CEG Gramat)

Fully automatic Segmentation

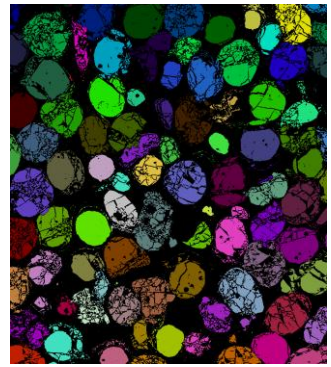




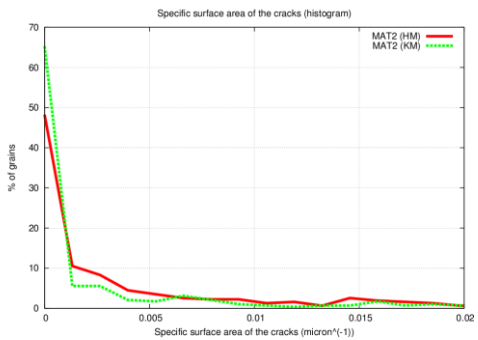
3D Reconstruction of fragmented granular materials
L. Gillibert and D. Jeulin, CEA Gramat

- Aim : to rebuild grains from fragments, then evaluation of damage for each grain
- Fragments gathering
- Tools : closing, distance map, watershed surface, k-means, h-minima

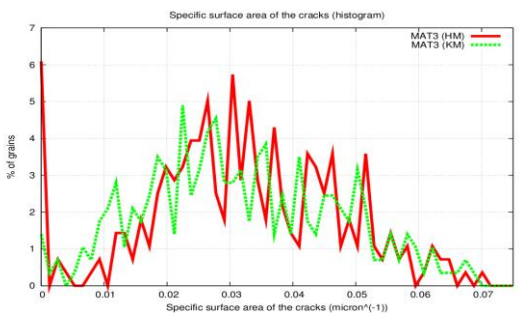
- *Gillibert L. Jeulin D. (2013) 3D Reconstruction and analysis of the FragmentedGrains in a Composite Material, Image Analysis and Stereology, Vol. 32, pp. 107-115*



Measurement of fragmentation
Specific Surface Area of cracks (MAT2)



Measurement of fragmentation
Specific Surface Area of cracks (MAT3)

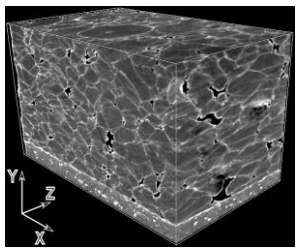


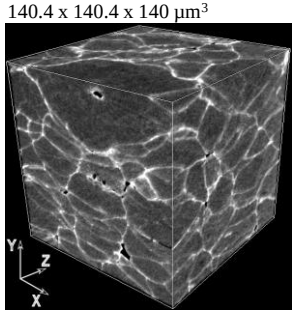
Images of Coldspray deposits

microtomography

- ESRF Images : 900 x 550 x 551 voxels^3 ; 0.35 micron per voxel
- (315 x 192.5 x 192.9 µm^3)

Region of interest :
401 x 401 x 400 voxels^3
140.4 x 140.4 x 140 µm^3

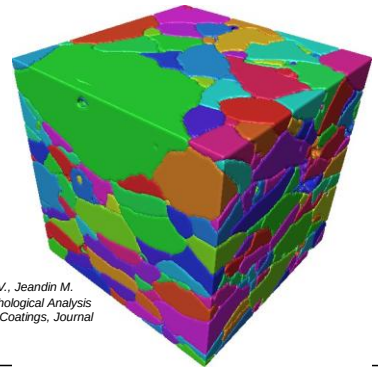




Combination of two segmentations Final Result

i) Multiscale Segmentation

ii) Stochastic watershed



Gillibert L., Peyrega Ch., Jeulin D., Guipont V., Jeandin M. (2012) 3D MultiscaleSegmentation and Morphological Analysis of X-ray Microtomography fromCold-sprayed Coatings, Journal of Microscopy, vol. 248, Part 2, pp. 187-199

Segmentation and Skeleton in 2D

A ball $Bx(r)$ with radius r and centre x is **maximal** for the set A $\Leftrightarrow$ there is no other ball $Bx\prime(r1)$ $(r1 \geq r)$, such that $Bx(r) \subset Bx\prime\,(r1) \subset A$

The **skeleton** of a set A according to a family of balls $B(r)$ is the union of centres of all its maximal balls (namely its **medial** axis)

- skeleton of a disc: its centre
- In the continuous case, the skeleton preserves the connectivity
- On digitized images special algorithms to preserve the connectivity, like sequences of homotopic thinnings

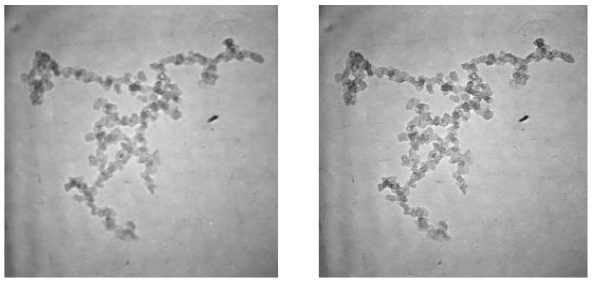
Segmentation and 3D reconstruction of aggregates from 2D TEM micrographs

Aggregates of carbon nanospheres (diameter 30nm)

- Sequence of images with 5^0 tilts
- Binary segmentation to extract characteristic points
- 3D coordinates of particles centres by stereographic reconstruction

Le Coënt A., Jeulin D., Strudel J.L. (1997) Carbon aggregate reconstruction from Transmission Electron Microscope images, Cell Vision, vol. 4 No.2, 219-221

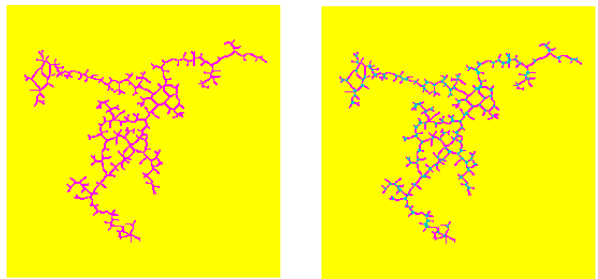
TEM micrographs



Carbon black aggregate (1500 nm); initial image and filtered image

Segmentation of TEM micrographs

Triple points of the skeleton: candidates for centres markers



Skeleton triple points

Basic Books

Matheron G. (1967) *Eléments pour une théorie des milieux poreux*, Masson, Paris
Matheron G. (1975) *Random Sets and Integral Geometry*, J. Wiley
Serra J. (1982) *Image analysis and Mathematical Morphology*, vol 1, Academic Press
Serra J. (ed.) (1988) *Image analysis and Mathematical Morphology*, vol 2, Academic Press
Coster M., Chermant J. L (1985) *Précis d'analyse d'images*, CNRS. *Paris*
Soille, P. (1999) *Morphological Image Analysis: Principles and Applications*, Springer Verlag
Najman L., Talbot, H. (eds.) (2013) *Mathematical morphology: from theory to applications*. John Wiley & Sons
Chiu S. N., Stoyan D., Kendall W. S., Mecke J. (2013) *Stochastic geometry and its applications,* John Wiley & Sons
Jeulin D. (2021) *Morphological models of random structures*, Springer Verlag

Morphological measurements for random structures

Morphological measurements for Random Structures

Morphological measurements

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dominique.jeulin@minesparis.psl.eu

November 2024

Introduction

Microstructure of materials: **heterogeneity**

Qualitative and **quantitative** Analysis of the morphology (**images analysis)**

⇓

Construction of **probabilistic models** of structures for porous or multicomponent media

⇓

Understanding and prediction of the behaviour of materials in service

Quantitative characterization of a random structure

- Random closed sets
- Multicomponent random sets
- Random functions

Quantitative characterization of a random structure

Main criteria

- Basic measures (volume fraction V_V, surface area S_V, integral of mean curvature M_V,...) $\longrightarrow$ stereology
- Size distribution (2D-3D)
- Distribution in space:
  - Clustering
  - Scales
  - Anisotropy
- Connectivity

Principle of morphological Measurements

Two steps (J. Serra):

- **morphological transformation** Φ applied to the structure (use of image analysis)
- **measurement** performed on the transformed object

Choice of measures and transformations according to morphological criteria

Principle of morphological Measurements

Experimental access $\longrightarrow$ **constraints** for allowed measurements:

- invariance by translation
- continuity (with respect to the mesh and to the sampling grid)
- local knowledge (study of the structure from bounded measure fields)
- additivity (averages)
- stereological properties (2D-3D)

Measurements and transformations respecting these constraints

Basic measurements and Minkowski functionals

In integral geometry, it is shown that in R^n, $n+1$ measures satisfy the constraints: the de Minkowski functionals, noted W_i, with W_i homogeneous and of degree $n-i$: $W_i(\lambda A) = \lambda^{n-i} W_i(A)$

In R

$$\begin{array}{c} W_0(A) = l(A) \text{ (length of } A) \\ W_1(A) = 2 \end{array} \qquad (1)$$

Basic measurements and Minkowski functionals

In R^2

$$\begin{array}{c} W_0(A) = A(A) \text{ (area of the set } A) \\ 2W_1(A) = L(A) \text{ (perimeter of } A) \\ 2W_2(A) = 2\pi N(A) \text{ (connectivity number in } R^2) \end{array} \qquad (2)$$

$N(A)$: difference between the number of connected components of A and the number of holes it contains

Basic measurements and Minkowski functionals

In R^3

$W_0(A) = V(A)$ (volume of the set A)
$3W_1(A) = S(A)$ (surface area of A)

$$3W_2(A) = M(A) = \frac{1}{2}\int_{\partial A}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)dS$$

(integral of the mean curvature) (3)

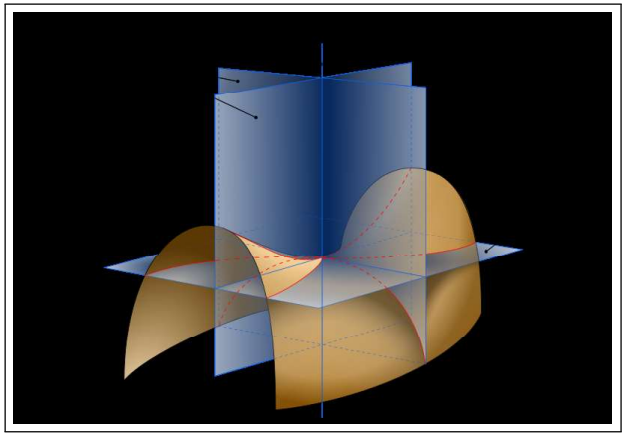
$$3W_3(A) = 4\pi(N-G) = N(A) = \int_{\partial A}\left(\frac{1}{R_1 R_2}\right)dS$$

(integral of the total curvature,
or connectivity number in R^3)

Basic measurements and Minkowski functionals

R_1 and R_2: principal radii of curvature in any point of the boundary ∂A of A
$(N-G)$: difference between the number of connected components of A, N and its genus G
Genus of A : maximal number of closed curves to be drawn on its boundary ∂A without disconnecting it into two parts: 0 for a sphere and 1 for a torus

Basic measurements and Minkowski functionals



Principal curvatures (Wikipedia)

Connectivity numbers

Connectivity numbers in R^2 and in R^3: **topological characteristics** describing the connectivity of an object
in R^2, $N(A)$ is obtained by the difference between two **convexity numbers** $C(A)$ and $C(A^c)$:

$$N(A) = C(A) - C(A^c) = \frac{1}{2\pi}\left(\int_{R>0} d\alpha - \int_{R<0} d\alpha\right) \quad (4)$$

R : radius of curvature in every point of ∂A and $d\alpha = \frac{ds}{R}$ rotation of the normal along the boundary, according to the arc ds

Connectivity numbers

in R^3, connectivity number related to the rotation of the normal to the boundary ∂A by:

$$N(A) = \frac{1}{4\pi}\left(\int_{R_1R_2>0} d\omega - \int_{R_1R_2<0} d\omega\right) \quad (5)$$

Morphological criteria

Basic Specific Measurements in R^3

For a random set A, the functionals W_i are random variables.
For a stationary random set (porous medium, mineralogical texture),with average properties invariant by translation, **specific values,** given per unit volume

- volume fraction V_V
- specific surface area S_V
- integral of the mean curvature M_V or of the total curvature $N_V - G_V$

estimation from slices (1D or 2D)

Specific Measurements and Stereology

Stereological	relationships	(Crofton)	for Slices	
R^3	V_V	S_V	M_V	$N_V - G_V$
	$\uparrow$	$\uparrow$	$\uparrow$	
	$V_V = \overline{A}_A$	$S_V = \frac{4}{\pi}\overline{L}_A$	$M_V = 2\pi\overline{N}_A$	
	$\uparrow$	$\uparrow$	$\uparrow$	
R^2	A_A	L_A	N_A	
	$\uparrow$	$\uparrow$		
	$A_A = \overline{L}_L$	$L_A = \pi\overline{N}_L$		
	$\uparrow$	$\uparrow$		
R	L_L	N_L		

Measures on RCS

For a closed random set, every measure μ must be a random variable $\mu(A)$
Example:

$$\mu(A) = \int 1_A(x)\mu(dx)$$

where $1_A(x)$ is the indicator function of A ($1_A(x) = 1$ if $x \in A$) and μ is a positive measure
$1_A(x)$ is **measurable:** $\mu(A)$ is a random variable with expectation $E[\mu(A)]$

$$E[\mu(A)] = \int P\{x \in A\}\mu(dx)$$

Measures on RCS

Application to the Lebesgue measure : $\mu(dx) = dx$
In R^n, volume

$$\mu_n(A) = \int 1_A(x)dx = W_0(A)$$

For a random closed set,

$$E[\mu_n(A)] = \int P\{x \in A\}dx$$

Estimation of the volume fraction of RCS

If the random closed set is not bounded, one can study $A \cap B$, where B is a bounded domain:

$$E[\mu_n(A \cap B)] = \int_B P\{x \in A\}dx$$

If the random closed set is **stationary**, $P\{x \in A\} = p$ and then

$$E[\mu_n(A \cap B)] = \int_B pdx = p\mu_n(B)$$

For every B in R^k $(0 < k \leq n)$

$$p = \frac{E[\mu_k(A \cap B)]}{\mu_k(B)}$$

And

$$V_V = A_A = L_L$$

Counting

Measurements estimated from images sampled on a **grid of points**, usually regular (square, or hexagonal in the plane), related to a graph

- To every node x of the grid is given the value 1 if $x \in A$, and 0 if $x \in A^c$ (complementary set of A)
- **Number of summits** v of the grid in $A \longrightarrow$ **area** in R^2 or **volume** in R^3
- **Number of intercepts** $N(01)$ in a given direction $\longrightarrow$ estimation of the number of chords generated by A in this direction

Counting

- **Connectivity Numbers** estimated by counting from the Euler relations
- In the plane, N, v, the number of faces (f) and the number of edges (e) are related by:

$$N = v - e + f \tag{6}$$

On a hexagonal or a square grid, (6) is deduced from counting the following configurations (where * means "not tested"):

$$N(A) = N\begin{pmatrix}00\\1\end{pmatrix} - N\begin{pmatrix}0\\11\end{pmatrix} \text{ on the hexagonal grid } C_6$$
$$N(A) = N\begin{pmatrix}10\\00\end{pmatrix} - N\begin{pmatrix}*1\\10\end{pmatrix} \text{ on the square grid } C_8 \tag{7}$$

Counting

In R^3, the Euler relation is expressed as a function of previous properties and of the number of blocks (b):

$$N - G = v - e + f - b \tag{8}$$

For an isotropic structure sampled on a cubic grid, specific connectivity number estimated from relation (8):

$$N_V - G_V = -\lim_{a \to 0} \left(\frac{C(0) - 3C(a) + 3P\binom{11}{11} - P(C)}{a^3} \right) \tag{9}$$

where C is the cube with side a and $C(h)$ is the covariance of the random set A

Minkowski Tensors

Symmetric tensors generated from symmetric tensor products of vector x and the normal vector n on ∂A
Restricting to tensors of rank two provides quadratic normal tensors (eigen vectors give information on local orientation)
In $\mathbb{R}^3$, noting $G_1 = 1$, $G_2 = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$, and $G_3 = \frac{1}{R_1}\frac{1}{R_2}$,

$$\begin{aligned} W_0^{2,0} &= \int_A x \otimes x dV \\ W_\nu^{r,s} &= \frac{1}{3}\int_{\partial A} G_\nu x^r \otimes n^s dS \end{aligned} \tag{10}$$

with $\nu = 1, 2, 3$ and $(r, s) = (2, 0), (1, 1), (0, 2)$. Obviously from definition of Eqs (10),

$$W_\nu^{r,s}(\lambda A) = \lambda^{3+s-\nu} W_\nu^{r,s}(A)$$

Jensen E. B. V., Kiderlen M. (eds.) (2017) Tensor valuations and their applications in stochastic geometry and imaging, Springer.

Minkowski Tensors

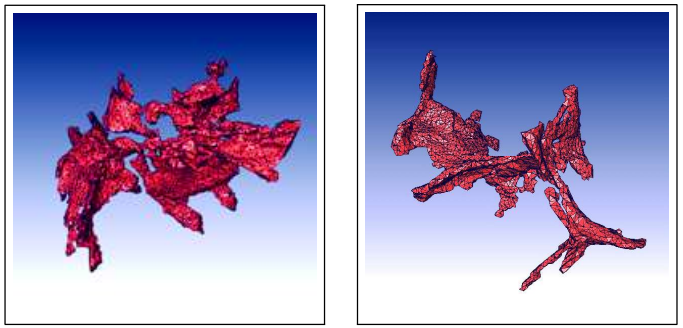
Tensors $W_1^{0,2}$ and $W_2^{0,2}$ invariant by translation
Scalar measurement of anisotropy obtained from the ratio of the smallest and the largest eigenvalues
Tensor $W_0^{2,0}$ related to the inertia tensor $I(A)$ used in to characterize particles of complex shapes and their orientations.The inertia tensor can be estimated for objects of any shape, I_d being the identity matrix:

$$\begin{aligned} I_{ij} &= \int (-x_i x_j + \delta_{ij}) dV \\ I(A) &= -W_0^{2,0}(A) + I_d\, tr(W_0^{2,0}(A)) \end{aligned}$$

Similarly $W_1^{2,0}$ is related to the tensor of inertia of a mass uniformly distributed over ∂K. Furthermore,

$$S(A) = 3tr(W_1^{0,2}(A))$$

Minkowski Tensors and Inertia features of particles with complex shapes



Microtomography of Intermetallic particles in an aluminium alloy

Reduced principal moments obtained from normalization of the principal components of inertia ($\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3$). For any set A (even non connected) :

$$\lambda_1 + \lambda_2 + \lambda_3 = 1;\ \lambda_i \leqslant 0.5\ (i = 1, 2, 3);\ \lambda_2 \geqslant 0.5(1 - \lambda_1)$$

Inertia features of particles with complex shapes

- Any object A represented by a point in the plane λ_1, λ_2 located inside a triangle typical of the shape
- For convex 3D objects, the summits of the triangle represent the three most distant types of mass distribution: spherical, flat, and thread type
- Between these summits there is a continuous change of shape. On the edges of the triangle, objects show typical shapes: prolate ellipsoid, oblate ellipsoid , and planar ellipse
- Orientation of the main axis of inertia (parallel to the eigen vector corresponding to the eigen value λ_1) allows us to study the in situ distribution of orientations of particles, and their evolution during during rolling

Parra-Denis E., Barat C., Jeulin D., Ducottet C. (2008) 3D complex shape characterization by statistical analysis: Application to aluminium alloys. Materials characterization, 59(3), 338-343

Basic Operations of Mathematical Morphology

Qualitative description of the morphology of objects: spherical or elongated pore, polyhedral grain...

- **Structuring element**: choice of an object K (compact set, like a point, sphere, segment) implanted in every point x of the euclidean space R^n
- **Answer to a question** on the mutual location of K_x and of the studied set A, for every point x
- For a **binary answer**, **indicator function** of $\Phi(A)$ obtained by transformation of the set A ($k(x) = 1$ for $x \in \Phi(A)$, else $k(x) = 0$)

Basic Operations of Mathematical Morphology

Dilation and erosion of A by K:

$$\begin{array}{l} \text{i) } K \text{ hits } A? \ (K \cap A \neq \emptyset)? \\ \text{ii) } K \text{ included in } A \ (K \subset A)? \end{array} \quad (11)$$

$$A \oplus \check{K} = \{x, K_x \cap A \neq \emptyset\} = \cup_{y \in K} A_{-y} = \cup_{x \in A, y \in K} \{x - y\} \quad (12)$$

$$A \ominus \check{K} = \{x, K_x \subset A\} = \cap_{y \in K} A_{-y} = (A^c \oplus \check{K})^c \quad (13)$$

$K_x = \{x + y, y \in K\}$: translated of K to point x; $\check{K}$ obtained by transposition of K: $\check{K} = \{-x, x \in K\}$

Basic Operations of Mathematical Morphology

Symbols $\oplus$ and $\ominus$: Minkowski **addition** and **substraction** :

$$A \oplus K = \cup_{x \in A, y \in K} \{x + y\} = \cup_{y \in K} A_y = \cup_{x \in A} K_x \quad (14)$$

$$A \ominus K = \cap_{y \in K} A_y = (A^c \oplus K)^c \quad (15)$$

Steiner Formulae

Relation between the Minkowski functionals of a convex set A dilated by a convex set K and the functionals of A and of K:

$$E\{W_i(A \oplus \check{K})\} = \frac{1}{C_{0n}} \sum_{k=0}^{k=n-i} \binom{n-i}{k} W_{k+i}(A) W_{n-k}(K) \qquad (16)$$

where $E\{W_i(A \oplus \check{K})\}$ is the average value of $W_i(A \oplus \check{K})$ over the rotations of K around the origin O.
K: ball with radius r, (polynomials with degree $n-i$)

Steiner Formulae

For $A \subset R^3$ and for $K \subset R^3$

$$E\{V(A \oplus \check{K})\} = V(A) + \frac{M(K)}{4\pi} S(A) + \frac{M(A)}{4\pi} S(K) + V(K)$$

$$E\{S(A \oplus \check{K})\} = S(A) + \frac{M(A)M(K)}{2\pi} + S(K) \qquad (17)$$

$$E\{M(A \oplus \check{K})\} = M(A) + M(K)$$

Steiner Formulae

For $A \subset R^3$ and for $K \subset R^2$, $M(K) = \frac{\pi}{2} L(K)$ and

$$E\{V(A \oplus \check{K})\} = V(A) + \frac{L(K)}{8} S(A) + \frac{M(A)}{2\pi} A(K)$$

$$E\{S(A \oplus \check{K})\} = S(A) + \frac{M(A)L(K)}{4} + 2A(K) \qquad (18)$$

For $A \subset R^3$ and for $K \subset R$ (for instance the segment: $K = l$)

$$E\{V(A \oplus \check{K})\} = V(A) + \frac{l}{4} S(A) \qquad (19)$$

Steiner Formulae

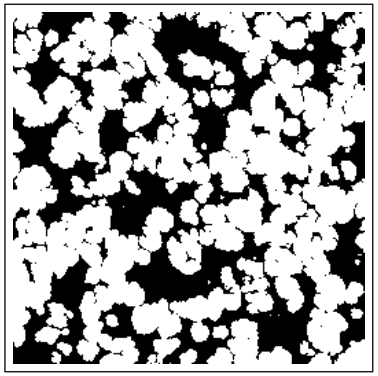
For $A \subset R^2$ and for $K \subset R^2$,

$$E\{A(A \oplus \check{K})\} = A(A) + \frac{L(A)L(K)}{2\pi} + A(K)$$

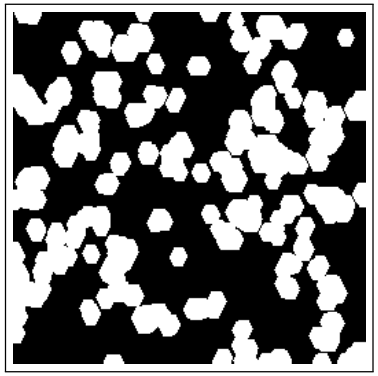
$$E\{L(A \oplus \check{K})\} = L(A) + L(K) \qquad (20)$$

Morphological Criteria

Granulometry



Fe-Ag



Separate objects

Granulometry

Size Distribution

- diameters of de cavities
- crack lengths
- Areas of objects
- Any increasing criterion W: $A \subset B \Longrightarrow W(A) \leq W(B)$

Access from 1D or 2D slices (spherical shape, or at least convex)
Size Distribution of any media (including connected networks) by **morphological opening** (erosion followed by a dilation) by convex structuring elements

Granulometry

Axiomatic of granulometries

Definition (G. Matheron) A granulometry is a family of set transformations Φ_λ depending on a positive parameter λ (the size), satisfying the following properties:

$$\text{i) } \Phi_\lambda(A) \subset A \text{ (Φ_λ is \textbf{anti-extensive})}$$

$$\text{ii) if } A \subset B, \Phi_\lambda(A) \subset \Phi_\lambda(B) \text{ (Φ_λ is \textbf{increasing})}$$

$$\text{iii) } \Phi_\lambda \circ \Phi_\mu = \Phi_\mu \circ \Phi_\lambda = \Phi_{\lambda \vee \mu} \text{ (\textbf{absorption for the composition})}$$

As a consequence of axiom iii), applied to $\lambda = \mu$, the transformation Φ_λ must be **idempotent:**

$$\Phi_\lambda \circ \Phi_\lambda = \Phi_\lambda$$

Granulometry

Size distribution by opening by convex sets

Openings of the set A by λK, $(\lambda > 0)$, for a **convex set** K, noted K_λ:

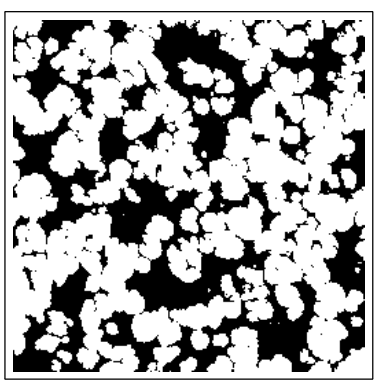
$$\Phi_\lambda(A) = (A \ominus \check{K}_\lambda) \oplus K_\lambda$$

Transformation to be applied to any set (isolated particles, or connected medium):

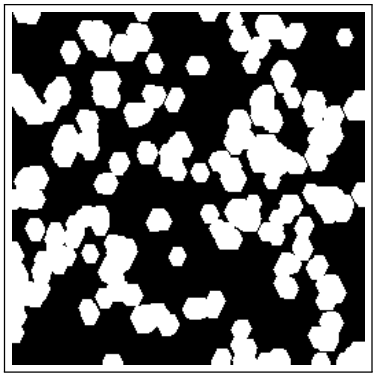
$$\Phi_\lambda(A) = A_{K_\lambda} = \{x \in A; \exists y \in A \text{ with } K_{\lambda,y} \subset A\} \quad (21)$$

A_{K_λ} is the set of points of A covered by K_λ translated in space, while remaining included in A

Granulometry



Fe



Hexagonal opening (6)

Granulometry

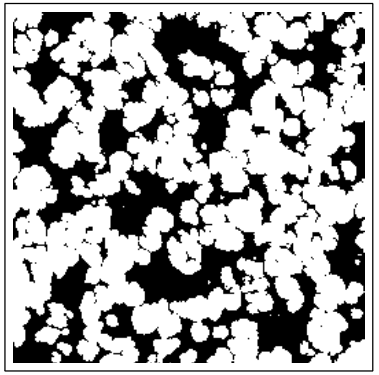
Size distribution by closing by convex sets

The closing operation is dual of the opening:

$$A^{K_\lambda} = (A \oplus \check{K}_\lambda) \ominus K_\lambda = (\Phi_\lambda(A^c))^c \quad (22)$$

By closing A by compact convex sets K, size distribution of the set A^c. The size distribution for a given granulometry is obtained from measures: counting numbers,or measure of a volume in the space R^n

Granulometry



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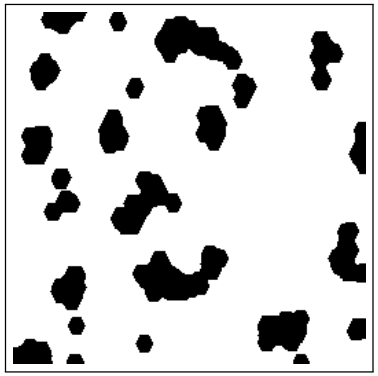


Hexagonal closing (6)

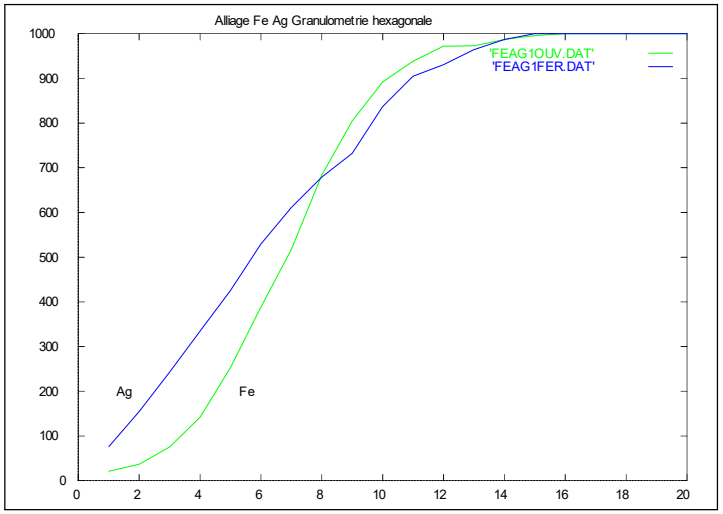
Two dimensional Granulometry

Openings or closings, followed by the measurement of an area (or volume for extension to 3D)
Cumulative measure distribution, from openings by convex structuring elements:

$$G(\lambda) = \frac{P\{x \in A\} - P\{x \in A_{K_\lambda}\}}{P\{x \in A\}} \tag{23}$$

From equation (23) are deduced in R^2 the moments of the area S of the largest K containing x and included in A. For a disc with radius r:

$$\begin{aligned} E\{S\} &= 2\pi \int_0^\infty (1 - G(r)) r \, dr \\ E\{S^n\} &= 2n\pi \int_0^\infty (1 - G(r)) r^{2n-1} \, dr \end{aligned} \tag{24}$$

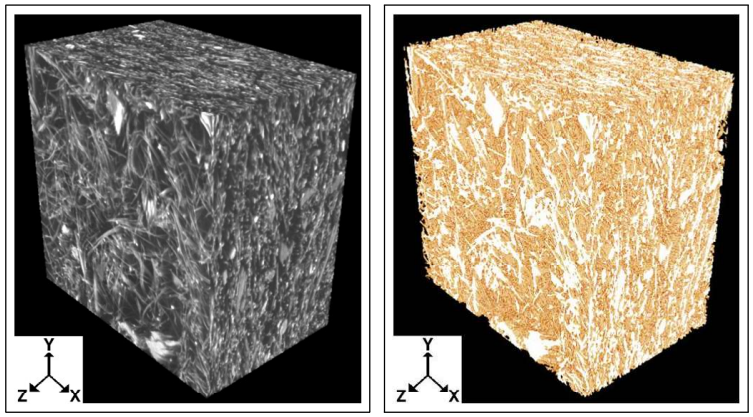


Hexagonal Granulometry

3D Granulometry of a fibrous medium

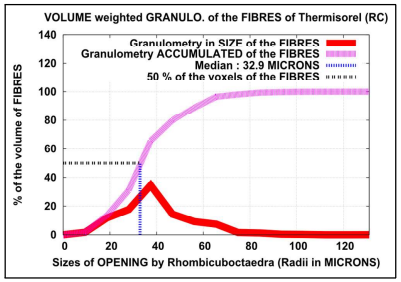
Project Silent Wall. Peyrega Ch., Jeulin D., Delisée Ch., Malvestio J. (2009) 3D Morphological modelling of a random fibrous network, Image Analysis and Stereology, Vol. 28, pp.129-141

3D Granulometry of a fibrous medium

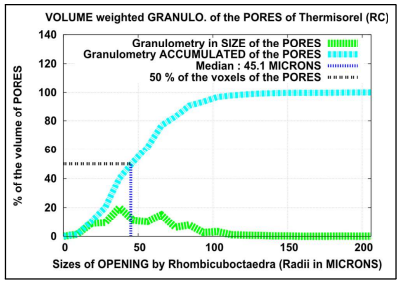


Microtomography US2B Binary image

3D Granulometry of a fibrous medium



Fibres



Pores

Linear granulometry

Available from linear erosions (depends on the orientation of l)
For a stationary random set A

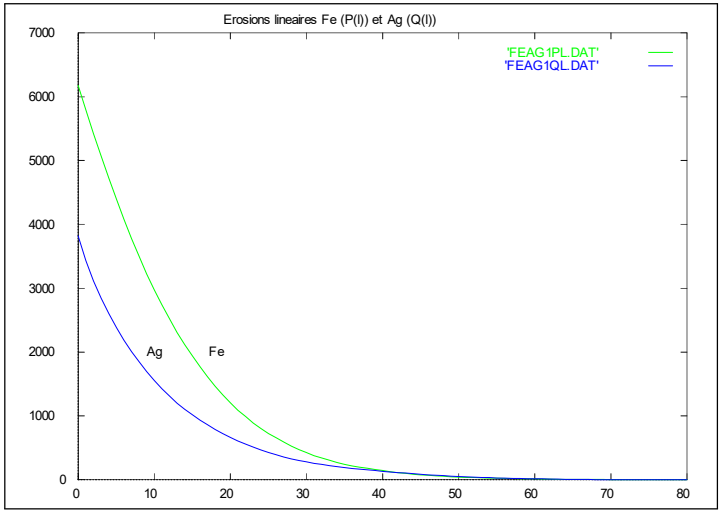
$$P(l) = P\{l \subset A\};\ Q(l) = P\{l \subset A^c\}$$

$P(l)$ and $Q(l)$ have a derivative. We have:

$$P\{x \in A_l\} = P(l) - lP'(l) \tag{25}$$

$$P\{x \in A_l^c\} = Q(l) - lQ'(l) \tag{26}$$

since $-P'(l)$ and $-Q'(l)$ are specific numbers of intercepts of A and of A^c , with lengths larger than l



Linear Erosions (Fe and Ag)

Linear granulometry

$P\{x \in A_l\}$ gives access to the **measure distribution** of intercepts, $G(l)$ (with the density $g(l)$ when it exists), every chord having a weight proportional to its length , (the probability for x to belong to a chord with length larger than l)

Linear granulometry

Measure Granulometry

For A^c (similar result for A after replacing $Q(l)$ by $P(l)$):

$$1 - G(l) = \frac{Q(l) - lQ'(l)}{Q(0)}$$
$$g(l) = \frac{lQ''(l)}{Q(0)} \quad (27)$$

Average measure intercept, L^*

$$L^* = \frac{2}{Q(0)} \int_0^\infty Q(l)\, dl \quad (28)$$

Linear granulometry

When $-Q'(0)$ is finite (specific number of intercepts), **number size distribution of intercepts** $\rightarrow$ counting the intercepts after length classification

Distribution function $F(l)$ and density (histogram) $f(l)$:

$$1 - F(l) = \frac{\text{number of d'intercepts with length} \geq l}{\text{total number of intercepts}} = \frac{Q'(l)}{Q'(0)}$$
$$f(l) = \frac{-Q''(l)}{Q'(0)} \quad (29)$$

Average "number" intercept $\overline{L}$

$$\overline{L} = \frac{-Q(0)}{Q'(0)} \quad (30)$$

Linear granulometry

Measure and number Size distributions

Two types of linear size distributions: $g(l)$ is proportional to $f(l)$ and to l (eqs (27,29)).

Example $f(l) = a\exp(-al)$ (exponential distribution),

$g(l) = a^2 l \exp(-al)$ (gamma distribution); then, $\overline{L} = \frac{1}{a}$ and $L^* = \frac{2}{a}$

For applications, choice of the most appropriate type of distribution

Size distribution of objects

For isolated objects, size distribution obtained from the distribution of a measurement (area,diameter) made on every object (and correction of edge effects by means of the Miles-Lantuejoul correction)

Size distribution of spheres

Estimation of the distribution of diameters of a population of spheres in R^3 from data obtained on sections: basic stereological problem ("**unfolding problem**")
Sometimes used for estimating a size distribution of objects with a non spherical shape (grains of a polycrystal seen in section)
Basic relations between properties in R^3 (number of spheres $N_V^{(3)}$ and distribution of diameters in number $F_3(D)$) and induced properties in R^2 (average number of discs $N_V^{(2)}$, distribution of diameters in number $F_2(D)$) and in R (average number of chords $N_V^{(1)}$ and distribution of lengths in number $F_1(D)$)

Size distribution of spheres

From R to R^3:

$$N_V^{(3)} = \frac{2}{\pi} N_V^{(1)} F_1''(0)$$
$$1 - F_3(D) = \frac{1}{D} \frac{F_1'(D)}{F_1''(0)} \tag{31}$$

From dimension i to $i+1$ $(i = 1, 2)$:

$$N_V^{(i+1)} = \frac{N_V^{(i)}}{\pi} \int_0^\infty \frac{F_i'(h)}{h}\, dh$$
$$N_V^{(i+1)}(1 - F_{i+1}(D)) = \frac{N_V^{(i)}}{\pi} \int_D^\infty \frac{F_i'(h)}{\sqrt{h^2 - D^2}}\, dh \tag{32}$$

Specific numerical techniques (regularization of an ill-posed problem)

Spatial Arrangement

Scales, clusters, or preferential associations between component of a microstructure:

- **covariance**
- **distance function**
- **anisotropy**

Spatial Arrangement: Covariance

Covariance $C(x, x+h)$ of a random set A

$$C(x, x+h) = P\{x \in A, x+h \in A\} \tag{33}$$

For a **stationary** random set, $C(x, x+h) = C(h)$
If in addition A is **ergodic**, $C(h)$ is estimated from the volume fraction of $A \cap A_{-h}$:

$$C(h) = V_V(A \cap A_{-h}) = V_V(A \ominus \check{h}) \tag{34}$$

Spatial Arrangement: Covariance

Estimation of the covariance from images (like plane sections) inside a mask X, by means of the **geometrical covariograms** of the sets $A \cap X$ ($K_{A\cap X}(h)$) and X ($K_X(h)$):

$$C^*(h) = \frac{A((A\cap X)\cap (A\cap X)_{-h})}{A(X\cap X_{-h})} = \frac{K_{A\cap X}(h)}{K_X(h)} \qquad (35)$$

Spatial Arrangement: Covariance

Result of the erosion by $\{x, x+h\}$, which depends on vector h (by its modulus $\mid h \mid$ and its orientation α) characteristic of the size and of the arrangement of connected objects building the set $A \rightarrow$ variations of $C(h)$ with h
Covariance $Q(h)$ of A^c (with $Q(0) = q = 1-p$):

$$Q(h) = P\{x \in A^c, x+h \in A^c\} = 1 - 2C(0) + C(h) \qquad (36)$$

The covariance characterizes simultaneously the two sets (A, A^c), while the two granulometries of (A, A^c) bring additional information

Spatial Arrangement: Covariance

Properties of the covariance of a random set in R^3

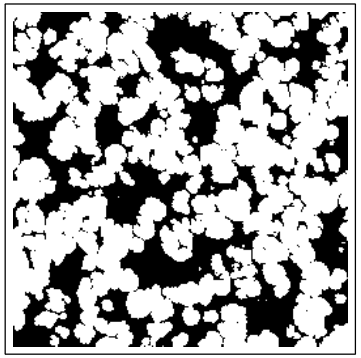
- $C(0) = P\{x \in A\} = p$
- $C(h) = C(-h)$
- $\frac{2}{\pi}\int_0^{2\pi} -\left(\frac{\partial C(h,\alpha)}{\partial h}\right)_{h=0} d\alpha = S_V(A)$ when the partial derivative remains finite.
- If $C(0) - C(h) \simeq h^\beta$ for $h \rightarrow 0$, with $0 < \beta < 1$, the boundary of A has a **non integer Haussdorf dimension** $d = 3 - \beta$ (A is a **fractal** set)
- $C(\infty) = p^2$ (the covariance of a stationary and ergodic random set reaches a sill)

Spatial Arrangement: Covariance

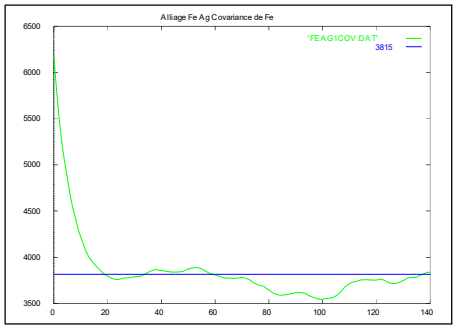
Properties of the covariance of a random set in R^3

- For the orientation α, $C(h)$ reaches a sill at the distance a_α, or **range** (characteristic length scale of the structure): $C(a_\alpha) = C(\infty) = V_V(A)^2 = p^2$
- Presence of various scales $\rightarrow$ inflections of the experimental covariance (nested structures, like clusters, clusters of clusters, etc.)
- Periodicity in images $\rightarrow$ periodicity of the covariance
- Anisotropic structures studied by roses of directions, from the derivative of the covariance in $h = 0$

Spatial Arrangement: Covariance

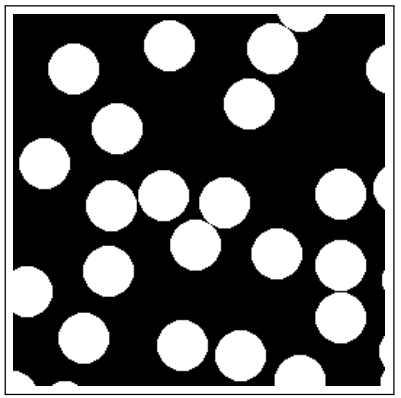


Binary image of *Fe*

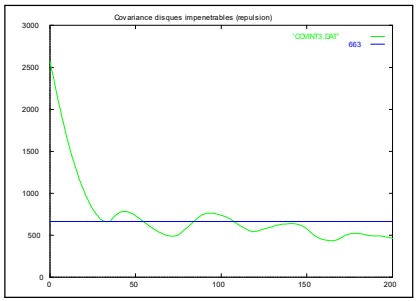


Covariance of Fe

Spatial Arrangement: Covariance

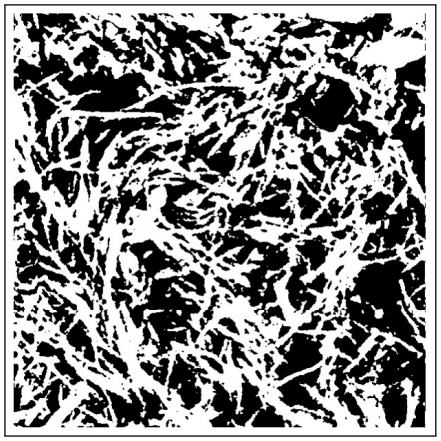


Non overlapping random discs



Covariance

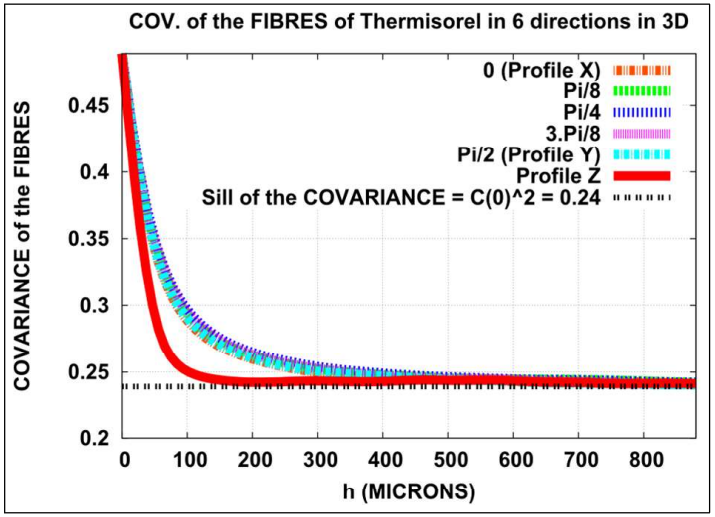
Covariance of a fibrous medium



Transverse section xOy

Axial section xOz

Covariance of a fibrous medium



Transverse isotropy of fibres $C(h)$

Random Functions: Characterization by the spatial law

RF $Z(x)$ $(E \rightarrow \overline{R})$ characterized for all $m \in N$ by (Kolmogorov):

$$F(x,z) = P\{Z(x_1) < z_1, ..., Z(x_m) < z_m\} \quad \text{where } x \in E^m \text{ and } z \in \overline{R}^m \tag{37}$$

- For $x = x_1$ and $x = \{x_1, x_2\}$, uni and bivariate distributions
- restriction to countable sets of points $\{x_1, x_2, ..., x_m\}$ in the space E

Random Functions: Characterization by moments

Covariance $C(x, x+h)$ and central correlation function $\overline{W}_2(x, x+h)$

$$C(x, x+h) = E\{Z(x)Z(x+h)\}$$
$$\overline{W}_2(x, x+h) = E\{Z(x)Z(x+h)\} - E\{Z(x)\}E\{Z(x+h)\} \tag{38}$$

Central correlation function of order m $\overline{W}_m(x)$, with $x \in E^m$:

$$\overline{W}_m(x) = E\{(Z(x_1) - E\{Z(x_1)\})...(Z(x_m) - E\{Z(x_m)\})\} \tag{39}$$

Random Functions: Characterization by moments

To study physical properties of random media, use of the correlation functions
Central correlation function deduced from the set covariance in the case of a two phase composite with properties $Z = Z_1$ when $x \in A$ and $Z = Z_2$ when $x \in A^c$:

$$\overline{W}_2(h) = E\{(Z(x+h) - E(Z))(Z(x) - E(Z))\}$$
$$= (Z_1 - Z_2)^2(C(h) - p^2) = (Z_1 - Z_2)^2(Q(h) - q^2) \tag{40}$$

Useful to solve sampling problems (estimation variance and statistical RVE) for any RF $Z(x)$

Integral range and estimation variance

Stationary RF $Z(x)$, with expectation $E\{Z\}$, and point variance D_Z^2

- Size of a RVE (representative volume element) for $Z(x)$
- variance $D_Z^2(V)$ of the average of $Z(x)$ in the volume V $(\bar{Z}(V) = \frac{1}{V}\int_V Z(x)dx)$:

$$D_Z^2(V) = \frac{1}{V^2}\int_V \int_V \overline{W}_2(x-y)\, dxdy, \tag{41}$$

- For $V \gg A_3$ (for finite A_3), to first order in $1/V$, A_3 being the **Integral Range**

$$D_Z^2(V) = D_Z^2 \frac{A_3}{V}$$

$$A_3 = \frac{1}{D^2[Z]}\int_{R3} \overline{W}_2(h)\, dh$$

Statistical RVE

Volume V made of $k = V/A_3$ subvolumes in which the average valuesof the RF $Z(x)$ are uncorrelated random variables
Absolute error ϵ_{abs} and relative error ϵ_{rela} of the average value obtained for n independant realizations of volume V, deduced from the confidence interval (to 95%)

$$\epsilon_{abs} = \frac{2D_Z(V)}{\sqrt{n}}; \epsilon_{rela} = \frac{\epsilon_{abs}}{E\{Z\}} = \frac{2D_Z(V)}{E\{Z\}\sqrt{n}} \quad (42)$$

RVE: volume V (obtained for instance for $n = 1$ realization in the ergodic case) for which $E\{Z\}$ is estimated with a given relative precision (for instance $\epsilon_{rela} = 1\%$). Applied to the volume fraction (with $Z(x) = 1_A(x)$, $E\{Z\} = p$ and $D^2[Z] = p(1-p)$), to a stress field $\sigma(x)$ or a strain field $\varepsilon(x) \rightarrow$ **RVE of effective properties**

Kanit T., Forest S., Galliet I., Mounoury V., Jeulin D. (2003). Determination of the size of the representative volume element for random composites: statistical and numerical approach. International Journal of solids and structures, 40(13-14), 3647-3679

Statistical RVE

Example: $2\sigma_E(V)$ and $2\sigma_E(S)$ ($\rightarrow$ interval of variation $q \pm 2\sigma$ of $q = 1 - p$ expected for observations) of a cube with volume $V = L^3$ and of a square of area $S = L^2$ for a Boolean model of spheres with diameter a with $a/L = 0.1$

q	0.05	0.1	0.2	0.3	0.5
$2\sigma_E(V)$	0.0054	0.00865	0.0134	0.0167	0.02
$2\sigma_E(S)$	0.021	0.032	0.049	0.0603	0.071

Spatial Agencement: Cross Covariances

Multicomponent random set with components A_i $(i = 1, 2, ..., m)$

Separate study for every covariance $C_{ii}(h)$, and mutual associations from the cross covariances $C_{ij}(h)$:

$$C_{ij}(x, x+h) = P\{x \in A_i, x+h \in A_j\} \quad (43)$$

Stationary and ergodic multicomponent random set:

$$C_{ij}(h) = V_V(A_i \cap A_{j-h}) \quad (44)$$

Spatial Agencement: Cross Covariances

Estimation of cross covariances image analysis from cross geometrical covariograms of the sets $A_i \cap X$ ($K_{A \cap X}(h)$) and X ($K_X(h)$):

$$C^*_{ij}(h) = \frac{A((A_i \cap X) \cap (A_j \cap X)_{-h})}{A(X \cap X_{-h})} = \frac{K_{A_i \cap X,\ A_j \cap X}(h)}{K_X(h)} \quad (45)$$

Spatial Agencement: Cross Covariances

Every $C_{ij}(h)$ brings its own information on the mutual association of the pair (A_i, A_j).
Morphological properties of cross covariances similar to those of covariances (case when the A_i build a space tessellation, such as $P\{x \in A_j \cap A_j\} = 0$)

Spatial Agencement: Cross Covariances

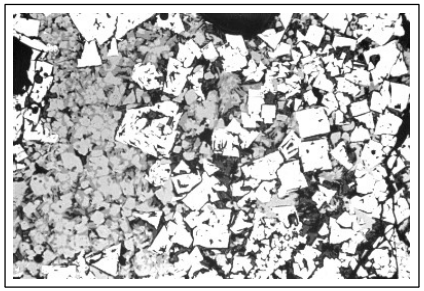
Properties of the cross covariance of a random set in R^3

- $C_{ij}(0) = 0$
- $\sum_{j=1}^{j=m} C_{ij}(h, \alpha) = C_{ii}(0, \alpha) = p_i$
- $C_{X_i \cup X_j}(j, \alpha) = C_{ii}(h, \alpha) + C_{jj}(h, \alpha) + C_{ij}(h, \alpha) + C_{ji}(h, \alpha)$
- $\frac{1}{\pi} \int_0^{4\pi} \left(\frac{\partial C_{ij}(h, \alpha)}{\partial h} \right)_{h=0} d\alpha = S_{V_{ij}}$ (specific surface area of contact between he components A_i and A_j), when the derivative is finite
- $C_{ij}(\infty) = p_i p_j$ (sill)

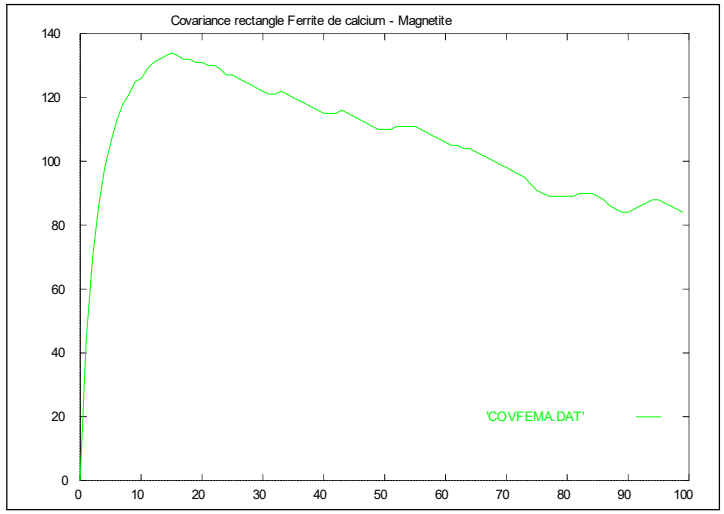
Spatial Agencement: Cross Covariances

For orientation α, $C_{ij}(h)$ reaches its sill at distance $a_{ij\alpha}$, or **range**:
$C_{ij}(a_{ij\alpha}) = C_{ij}(\infty) = p_i p_j$
For $h \geq a$ the events $\{x \in A_i\}$ and $\{x + h \in A_j\}$ are uncorrelated

Spatial Agencement: Cross Covariances



Micrograph of an iron ore sinter (with decreasing reflecting power: Hematite, magnetite, calcium ferrites , slag, pores)



Cross covariance calcium ferrites-magnetite

Spatial Agencement: Cross Covariances

Characterization of multicomponent textures from surface areas of contact

Behavior in $h = 0$ of the cross covariances $C_{ij}(h) \longrightarrow$ specific surfaces area of contact $S_{V_{ij}}$:

- **Distribution of contacts** of the component A_i with the components A_j $(j \neq i)$, or transition probability from A_i to A_j for a random point over ∂A_i:

$$p_{ij} = \frac{S_{V_{ij}}}{S_{V_i}} \tag{46}$$

  In general, $p_{ij} \neq p_{ji}$, as $S_{V_i} \neq S_{V_j}$

Spatial Agencement: Cross Covariances

Characterization of multicomponent textures from surface areas of contact

- **Indexes of coordination** $i_c(ij)$, symmetrical in i and j:

$$i_c(ij) = \frac{S_{V_{ij}}}{S_{V_i}} \frac{S_V}{S_{V_j}} \tag{47}$$

  with $S_V = \sum_{j=1}^{j=m} S_{V_j}$ (total surface area of contact of the medium)

- Compare p_{ij} to $\frac{1}{m-1}$ and $i_c(ij)$ to $\frac{m}{m-1}$, to test a mutual attraction or a repulsion between components i and j

Spatial arrangement from distance distributions

Clustering

- Distribution of distances of any point x outside objects to their boundary (estimated by the volume fraction measurements after dilation by balls with increasing radius)
- Distances of any point x located in one component (phase of a composite) to boundaries of another component (cracks)
- Construction of zones of influence of objects (or Voronoï cells); estimation of the distribution of its area (or of its inverse, equivalent to an intensity) $\longrightarrow$ 2D-3D transition to be done from models

Spatial arrangement from distance distributions

For a random set A and a ball with radius r, $B(r)$:

$$T(B_x(r)) = P\{x \in A \oplus B(r)\}$$

$T(B_x(r))$ allows us to estimate the distribution of the random variable $R(x, A)$, or distribution of the first point of contact, where the symbol $\vee$ means the upper bound (sup):

$$R(x, A) = \vee\{r; B_x(r) \subset A^c\}$$

$$F_x(r) = P\{R(x, A) \leq r \mid x \in A^c\} = \frac{T(B_x(r)) - T(x)}{1 - T(x)}$$

Spatial arrangement from distance distributions

For a stationary random set A,

$$T(r) = 1 - F(r) = \frac{1 - P\{x \in A \oplus B(r)\}}{1 - p} \quad (48)$$

The moments of $F(r)$ allow us to summarize this distribution.
For a **fractal** random set A (with irregular boundaries with non integer dimension d), $F(r)$ behaves as r^β when $r \longrightarrow 0$ with $d = n - \beta$

Spatial arrangement from distance distributions

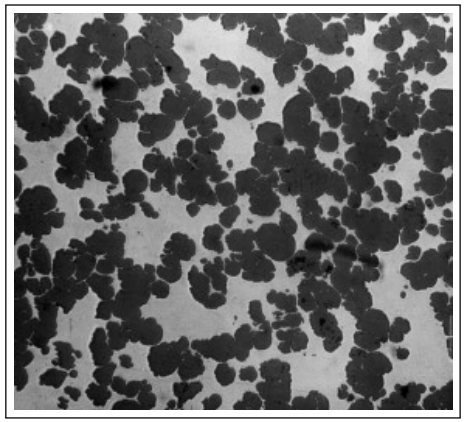
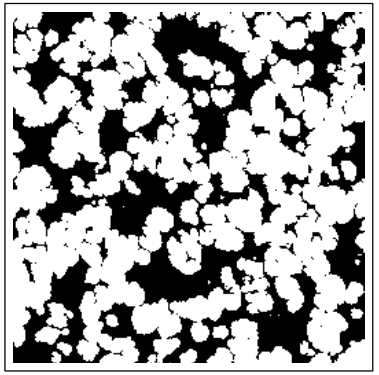
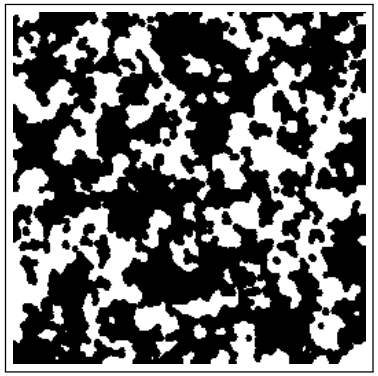


Image of a two-phase alloy Fe (black) Ag (grey)

Spatial arrangement from distance distributions

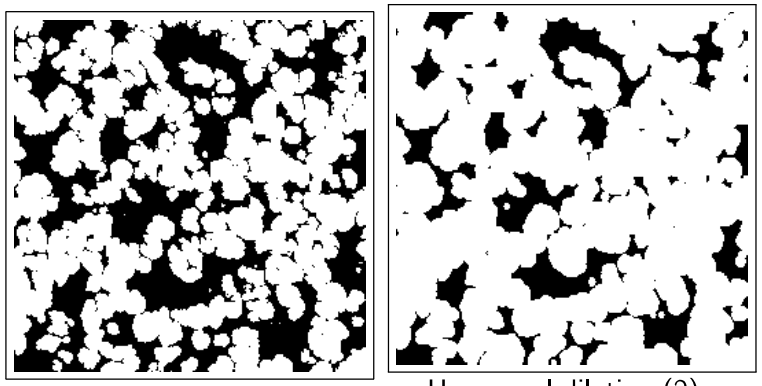


Fe

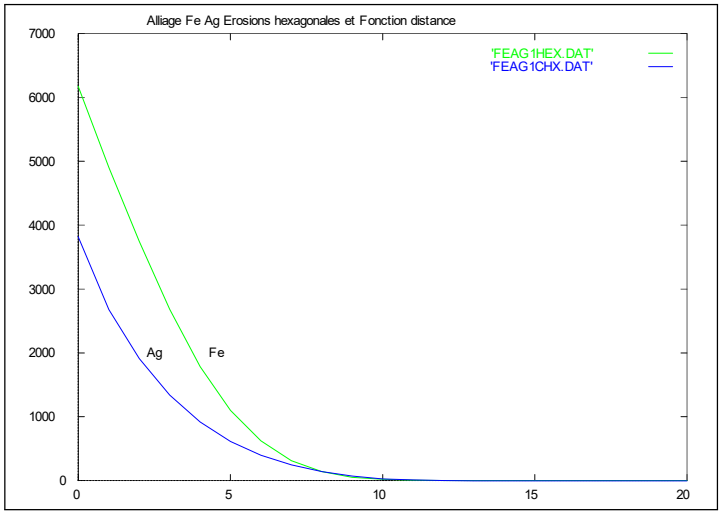


Hexagonal erosion (2)

Spatial arrangement from distance distributions



Fe Hexagonal dilation (2)



Hexagonal Distance

Spatial arrangement: increasing neighbourhoods

For **multi component systems,** possible generalization of cross-covariances $C_{ij}(h)$ to degenerate components (with dimension $d < n$), as cracks, boundaries, or points, for which covariances are equal to zero

Comparison of microstructure in neighborhoods with increasing sizes of a given component, to the overall microstructure $\longrightarrow$ measurement of specific properties after dilation by balls with increasing radius

Spatial arrangement: increasing neighbourhoods

$$\Phi_j(k,r) = P\{x \in (A_k \oplus B(r)) \cap A_j\} \tag{49}$$

$$\rho_j(k,r) = \frac{\Phi_j(k,r)}{P\{x \in (A_k \oplus B(r))\} - p_k} \frac{1-p_k}{p_j} \tag{50}$$

When $\rho_j(k,r) > 1$, a preferential association is expected between components A_k and A_j, on scale r

For $\rho_j(k,r) < 1$, a repulsion effect is detected between the two components at this scale

Spatial arrangement: increasing neighbourhoods

The functions $\Phi_j(k,r)$ and $\rho_j(k,r)$ are related to the distributions of distances:

- of a random point x in A_k^c to the boundary of A_k:

$$F(k,r) = P\{d(x,A_k) < r \mid x \in A_k^c\} = \frac{P\{x \in (A_k \oplus B(r))\} - p_k}{1 - p_k} \tag{51}$$

- of a random point x in A_j to the boundary of A_k:

$$F_j(k,r) = P\{d(x,A_k) < r \mid x \in A_j\} = \frac{\Phi_j(k,r)}{p_j} \tag{52}$$

We have:

$$\rho_j(k,r) = \frac{F_j(k,r)}{F(k,r)} \tag{53}$$

Spatial arrangement: increasing neighbourhoods

For structures in R^3, probability P estimated by the volume fraction V_V
Other information obtained by replacing the measure V_V by other specific properties as the surface area S_V, or the connectivity number N_A

Spatial arrangement: increasing neighbourhoods

Distance to	**a random point in**	**Measure**
component A_i	crack A_k	$\frac{S_V((A_i \oplus B(r)) \cap A_k)}{S_V(A_k)}$
crack A_k	component A_i	$\frac{V_V((A_k \oplus B(r)) \cap A_i)}{V_V(A_i)}$
points E_k (in R^2)	component A_i	$\frac{A_A((E_k \oplus B(r)) \cap A_i)}{A_A(A_i)}$
component A_i (in R^2)	points E_k	$\frac{N_A((A_i \oplus B(r)) \cap E_k)}{N_A(E_k)}$

(54)

Morphological criteria

Spatial arrangement: Anisotropy

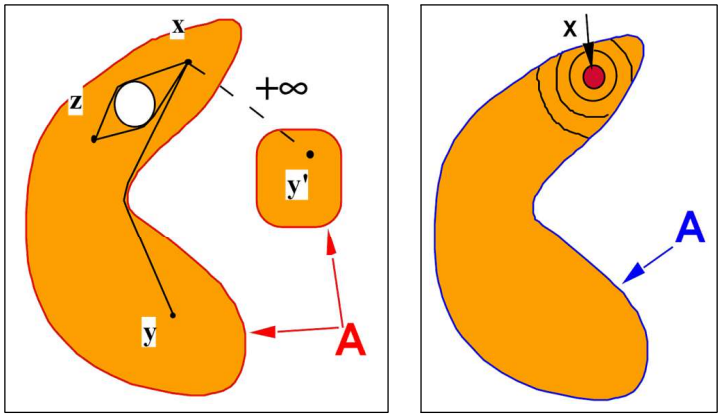
- **Directional Measurements** : variation with orientation of the covariance or of the histogram of chord lengths
- **Roses of orientation**, obtained by counting in the direct space or by directional filtering in the Fourier plane $\longrightarrow$ characterization of damage with respect to a coordinate system (orientation off the applied stress, or crystallographic orientation)

Connectivity

Important morphological aspect for the prediction of physical properties of composites with components presenting a high contrast (microcracks, voids)

- Connectivity number N in R^3: topological measurements describing the overall connectivity of a given medium
- Heterogeneous propagation phenomena, based on paths across a specimen (**percolation** effect) $\longrightarrow$ determination of a **geodesic distance** (length of shortest paths)

Connectivity: Geodesic Distance



Connectivity: Geodesic Distance

Propagation phenomena, with different propagation velocities in heterogeneous media:

- propagation of light in optics (principle of Fermat)
- sound in acoustics
- fluid in a porous medium
- diffusion of a constituent
- advance of a crack front

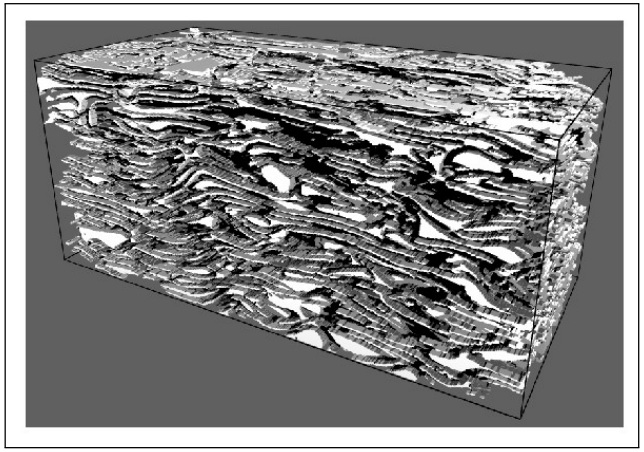
Connectivity: Geodesic Distance

For a valued planar graph, calculation of distances to a source on the valued graph, the **geodesic distance** (or length of shortest paths)

- geodesic distance between W_s and W_d (or length of shortest paths linking the source and the destination)
- distribution of the geodesic distances in the image $\longrightarrow$ sequence of propagation of the front and estimation of the **tortuosity** of a network
- extraction of shortest paths connecting W_s and W_d
- Points not be accessed during the propagation are located at the distance $+\infty$ from the source $\longrightarrow$ **detection of closed pores** in a specimen

Applications: fracture of polycrystalline graphite, diffusion in polymers and in porous media, fracture of simulated random media at different scales (porous media, polycrystals), fluid flow between rough surfaces, sound propagation in porous media

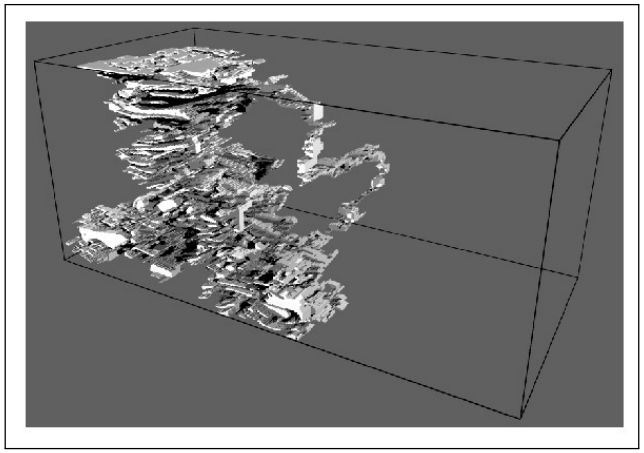
Connectivity of a porous medium



Interconnected pores

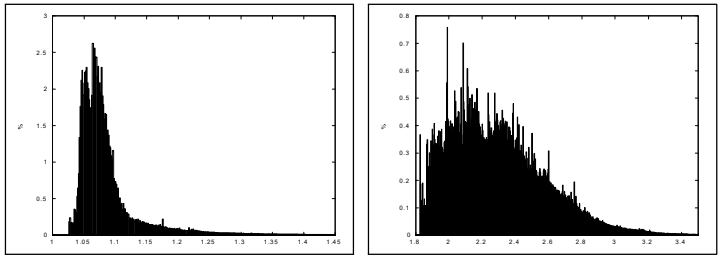
3D serial sections (*L. Decker,D. Jeulin, I. Tovena; with CEA*)

Connectivity of a porous medium: Shortest paths



Shortest paths (tortuosity: $1.6 - 2.2$)

Connectivity: Geodesic Distance and Tortuosity



(a) (b)

Distribution of the tortuosity of paths (cuboctahedron) in the x (a) and y (b) directions

Conclusion

Morphological Characterization of a microstructure $\longrightarrow$ **large number of parameters** according to various criteria
Every type of measurement $\longrightarrow$ specific aspect of the structure

- Possible reduction of data by multivariate data analysis $\longrightarrow$ **texture learning**
- **Synthesis from probabilistic models**
- Use of these morphological measurements to **predict the macroscopic behavior** of materials (change of scale models)

Introduction to basic models of random structures

Intoduction to basic models of random structures

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November 2024

Modelling random media

In materials, geological and biological structures: heterogeneous properties at various scales → **probabilistic approach** → estimation of spoiled or missing data, or physical properties of heterogeneous media

- Random structure characterization
- Introduction of the **variability** by random morphological models at the **micro scale**
- Prediction of the average **macroscopic** response of random media from their microstructure

Characterization of a random set

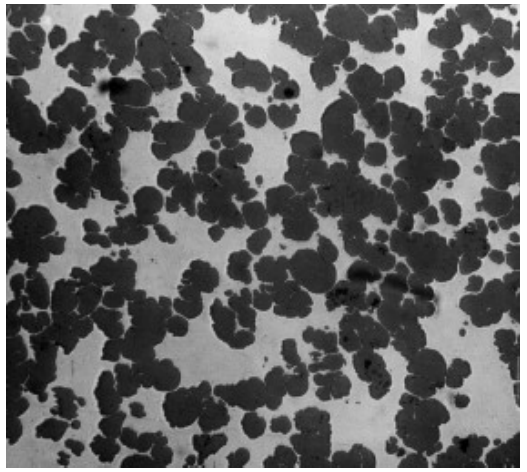
- Models derived from the **theory of Random Sets by G. MATHERON**
- For a random closed set A (RACS), characterization by the **CHOQUET capacity** $T(K)$ defined on the compact sets K

$$T(K) = P\{K \cap A \neq \emptyset\} = 1 - P\{K \subset A^c\} = 1 - Q(K)$$

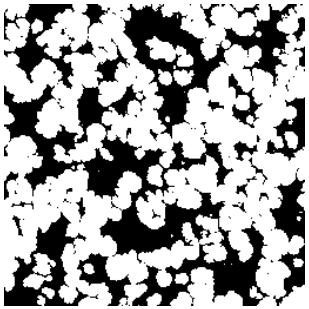
- In the Euclidean space R^n, CHOQUET capacity and **dilation operation**

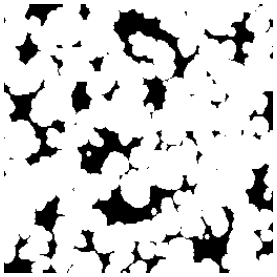
$$T(K_x) = P\{K_x \cap A \neq \emptyset\} = P\{x \in A \oplus \check{K}\}$$

Binary Morphology (Fe-Ag)

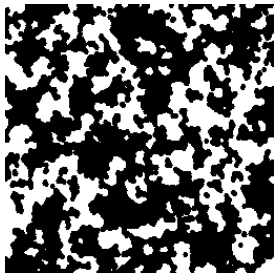


Basic Operations of Mathematical Morphology





Dilation by hexagon (2)



Erosion by hexagon (2)

Morphological interpretation

Every compact set K brings its own information on the set A

- $K = \{x\}$

$$T(x) = P\{x \in A\}$$

In R^3 $T(x) = p = V_V(A)$

- $K = \{x, x+h\}$

$$T(x, x+h) = P\{x \in A \cup A_{-h}\}$$

$$Q(x, x+h) = P\{x \in A^c \cap A^c_{-h}\}$$

$Q(x, x+h)$ is the covariance of A^c. It depends only on h for a stationary random set.
Useful to study the spatial distribution of A, to measure its scale (correlation length, integral range) or its anisotropy

Morphological interpretation

Experimental estimation of $T(K)$ by image analysis, using realizations of A, and dilation operation

- General case: several realizations and estimation of a frequency for every point x
- For a **stationary** random set, $T(K_x) = T(K)$;
- For an **ergodic** random set, $T(K)$ estimated from a single realization

$$T(K)^* = P\{x \in A \oplus \check{K}\}^* = V_V(A \oplus \check{K})^*$$

Every compact set K (points, ball...) brings its own information on the random set A

Morphological interpretation

Spatial law of A

- $K = \{x_1, x_2, ..., x_n\}$

$$T(K) = 1 - P\{x_1 \in A^c, x_2 \in A^c, ..., x_n \in A^c\}$$

- Cannot completely characterize the RACS A: $T(K) = 0$ for a stationary point process... ($\rightarrow$ use of $B(r)$, closed ball with radius r, undenumberable set of points)
- Can be used to estimate bounds of the effective physical properties

Characterization of a random model

- The CHOQUET capacity of a RACS is equivalent to the **distribution function of a random variable**
- Two models (RACS, usc RF, lsc RF) with the same functional $T(K)$, $T(g)$, $P(g)$ cannot be distinguished (theoretically as well as experimentally)
- The functional $T(K)$, $T(g)$, $P(g)$ connects theory and experimentation; it is used to estimate the parameters of a model and to test its validity
- Possible **predictive implementation** ($T(K)$ for K not used during the identification step). Examples: 3D properties deduced from 2D observations (stereology); change of support by $\vee$ or $\wedge$ in the case of a change of scale in fracture statistics

Calculation of the CHOQUET capacity

For a given model, the functional T is obtained:

- by **theoretical calculation**
- by **estimation**
  - on simulations
  - on real structures (possible estimation of the parameters from the "experimental" T , and tests of the validity of assumptions)

General properties of the proposed models

- Most random structure models defined in the **Euclidean space** R^n :
  - more general than stochastic processes limited to the 1D space R, where the order relation is used
  - different from discrete models defined on a grid, even if the discrete counterpart of the euclidean models is easily defined
- Models depending on a **few number of parameters**, not to ask too much from the available data, and for realistic experimental identification and test

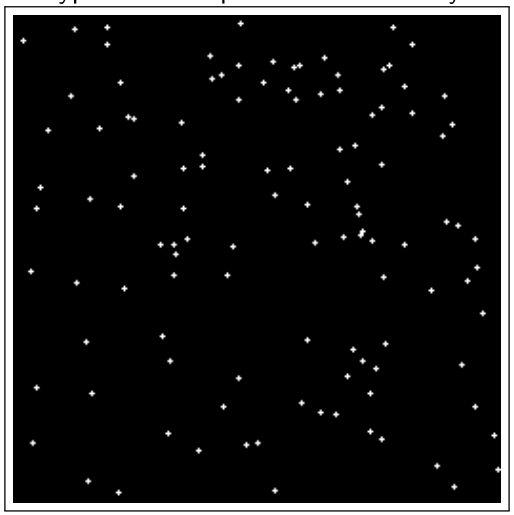
Point Processes

Most simple kind of random structure $\longrightarrow$ very small defects isolated in a matrix

- Particular RACS $\longrightarrow$ Choquet capacity $T(K)$
- Probability generating function $G_K(s)$ of the random variable $N(K)$ representing the number of points of the process contained in K

Poisson Point Process

Prototype **random** process without any order



Poisson Point Process

Non homogeneous Poisson point process in R^n with a regionalized intensity $\theta(x)$ ($x \in R^n; \theta \geq 0$): the numbers $N(K_i)$ are **independent random variables** for any family of **disjoint** compact sets K_i
$N(K)$ is a Poisson random variable with parameter $\theta(K)$:

$$\theta(K) = \int_K \theta(dx) \tag{1}$$

$$P_n(K) = P\{N(K) = n\} = \frac{\theta(K)^n}{n!} \exp(-\theta(K)) \tag{2}$$

Poisson Point Process

As a **random set**, Choquet capacity of the Poisson point process

$$T(K) = 1 - Q(K) = 1 - P_0(K) = 1 - \exp(-\theta(K)) \tag{3}$$

In the stationary case

$$T(K) = 1 - \exp(-\theta\mu_n(K)) \tag{4}$$

Cox Point Process

- Non homogenous intensity $\theta(x)$ generated by a positive random function (RF), useful to generate multi-scale random processes
- For any realization of this RF, generation of a Poisson point process with intensity $\theta(x)$
- Number of points in a domain D: for a given realization, Poisson distribution with average $\theta(D) = \int_D \theta(dx)$
- Conditionnaly to a given realization, or for a deterministic intensity $\theta(x)$, $T(K) = 1 - \exp(-\theta(K))$

Cox Point Process

Probabilistic properties of the Cox point process derived from those of the Poisson point process by randomization of the random measure θ
Let $\Phi_K(\lambda)$ be the Laplace transform of the positive random variable $\theta(K)$

Proposition

The probability generating fonction of the Cox point process is given by

$$G_K(s) = E\{\exp(\theta(K)(s-1))\} = \Phi_K(1-s) \quad (5)$$

and the Choquet capacity of the Cox point process is

$$T(K) = 1 - G_K(0) = 1 - E\{\exp(-\theta(K))\} = 1 - \Phi_K(1) \quad (6)$$

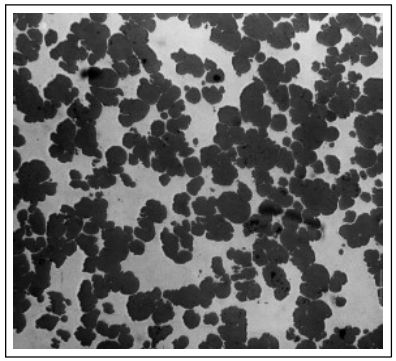
Random sets and Random Functions Models

Starting from a point process, more general models, called **grain models**:

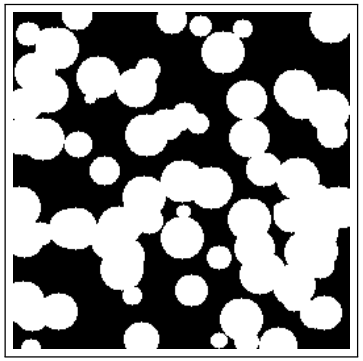
- The Boolean model
- The dead leaves model
- Random function models

Boolean Model

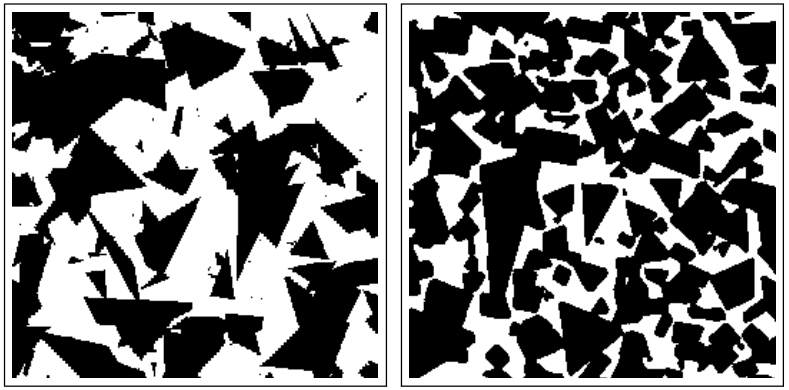
The **Boolean model** (G. Matheron) is obtained by implantation of random primary grains A' (with possible overlaps) on Poisson points x_k with the intensity θ: $A = \cup_{x_k} A'_{x_k}$
Any shape (convex or non convex, and even non connected) can be used for the grain A'



Fe (black) Ag (grey) alloy



Boolean spheres $V_v = 0.5$



(a) (b)

Boolean model with Poisson polyhedra (a); WC (black) -Co (white) composite (b) (*J.L. Quenec'h, J.L. Chermant, M. Coster, D. Jeulin*)

Boolean Model

Choquet capacity, with $q = P\{x \in A^c\}$:

$$T(K) = 1 - Q(K) = 1 - \exp\left(-\theta \overline{\mu}_n(A' \oplus \check{K})\right) = 1 - q^{\frac{\overline{\mu}_n(A' \oplus \check{K})}{\overline{\mu}_n(A')}} \quad (7)$$

The number of primary grains hit by K follows a Poisson distribution with average $\theta \overline{\mu}_n(A' \oplus \check{K})$
Ex.: Covariance, Three points probability...
Percolation threshold obtained from simulations: 0.2895 ± 0.0005 for spheres with a single diameter

Boolean Model

- **Covariance**

$$\begin{aligned} Q(h) &= P\{x \in A^c, x + h \in A^c\} = q^2 \exp\left(\theta K(h)\right) \\ &= q^{2 - r(h)} \end{aligned}$$

with the geometrical covariogram

$$K(h) = \overline{\mu}_n(A' \cap A'_{-h}) \text{ et } r(h) = \frac{K(h)}{K(0)}$$

Boolean Model

- **Three points Probability**

$$\begin{aligned} Q(h_1, h_2) &= P\{x \in A^c, x + h_1 \in A^c, x + h_2 \in A^c\} \\ &= \exp\left(-\theta \overline{\mu}_n(A' \cup A'_{-h_1} \cup A'_{-h_2})\right) \\ &= q^{3 - r(h_1) - r(h_2) - r(h_2 - h_1) + s(h_1, h_2)} \end{aligned}$$

with

$$s(h_1, h_2) = \frac{\overline{\mu}_n(A' \cap A'_{-h_1} \cap A'_{-h_2})}{K(0)}$$

Boolean Model

- **convex primary grains** $\longrightarrow$ application of the Steiner formula: $\overline{\mu}_n(A' \oplus \lambda \check{K})$ is a polynom with degree k in λ, where K is a convex set in R^k , with coefficients depending on the average Minkowski fonctionnals of A'
- **Segment**

$$\begin{aligned} Q(l) &= \exp\left(-\theta \overline{\mu}_n(A' \oplus l)\right) = \exp\left(-\theta(K(0) - lK'(0)\right) \\ &= q^{1-lr'(0)} \end{aligned}$$

Boolean Model: identification

Unknowns

- intensity θ
- random set A'
- Use of a family of compact **convex** sets K $(l, C(r), H(r), B(r))$

$$\downarrow$$

Expectation of the **Minkowski Fonctionnals of** A'

- Use of **non connected** compact sets K $(\{x, x+h\}, \{x, x+h_1, x+h_2\})$

$$\downarrow$$

$$\overline{\mu}_n(A' \ominus \check{K})$$

Boolean Model: identification

Example of the covariance $Q(h)$

$$Q(h) = q^{2-r(h)}$$

$$r(h) = \frac{K(h)}{K(0)} = 2 - \frac{\log\left(Q(h)\right)}{\log\left(q\right)}$$

One can estimate $r(h)$ from the covariance for any kind of primary grain (convex or non convex)

Boolean Model: examples of application

- Boolean model with **spherical** primary grains
  - Fe Ag Alloys (Th. Bretheau, D. Jeulin)
- Boolean model with **Poisson** primary grains
  - Porous media: formed coke (D. Jeulin)
  - WC-Co sintering (J.L. Quenec'h, M. Coster, J.L. Chermant, D. Jeulin)

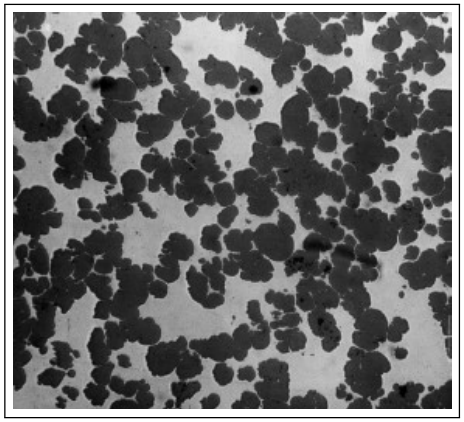
Models with 2 parameters; identification from the covariance $Q(h)$

Bretheau T., Jeulin D. (1989,. Caractéristiques morphologiques des constituants et comportement à la limite élastique d'un matériau biphasé Fe/Ag, Revue de Physique Appliquée, 24(9), 861-869.

Quénec'h J. L., Chermant J. L., Coster M., Jeulin D. (1996) Example of application of probabilistic models: Determination of the kinetics parameters during liquid phase sintering, Microscopy Microanalysis Microstructures, 7(5-6), 573-580.

Fe Ag Alloys

- Influence of the **spatial distribution** of Fe and Ag on the macroscopic plastic behaviour of composites obtained from spherical powders (hot isostatic compression)
- Propagation of plastification fronts (Piobert-Lüders shear bands) for some compositions (percolation effect): **propagation for continuous paths** in phase Fe $\longrightarrow$ **connectivity problem** in R^3
- **Microstructural data**: 2D slices $\longrightarrow$ 3D data estimated from a probabilistic model in R^3



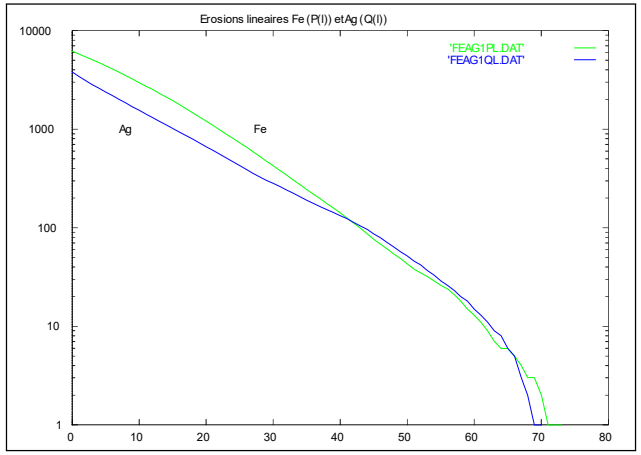
Fe Ag Alloy

Morphological criterion: $\longrightarrow$ connectivity number (Euler-Poincaré caracteristic) $N_V - G_V$ estimated from the Boolean model with spherical primary grains

Fe Ag Alloys

- Test of the Boolean model with convex primary grains from $Q(l)$ (exponential function)
- **Identification** of the model from the Covariance $Q(h)$ (**stereological aspect**)

Fe Ag Alloys



Spherical primary Grain

↓

Population of spheres known from $K(h)$

↓

Identification from a Boolean Model from its covariance $Q(h)$

For any sphere with diameter a:

$$K(h) = \frac{\pi a^3}{6}\left(1 - \frac{3}{2}\frac{h}{a} + \frac{1}{2}\frac{h^3}{a^3}\right) = \frac{\pi}{6}\left(a^3 - \frac{3}{2}ha^2 + \frac{1}{2}h^3\right) \text{ if } h \leq a$$
$$= 0 \text{ if } h \geq a$$

Spherical primary Grain

$$\left(\frac{\partial^3}{\partial h^3}K(h)\right)_{h=0} = \frac{\pi}{2}$$

(true for any population of spheres)

↓

$$\theta = \frac{2}{\pi}\left(\frac{\partial^3}{\partial h^3}\log Q(h)\right)_{h=0}$$

→ **Numerical unstability**

Primary Grain: population of spheres

Geometrical covariogram of a population of spheres with random diameters following a discrete distribution ($p_i = P\{a = a_i\}$)

$$\overline{K}(h) = \frac{\pi}{6}\left(\sum_{a_i>h} p_i a_i^3 - \frac{3}{2}h\sum_{a_i>h} p_i a_i^2 + \frac{1}{2}h^3\sum_{a_i>h} p_i\right)$$

continuous diameter distribution $f(x)$ (with cumulative distribution function $F(x)$)

$$\overline{K}(h) = \frac{\pi}{6}\left(\int_h^\infty a^3 f(a)\,da - \frac{3}{2}h\int_h^\infty a^2 f(a)\,da + \frac{1}{2}h^3\int_h^\infty f(a)\,da\right)$$

Primary Grain: population of spheres

$F(x)$ deduced from $K(h)$ by an inversion relation, obtained from $Q(h)$ and from $\overline{K}(h)$:

$$\frac{1}{2\pi h}\frac{\partial^2}{\partial h^2}\log Q(h) = \theta(1 - F(h))$$

→ **Numerical unstability** → use of prior distributions $F(r)$: uniform, exponential, lognormal, Gamma,....

Boolean model: Connectivity

Connected morphology for a high intensity θ (possible overlaping grains)

Connectivity number in R^3 for the Boolean model

$N_V - G_V$, for a 3D Boolean model with isotropic primary grains, with surface area $\overline{S}(A')$, and integral mean curvature $\overline{M}(A')$, (R.E. Miles):

$$N_V(A) - G_V(A) = q\left(\theta - \theta^2 \frac{\overline{M}(A')\overline{S}(A')}{4\pi} + \frac{\pi}{6}\theta^3 \left(\frac{\overline{S}(A')}{4}\right)^3\right)$$

Percolation threshold obtained from simulations: 0.2895 ± 0.0005 for spheres with a single diameter

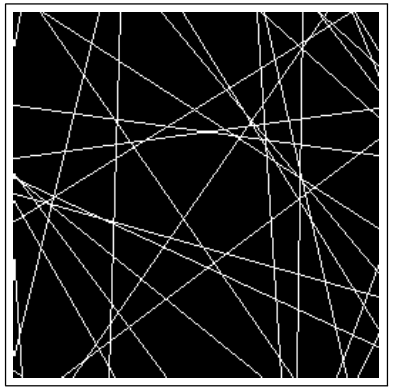
Boolean model: Connectivity

- Model fit in a limited range of composition (9-18% Ag)
- Change of composition obtained by the intensity θ, keeping constant the population of primary grains
- Theoretical calculation of $N_V - G_V$ (use of the model in a predictive way)
- Domains of transition of $N_V - G_V$ corresponding to changes of the rheological behaviour (behaviour of pure Fe or of pureAg)
- Same type of variation (positions of 0 and of extrema) expected for similar grains $\rightarrow$ possible experimental control on powders with different sizes

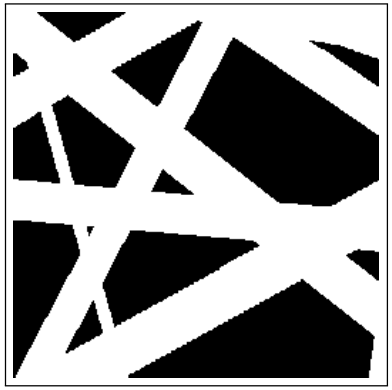
Boolean model

- Point process with a limited extent as a primary grain A', $\rightarrow$ Neyman-Scott point process, with **clusters**
- **Anisotropic models**
- Replacing the Poisson points by **Poisson varieties** $\rightarrow$ generate random sets models with fiber or strata textures (exemple, random fibers network of wood composites: *Ch. Delisée, D. Jeulin, F. Michaud, F. Bos, P. Castera*)

Delisée Ch., Jeulin D., Michaud F. (2001) Caractérisation morphologique et porosité en 3D de matériaux fibreux cellulosiques, C.R. Académie des Sciences de Paris, t. 329, Série II b, pp. 179-185, 2001.

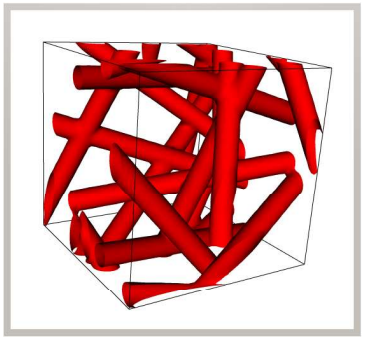


Poisson lines



Poisson fibers or strata

Boolean model on Poisson lines in 3D



Isotropic Poison fibres

Faessel M., Jeulin D. (2011) 3D Multiscale Vectorial Simulation of Random Models, Proc. ICS 13, Beijing.

Schladitz K., Peters S., Reinel-Bitzer D., Wiegmann A., Ohser J., Design of acoustic trim based on geometric modeling and flow simulation for non-woven, Computational Materials Science 38 (2006) 56–66.

Multi Scale Models - Motivation

- Non-homogeneous dispersion of a charge in a matrix: arrangement of aggregates (like carbon black) at different scales
- Prediction of the effective properties of such composites (dielectric permittivity or the elastic moduli), from the properties of the two components (charge and matrix), and their spatial distribution
- General methodology, based on the theory of random sets : morphology summarized and simulated by multi-scale random models accounting for the heterogeneous distribution of aggregates. Identification of the model from image analysis

Multi Scale Models - Combination of basic random sets

Starting from the basic models $\longrightarrow$ more complex structures, such as superposition of scales, or fluctuations of the local volume fraction p of one phase

Union or intersection of random sets

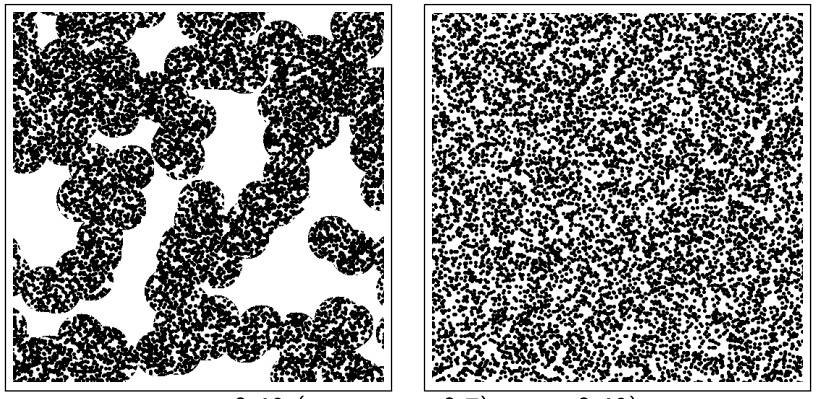
$$A = A_1 \cap A_2$$

for two **independent** random sets A_1 and A_2

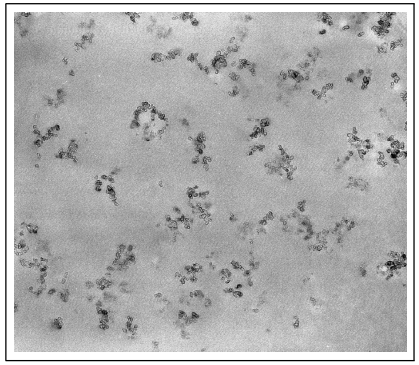
$$P(K) = P\{K \subset (A_1 \cap A_2)\} = P\{K \subset A_1\} P\{K \subset A_2\}$$

$$p = p_1 p_2$$

Multi Scale Models - Intersection of random sets



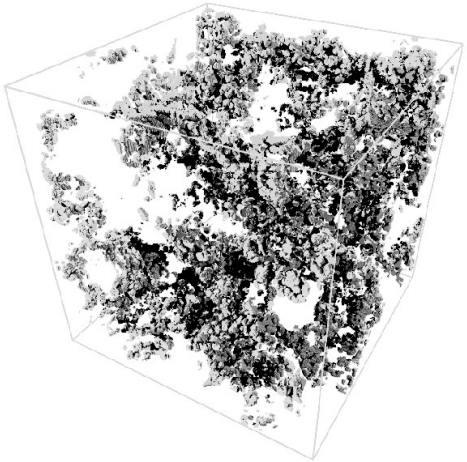
$p = 0.49$ ($p_1 = p_2 = 0.7$); $p_1 = 0.49$)



Composite carbone black resin (TEM)

Transmission micrographs (*L. Savary, D. Jeulin, A. Thorel*)

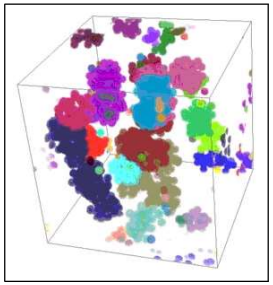
Savary, L., Jeulin, D., & Thorel, A. (1999). Morphological analysis of carbon-polymer composite materials from thick sections. Acta Stereologica.



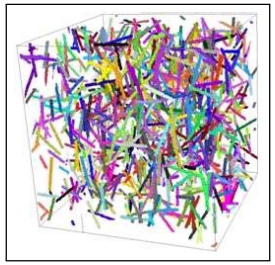
Simulation of a Carbon black composite: Intersection of 3 scales Boolean models of spheres (identification from thick sections)

Percolation of 3D multi-scale aggregates in Nanocomposites (D.Jeulin, M. Moreaud)

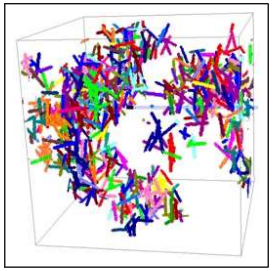
For several well separate scales, $p_c \simeq p_{1c} p_{2c} ... p_{rc}$



(a) Two scales spheres



$l/r = 100$

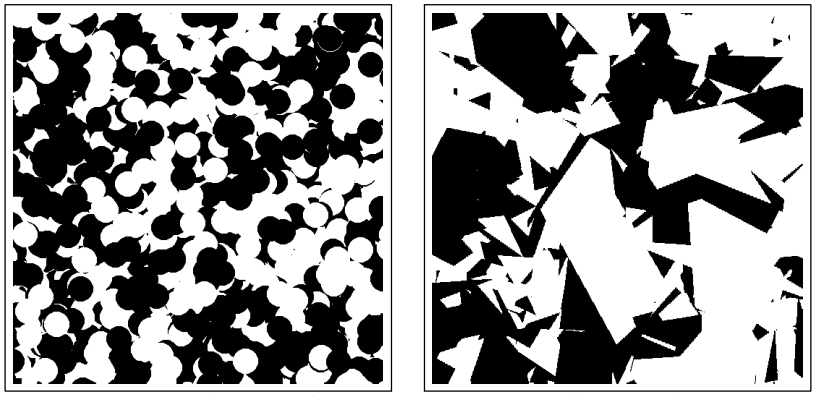


Carbon black (a): ($p_c \simeq 0.0856 \simeq 0.3^2$; **Carbon nanotubes** (b) $p_c \simeq 0.01$; **2 scales CNT** (c) $p_c \simeq 0.00049$

Color Dead Leaves Model

- Sequential implantation of random primary grains $A'(t)$ on a Poisson point process
- In point x is kept the last occurring color during the sequence
- Non symmetric random sets obtained if two different families of primary grains are used for A and for A^c

Color Dead Leaves Model



Discs ($p = 0.5$); Poisson polyhedra ($p = 0.5$)

Color Dead Leaves Model

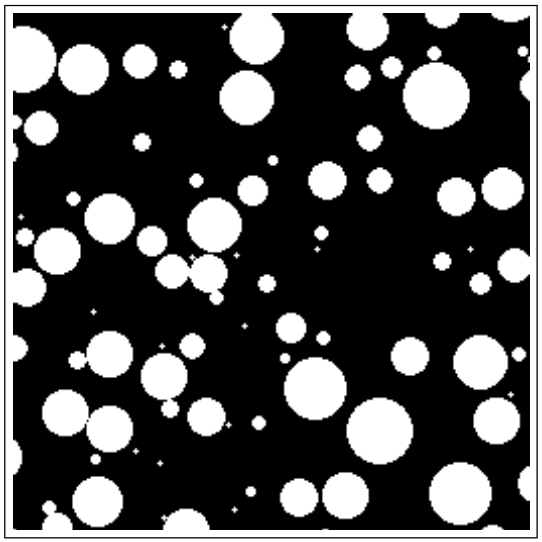
Covariances $C_{ii}(h)$ for primary grains A'_i independent on time t, implanted with a constant intensity θ_i (asymptotic value for $t \to \infty$)

$$C_{ii}(h) = \frac{P_i r_i(h) + 2P_i^2(1 - r_i(h))}{2 - r(h)}$$

where $r_i(h) = K_i(h)/K_i(0)$, $\theta K(h) = \sum_{i=1}^{i=n} \theta_i K_i(h)$
Volume fraction P_i

$$P_i = \frac{\theta_i K_i(0)}{\theta K(0)}$$

Intact grains: a random set with non overlapping objects



Dead leaves: intact grains

Intact grains: a random set with non overlapping objects

- **Size distribution** of grains of the random packing, given by **the pdf** $\varphi_{A'_i}$ **of intact grains** A'_i ($f_i(r)$ in the case of spheres), and **volume fraction**
- For a single type of convex grain (e.g. sphere with the radius r, or polyhedron) and a homogeneous model in R^n

$$P\{x \in A\} = \frac{1}{2^n}$$

volume fraction in space: 0.125 (0.25 in the plane, and 0.5 on the line)

- Multivariate distribution function of the centres of grains building a **hard-core point process**

Random Functions and Random Sets

For a **Random Function** $Z(x)$,

- Continuous function too restrictive
- Semi-continuous (upper, lower) for which the changes of supports by $\vee$ or by $\wedge$ provide random variables:

$$Z_{\vee}(K) = \vee_{x \in K}\{Z(x)\}$$

$$Z_{\wedge}(K) = \wedge_{x \in K}\{Z(x)\}$$

Applications to **Fracture Statistics Models** based on the weakest link assumption, and to the statistics of extremes

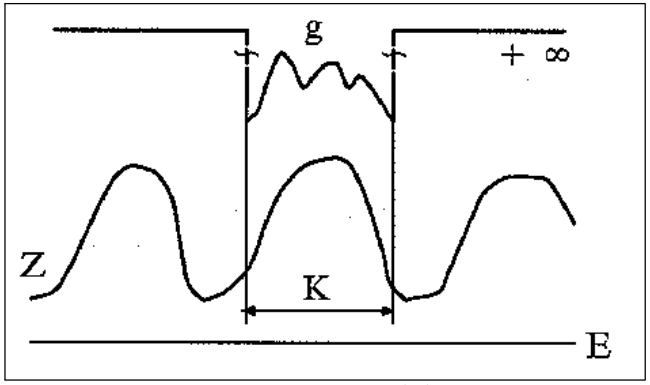
Characterization of a random function model

For an **upper semi continuous random functions (usc RF)**, characterization by the **CHOQUET capacity** $T(g)$ defined on the lower semi continuous functions (lsc) g with compact support K

$$T(g) = P\{x \in D_Z(g)\}; D_Z(g)^c = \{x, Z(y) < g(y-x), \forall y \in K\}$$

Particular case: **spatial law** for a support K made of a finite number of points

Characterization of a random function model



usc RF: $x \in D_Z(g)^c$

Characterization of a random function model

$g(x_i) = z_i$ for x_i $(i = 1, 2, ..., n)$, else $g(x) = +\infty$:

$$T(g) = 1 - P\{Z(x_1) < z_1, ..., Z(x_n) < z_n\}$$

$1 - T(g)$ gives the **spatial law**.
For $g(x_i) = z_i$, else $g(x) = +\infty$:

$$1 - T(g) = Q(g) = P\{Z(x) < z\}$$

Characterization of a random function model

For the function $g(x) = z$ if $x \in K$, and $g(x) = +\infty$ if $x \notin K$,

$$D_Z(g)^c = \{x, Z(y) < z, \forall y \in K\}$$

Therefore

$$T(g) = P\{x \in D_Z(g)\} = 1 - Q(g) = 1 - P\{Z_\vee(K) < z\}$$

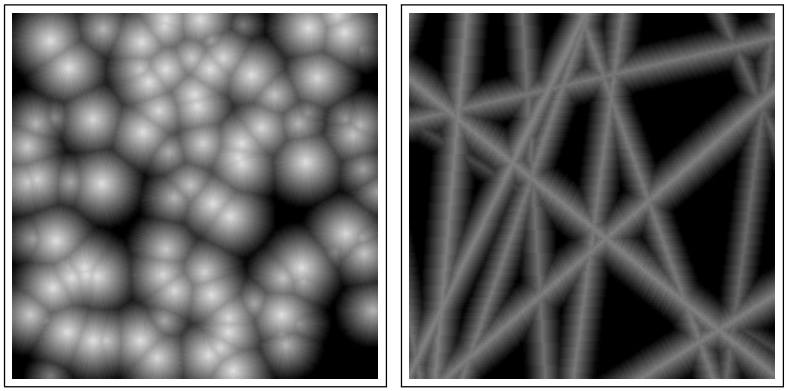
The Choquet capacity gives the probability distribution of the RF $Z(x)$ after a **change of support by** $\vee$ over the compact set K

Random Function Models

Continuous version of random sets models

Boolean RF: random implantation of primary random functions on points of a Poisson point process
The $\cup$ operation for overlapping grains is replaced by the supremum ($\vee$) or by the infimum ($\wedge$)

Random Function Models: Boolean RF



(a) (b)

Boolean RF with cone primary grains (a) and Boolean Variety RF (b)

Boolean random functions

Choquet Capacity

Starting from a sequence of primary RF's implanted according to the supremum, the Choquet capacity of the BRF $Z(x)$ is given by:

$$1 - T(g) = Q(g) = \exp\left(-\int_R \overline{\mu}_n(D_{Z'_t}(g))\,\theta(dt)\right)$$

If $g(x_i) = z_i$ in points x_i $(i = 1, 2, ..., n)$, else $g(x) = +\infty$, we get the **spatial law** for $A_{Z'_t}(z) = \{x, Z'_t(x) \geq z\}$

$$1 - T(g) = P\{Z(x_1) < z_1, ..., Z(x_n) < z_n\}$$
$$= \exp\left(-\int_R \overline{\mu}_n(A_{Z'_t}(z_1)_{x_1} \cup ... \cup A_{Z'_t}(z_n)_{x_n})\,\theta(dt)\right)$$

Boolean random functions

Change of support by $\vee$

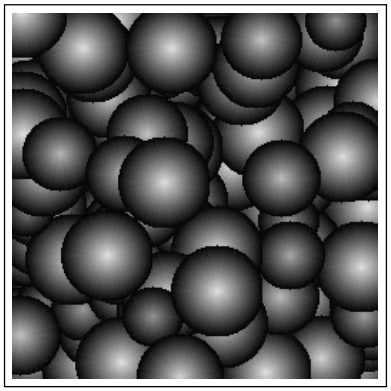
If $g(x) = z$ for $x \in K$ and $g(x) = +\infty$ otherwise (K compact set), we get the change of support of $Z(x)$ by operator $\vee$ over the compact set K ($Z_\vee(K) = \vee_{x \in K}\{Z(x)\}$)
In that case $D_{Z'_t}(g) = A_{Z'_t}(z) \oplus \check{K}$, and

$$P\{Z_\vee(K) < z\} = \exp\left(-\int_R \overline{\mu}_n(A_{Z'_t}(z) \oplus \check{K})\ \theta(dt)\right)$$

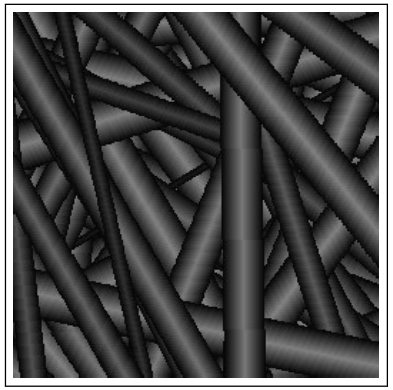
Dead Leaves Random Functions

- Sequential implantation of random primary functions $Z'_t(x)$ on a Poisson point process
- In point x is kept the last occurring value during the sequence

Dead Leaves Random Functions



Dead Leaves RF with cones

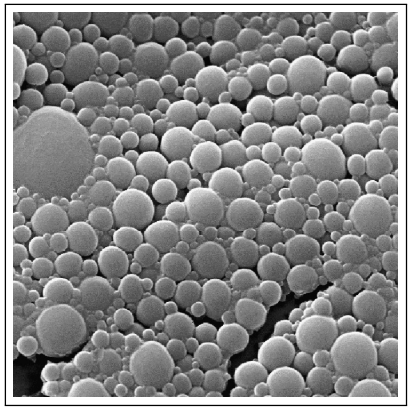


Dead Leaves Variety

Dead leaves and powders

Morphological analysis of powders with the help of the SEM, from the dead leaves model$\longrightarrow$ possible control of the quality of a product or of a process (*Ivan Terol & D. Jeulin in connexion with Pechiney, Calgon and Isover*)

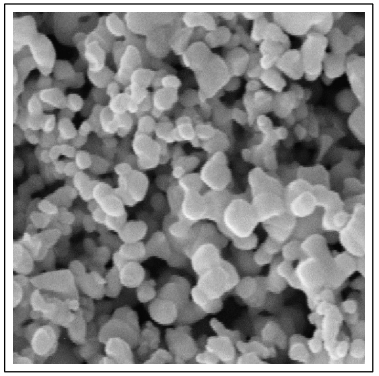
Size distribution of a spherical powder



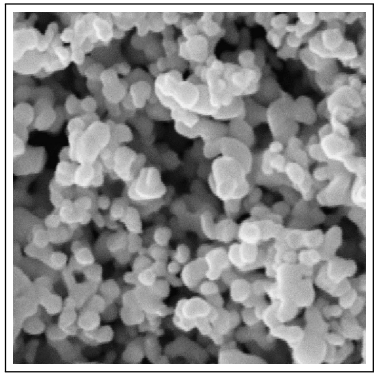
SEM Image

$15.5\mu m$; $a = 0.03\mu m$

UO2 powder: composition



Mix C ($5.12\mu m$)



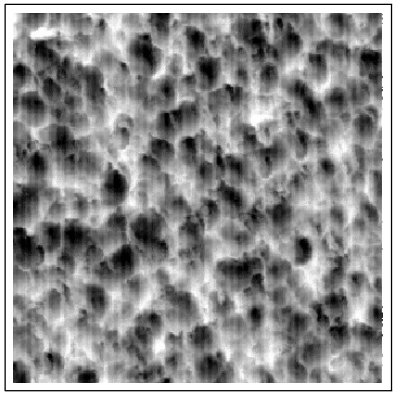
Mix C $a = 0.01\mu m$

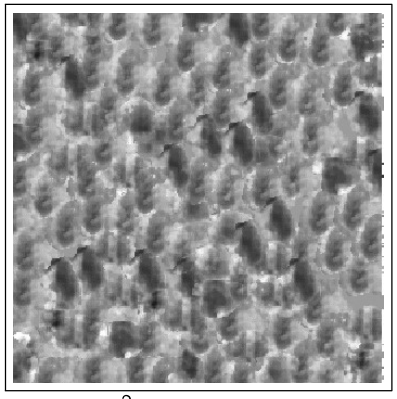
Roughness

Application of random functions and of signal processing, and from numerical image processing $\rightarrow$ plate users (steel, car industry) and mechanical parts users (tribology)

- **Theoretical models of random surfaces** (boolean functions, sequential alternate RF, reproducing depositions and abrasions)
- **Application to steel products** : modeling the EDT roughness and its transfer to the plate in skin-pass by simulation of the indentation (*P. Laurenge and D. Jeulin, in relation to IRSID*)

Simulation of the EDT Roughness





(IRSID) 1.28x1.28 mm^2

Jeulin D., Laurenge P. (1996) Probabilistic model of rough surfaces obtained by electro-erosion, Mathematical Morphology and Its Applications to Image and Signal Processing, 289-296.

The Mosaic model

Mosaic or cell model (Miller, Hori, Matheron) built in two steps:

- random tessellation of space into cells C_i (Voronoï, Johson-Mehl, Poisson...)
- to every cell C_i is affected independently a realization of the random variable Z

Parameters of this model: distribution function of Z and probabilistic properties of the tessellation (separation of the space and of the physical variables)

The Mosaic model

Centred Covariance $\overline{W}_2$

$$\overline{W}_2(h) = \overline{C}(h) = E\{(Z(x+h) - E(Z))(Z(x) - E(Z))\} = D^2[Z]\, r(h)$$

where $D^2[Z]$ is the variance of the RV Z, and $r(h) = \dfrac{K(h)}{K(0)}$ is the probability that two points are in the same cell with the geometric covariogram $K(h)$

Third order moment $\overline{W}_3$

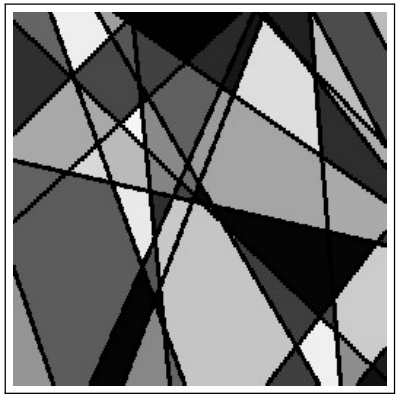
$$\begin{aligned} & \overline{W}_3(h_1, h_2) \\ = \; & E\{(Z(x) - E\{Z\})(Z(x+h_1) - E\{Z\})(Z(x+h_2) - E\{Z\})\} \\ = \; & s(x, y, \theta) E\left\{(Z - E\{Z\})^3\right\} \end{aligned}$$

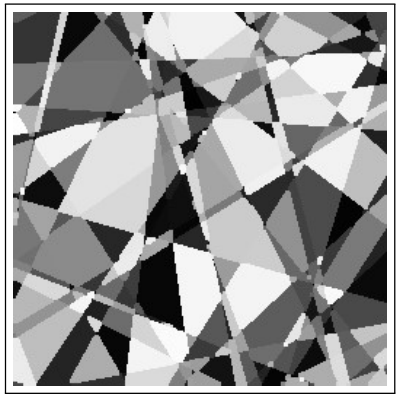
The Mosaic model

$s(|h_1|, |h_2|)$ is the probability that the three points $\{x\}$, $\{x+h_1\}$, $\{x+h_2\}$ belong to the same random cell A' (μ_d: Lebesgue measure in R^d, $\overline{\mu}_d$ its average over the realizations of A', and A'_h obtained by translation of A' by the vector h):

$$s(|h_1|, |h_2|, \theta) = \frac{\overline{\mu}_d(A' \cap A'_{h_1} \cap A'_{h_2})}{\overline{\mu}_d(A')}$$

The Poisson Mosaic model





The Poisson Mosaic model

Poisson mosaic

Poisson random planes in R^3 (with the intensity λ), and Poisson lines in R^2 (with the intensity λ)
Cells: Poisson polyhedra and Poisson polygons. In R^2 we have:

$$\begin{aligned} r(h) &= \exp(-2\lambda\,|h|) \\ s(h_1,h_2) &= \exp\left(-\lambda\left(|h_1|+|h_2|+|h_2-h_1|\right)\right) \end{aligned}$$

Reaction-Diffusion Models

L. DECKER and D. JEULIN

Reaction-diffusion RF are solutions of stochastic PDE, modeling competion between chemical reactions between species, and transport by diffusion.
$\rightarrow$ Textures reproducing various morphologies (example: interconnected random set obtained by thresholding a simulation of the Ginzburg-Landau model)
Lack of knowledge of probabilistic properties

Decker L., Jeulin D. (1999) 3D spatial time structure simulations by reaction-diffusion models, Acta Stereologica.

Reaction-Diffusion Models

- Space-time dependent, multi-variate
- Coupling of a physical phenomenon of diffusion (transport) with a physical phenomenon of reaction (generation of species)
- Complex systems: generation of heterogeneous media, self-organization
- Random models: starting from a white noise as initial conditions

Reaction-Diffusion Equations

- N chemical species (components) : $1, 2, ..., N$
- Variables = concentrations (densities)
  $Z_1(x,t),\ Z_2(x,t), ...,\ Z_N(x,t)$
- Set of N non linear partial differential equations :

$$\frac{\partial Z_i(x,t)}{\partial t} = D_i \Delta Z_i(x,t) + F_i(Z_1, Z_2, ..., Z_N)$$

- Example: for the reaction $X_1 + 3\,X_2 \underset{k_b}{\overset{k_a}{\rightleftarrows}} 2\,X_3 + X_4$

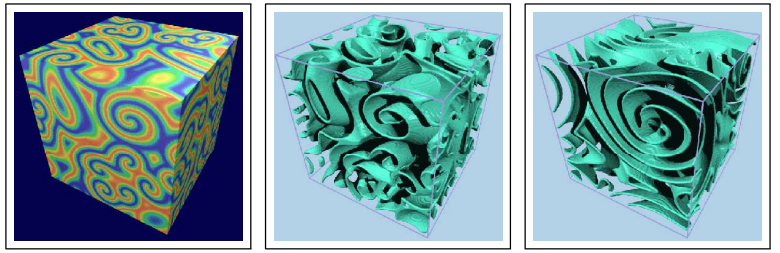
$$F_1(Z_1,Z_2,Z_3,Z_4) = -k_a\,Z_1\,{Z_2}^3 + k_b\,{Z_3}^2\,Z_4$$

Ginzburg-Landau Model

- Reactive system with two species of concentrations $Z_1(x, y, z, t)$ and $Z_2(x, y, z, t)$, having a representation in the complex plane $Z(Z_1, Z_2)$
- Partial differential equations of reaction-diffusion:

$$\frac{\partial Z}{\partial t} = D\Delta Z + A\,Z - B\,|Z|^2\,Z$$
$$\text{with} \begin{cases} Z = Z_1 + i\,Z_2 \\ A = \alpha + i\,\gamma \\ B = \beta + i\,\delta \end{cases}$$

- Genesis of a population of coupled and rotating spirals, delimiting convective cells



(a) (b) (c)

Reaction Diffusion model built from the simulation of Ginzburg-Landau equations; component Z_1(a); binary images obtained by thresholding ($Z_1(x, t) > 0.9$) after 2000 (b) and 10000 (c) iterations

Change of scale in random media

Applications of the models of random media to the **prediction of the macroscopic behavior of a physical system from its microscopic behavior**.

- **Estimation of the effective properties** (namely the overall properties of an equivalent homogeneous medium) of random heterogeneous media from their microstructure. From variational principles, **bounds** of the effective properties for linear constitutive equations
- **Estimation of the effective behaviour** from numerical simulations on random media
- **Fracture statistics models**

Probabilistic models of structures

- **Tools of simulation** to generate realizations of random and periodic structures (polycristals, multicomponents) → input in a finite element code (*L. Decker & D. Jeulin*)
- **Estimation from 3D simulations of transport properties** (permeability, coefficient of macroscopic diffusion) of heterogeneous media (e.g. porous media) (*L. Decker, D. Jeulin & I. Tovena*)

Decker L., Jeulin D. (2000) Simulation 3D de matériaux aléatoires polycristallins, Revue de Métallurgie, 97(2), 271-175.

Decker L., Jeulin D., Tovena I. (1998) 3D morphological analysis of the connectivity of a porous medium, Acta Stereologica.

- **Estimation by FFT** of dielectric properties (*D. Eyre, G. Milton, A. Delarue, D. Jeulin*) and of elastic and nonlinear properties (*H. Moulinec, P. Suquet, F. Willot*) of random media

- Delarue A., Jeulin D. (2002) Homogenization of dielectric properties of random nanocomposites, "Matériaux 2002" (Tours, 21-25 October 2002), N-12/02/MM , Paris School of Mines.
- Eyre D. J., Milton G. W. (1999) A fast numerical scheme for computing the response of composites using grid refinement, The European Physical Journal-Applied Physics, 6(1), 41-47.
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- Willot F., Pellegrini Y. P. (2008) Fast Fourier Transform Computations and Build-Up of plastic deformation in 2D, elastic-perfectly plastic, pixelwise disordered porous media, Continuum Models and Discrete Systems CMDS11 (ed. D. Jeulin and S. Forest), Presses Mines ParisTech.

Conclusion

- Random models of structures → simulate **complex morphology** of microstructures
- Approach, based on measurements obtained by **image analysis** → test and select appropriate models, estimate their parameters
- **Prediction** of **overall physical properties** of materials, and modeling **fracture statistics** behaviour
- Possible use in the **synthesis of textures**
- Simulations to feed deep learning tools like CNN (Convolutional Neural Network)

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