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Reconstructing Indistinguishable Solutions Via a Set-Valued KKL Observer

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Abstract

KKL observer design consists in finding a smooth change of coordinates transforming the system dynamics into a linear filter of the output. The state of the original system is then reconstructed by implementing this filter from any initial condition and left-inverting the transformation, under a *backward-distinguishability* property. In this paper, we consider the case where the latter assumption does not hold, namely when distinct solutions may generate the same output, and thus be indistinguishable. The KKL transformation is no longer injective and its “left-inverse” is thus allowed to be set-valued, yielding a set-valued KKL observer. Assuming the transformation is full-rank and its preimage has constant cardinality, we show the existence of a globally defined set-valued left-inverse that is Lipschitz in the Hausdorff sense. Leveraging on recent results linking this left-inverse with the *backward-indistinguishable sets*, we show that the set-valued KKL observer converges in the Hausdorff sense to the backward-indistinguishable set of the system solution. When, additionally, a given output is generated by a specific number of solutions not converging to each other, we show that the designed observer asymptotically reconstructs each of those solutions. Finally, the different assumptions are discussed and illustrated via examples.

Key words: KKL observer, set-valued observer, indistinguishability, p -valued maps, Lipschitz extension

1 Introduction

Consider a dynamical system of the form

$$\dot{x} = f(x), \quad y = h(x), \quad (1)$$

where $x \in \mathbb{R}^{n_x}$ is the state, $y \in \mathbb{R}^{n_y}$ is the output, $f : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x}$ and $h : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_y}$ are continuously differentiable. We assume that the trajectories of interest remain in a compact subset $\mathcal{X} \subset \mathbb{R}^{n_x}$. A typical problem in many engineering applications is to estimate online the current state $x(t)$ of (1) based on the knowledge of the output y on the interval $[0, t]$, in the sense that, the error between the estimate, denoted by $\hat{x}(t)$, and $x(t)$ asymptotically converges to zero. To do so, we usually design an *observer*; namely, a dynamical system

$$\dot{z} = F(z, y), \quad \hat{x} = \mathcal{T}(z, y), \quad (2)$$

where $z \in \mathbb{R}^{n_z}$ is the state and $\hat{x} \in \mathbb{R}^{n_x}$ is the output. System (2) is fed with the known output y to (1), and

provides as output the estimate \hat{x} such that (see [8])

$$\lim_{t \rightarrow \infty} (\hat{x}(t) - x(t)) = 0; \quad (3)$$

Since the observer cannot distinguish solutions producing the same output, the possibility of achieving asymptotic estimation, implicitly, requires that *indistinguishable* solutions, i.e., solutions producing a same output, at least asymptotically converge to each other. In other words, the system must be *detectable*. In general, nonlinear observers are designed under stronger *observability* conditions saying that there is no indistinguishable solutions, or, in other words, that the output signal carries enough information to determine the state uniquely.

1.1 Background on KKL-Observer design

One possible route to design (2) that guarantees (3) is the *nonlinear Luenberger* (or *Kazantzis-Kravaris-Luenberger (KKL)*) approach, initially introduced in [20] for single-output linear systems. The idea is to look for a continuous and injective change of coordinates $T : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_z}$ such that $z := T(x)$ is governed by

$$\dot{z} = Az + By, \quad (4)$$

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for a pair $(A, B) \in \mathbb{R}^{n_x \times n_x} \times \mathbb{R}^{n_x}$ to be chosen with A Hurwitz. In particular, when T is smooth, it must verify

$$\frac{\partial T}{\partial x}(x)f(x) = AT(x) + Bh(x) \quad \forall x \in \mathcal{X}. \quad (5)$$

Given a solution $t \mapsto x(t)$ to (1), since A is Hurwitz, any solution to (4) subject to $t \mapsto y(t) = h(x(t))$, exponentially estimates $t \mapsto T(x(t))$; thus, an estimate of x can be recovered through a left-inversion of T .

Originally, when $n_y = 1$, D. Luenberger showed in [20] that the map T always exists for linear observable systems as long as the pair (A, B) is picked controllable, $n_z = n_x$, and A does not share any eigenvalue with the system's dynamics. In the context of nonlinear systems, the existence of the map T was first established around an equilibrium point in [26], [16] and [18]. Then, this local result was extended in [17] to a global one using a strong observability assumption. This result, however, does not provide an indication on the necessary dimension n_z . This problem was addressed in [4] under a weak *backward-distinguishability* assumption. That is, the injective map T is shown to exist, for each matrix A complex, diagonal, of dimension $n_z = n_x + 1$, and with $n_x + 1$ *distinct* complex eigenvalues. The aforementioned result is generalized in [10], for *almost any* real controllable pair (A, B) of dimension $n_z = 2n_x + 1$ with A diagonalizable. In terms of implementation, running (4) online from any initial condition, provides an estimate of $t \mapsto T(x(t))$. A left-inverse T^{inv} of T must thus be available to compute an estimate $\hat{x} = T^{\text{inv}}(z)$ of the system solution. An expression of the map T can sometimes be found analytically by solving (5) (e.g., for parameter identification [1] or state/parameter estimation in electrical machines [15,9]). Otherwise, when an expression for T or its left inverse T^{inv} is not available, numerical approximation methods based on neural networks are being developed [25,13,24]. But in any case, the distinguishability property assumed in this literature [4,10] requires that the backward solutions from any distinct states x_a, x_b in \mathcal{X} generate distinct past outputs y_a, y_b ; hence, we say that x_a and x_b are *backward-distinguishable*. Said differently, any given state in \mathcal{X} is uniquely determined by its past output. The goal of this paper is to study situations where this assumption is not verified.

1.2 Motivation

Backward-distinguishability is not always verified in applications, where some systems may exhibit indistinguishable states. This is the case of induction motors [22,28] and permanent magnet synchronous motors (PMSMs) [9]. Estimating the state from the outputs is then impossible. However, strategies to recover all the possible trajectories corresponding to a given output, are still of great interest, in particular for monitoring, or when an extra-knowledge or expertise can be used to decide on the actual-solution estimate, among all the possibilities provided by the set-valued observer. More pre-

cisely, when two solutions, not asymptotically tending to each other, generate a same output, there is no hope to design an observer producing a single asymptotically-correct estimate. However, one could imagine having an algorithm producing a set of estimates that *converge* asymptotically — for a certain distance to be specified — to the set of solutions generating that same output, or producing one estimate converging asymptotically to one of the several possible solutions. This is motivated by classes of nonlinear systems producing *finite* numbers of indistinguishable solutions, as in induction motors [22,28] or PMSMs [9]. Indeed, when the number of possible indistinguishable solutions corresponding to a given output signal is finite, one could hope to design an observer whose output \hat{x} reconstructs one of (or all) the possible indistinguishable trajectories. This is done in [22,23,21], through sliding mode tools, for systems that can be written in a “non-observable cascade” form, made of observable triangular forms interconnected via a non-injective mapping. Instead, in [9], the KKL design is used on a particular application featuring a PMSM with unknown resistance. It is shown that indistinguishable solutions exist, but there are always less than six, and that there exists a map $T : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_z}$ transforming the dynamics into (4) and whose inversion enables to reconstruct all the possible states. The correct value of the resistance can then be monitored via extra physical knowledge. A preliminary work in [2] attempted to generalize this idea by showing that a KKL observer is able to extract from a given output signal $t \mapsto y(t)$ all the possible information about the corresponding (indistinguishable) solutions. More precisely, it is proved that, for an appropriate choice of the pair (A, B) , there exists a continuously differentiable transformation $T : \mathcal{X} \rightarrow \mathbb{R}^{n_z}$ that transforms (1) into the form of (4) and that the map T *characterizes the distinguishable states*, in the sense that, its preimage gives exactly the set of indistinguishable states. This suggests that, when the preimage map is continuous (in the sense of set-valued maps), then it might be possible to online reconstruct, from a solution z to (4) subject to an output y , all the possible indistinguishable solutions generating y . Although this was confirmed on a particular simulation example, unless further assumptions are made, the preimage map is only defined on the image set $T(\mathcal{X})$ and it is only upper semicontinuous in general, which prevents from stating a general convergence result.

1.3 Contribution

In this paper, we push forward the theory KKL observers when the backward-distinguishability assumption is not satisfied. More specifically, we consider a general nonlinear system (1) having a finite and constant number of indistinguishable states on \mathcal{X} (resp. solutions). A smooth map T transforms it into the form in (4). As a result, the (set-valued) preimage map of T , denoted by $T^- : T(\mathcal{X}) \rightrightarrows \mathcal{X}$, allows us to generate what we call a set-valued KKL observer for (1). Our goal is to

propose sufficient conditions to guarantee that the designed observer asymptotically reconstructs the possible solutions generating a same output. The paper's contributions can be listed as follows:

1- We show that the map T^- is Lipschitz continuous provided that the Jacobian of T is full rank on \mathcal{X} and T^- has a constant cardinality on $T(\mathcal{X})$.

2- Since the solutions to (4) are not guaranteed to remain in $T(\mathcal{X})$, we prove the existence of a Lipschitz continuous set-valued map $T^{\text{inv}} : \mathbb{R}^{n_z} \rightrightarrows \mathbb{R}^{n_x}$ that is an extension of T^- to \mathbb{R}^{n_z} .

3- As a consequence of the latter two items, we show that, for any x solution to (1) generating the output y and for any z solution to (4) subject to y , the Hausdorff distance between $T^{\text{inv}}(z(t))$ and the backward-indistinguishable set of $x(t)$ converges to zero.

4- We provide further assumptions, under which, any continuous selection $t \mapsto \hat{x}(t) \in T^{\text{inv}}(z(t))$ converges asymptotically to a solution to (1) generating the output y . In particular, we consider the case where the cardinality of T^- equals the number of solutions generating the output signal y and eventually remaining in \mathcal{X} .

5- Finally, we discuss the necessity and establish some connections between the different assumptions used in this paper. In particular, we discuss the necessity of a full-rank Jacobian of T to conclude Lipschitz continuity of T^- , and the necessity of the constant cardinality of T^- to conclude continuity of T^- . Furthermore, we investigate the link between the existence of a finite number of indistinguishable trajectories and the constant and finite cardinality of the indistinguishable sets. We also relate the rank of T with the rank of a *differential observability map*, that can be more easily computed.

The rest of this paper is organized as follows. After stating the problem in Section 2, the Lipschitz continuity and Lipschitz extension of the preimage map T^- are then analysed in Section 3, while convergence properties of the proposed set-valued KKL observer are presented in Section 4. The necessity and the link between the different assumptions is finally investigated in Section 5. Examples are presented all along the paper to illustrate our results.

Notations. For $x \in \mathbb{R}^{n_x}$, $|x|$ denotes the Euclidean norm of x and for $\varepsilon > 0$, $\mathcal{B}(x, \varepsilon)$ denotes the open ball centered at x whose radius is ε . For a set $K \subset \mathbb{R}^{n_x}$, we use $\text{int}(K)$ to denote its interior, ∂K its boundary, $\text{card}(K)$ the number of elements in the set K , and, for a point $x \in \mathbb{R}^{n_x}$, $d(x, K)$ denotes the distance from x to the set K . For $O \subset \mathbb{R}^{n_x}$, $K \setminus O$ denotes the subset of elements in K that are not in O . A point $x \in K$ is *isolated* in K if there exists $\varepsilon > 0$ such that $\mathcal{B}(x, \varepsilon) \cap K = \{x\}$. The set K is simply connected if any loop in K can be continuously contracted to a point. The *Dubovitsky-Milutin*

cone of K at x is given by

$$D_K(x) := \{v \in \mathbb{R}^n : \exists \epsilon_0 > 0 : x + \epsilon(v + w) \in K \forall \epsilon \in (0, \epsilon_0], \forall w \in \epsilon_0 \mathbb{B}\}, \quad (6)$$

and the *contingent* cone of K at x is given by

$$T_K(x) := \left\{ v \in \mathbb{R}^n : \liminf_{h \rightarrow 0^+} \frac{d(x + hv, K)}{h} = 0 \right\}. \quad (7)$$

For a differentiable map $x \mapsto T(x) \in \mathbb{R}^{n_z}$, $\frac{\partial T}{\partial x}(x)$ denotes the Jacobian of T at the point x . By $F : \mathbb{R}^{n_z} \rightrightarrows \mathbb{R}^{n_x}$, we denote a set-valued map associating to each element $z \in \mathbb{R}^{n_z}$ a subset $F(z) \subset \mathbb{R}^{n_x}$, and, for some $\mathcal{Z} \subset \mathbb{R}^{n_z}$, we let $F(\mathcal{Z}) := \cup_{z \in \mathcal{Z}} F(z)$. The set-valued map F is said to be p -valued, for some $p \in \{1, 2, \dots\}$, if for all $z \in \mathbb{R}^{n_z}$, $\text{card}(F(z)) = p$. Furthermore, we denote $\mathcal{A}_p(\mathbb{R}^{n_x})$ the space of unordered p -tuples in \mathbb{R}^{n_x} . Note that multiplicity is allowed in $\mathcal{A}_p(\mathbb{R}^{n_x})$, i.e., a map $\tilde{F} : \mathbb{R}^{n_z} \rightarrow \mathcal{A}_p(\mathbb{R}^{n_x})$ associates to each $z \in \mathbb{R}^{n_z}$ a tuple of p points in \mathbb{R}^{n_x} that may not be distinct; in which case, we say that \tilde{F} is an *Almgren p -valued map* [3]. Distances $d_{\mathcal{H}}$, δ , \mathcal{G} , and continuity notions for set-valued and p -valued maps are recalled in Appendix A. Given $x_0 \in \mathbb{R}^n$, we denote by $\omega(x_0)$ the ω -limit set of x_0 for (1), i.e.,

$$\omega(x_0) := \{y : \forall \bar{t} > 0, \forall \varepsilon > 0, \exists t > \bar{t} : |\phi(t) - y| < \varepsilon\},$$

where ϕ is the unique maximal solution to (1) starting from x_0 . We recall that a (backward) solution ϕ to (1) starting from x_0 is (backward) maximal if the domain of definition of every (backward) solution to (1) starting from x_0 is subset of the domain of definition of ϕ .

2 Problem Statement

2.1 Indistinguishability

Before stating the problem tackled in this paper, we recall some important definitions.

Definition 1 (Backward indistinguishable points)

Two points $x_a, x_b \in \mathbb{R}^{n_x}$ are said to be backward indistinguishable for (1), which we denote $x_a \sim x_b$, if the maximal backward solutions $\phi_a : (-\sigma_a, 0] \rightarrow \mathbb{R}^{n_x}$, $\phi_b : (-\sigma_b, 0] \rightarrow \mathbb{R}^{n_x}$ to (1) initialized at x_a, x_b respectively, i.e., with $\phi_a(0) = x_a$ and $\phi_b(0) = x_b$, verify

$$h(\phi_a(t)) = h(\phi_b(t)) \quad \forall t \in (-\min\{\sigma_a, \sigma_b\}, 0].$$

In other words, two points x_a and x_b are backward indistinguishable for (1) if they cannot be distinguished from the past output of (1). Indistinguishability may be seen as an equivalence relation whose classes of equivalence define the *indistinguishable sets*.

Definition 2 (Indistinguishability set) Given $\mathcal{O} \subset \mathbb{R}^{n_x}$ and $x \in \mathbb{R}^{n_x}$, the backward indistinguishable set with respect to \mathcal{O} for (1) is given by

$$\mathcal{I}_{\mathcal{O}}(x) := \{x' \in \mathcal{O}, x' \sim x\}. \quad (8)$$

In other words, $\mathcal{I}_{\mathcal{O}}(x)$ contains all the states in \mathcal{O} that cannot be distinguished from x based on the knowledge of the past output. To ease the notation, we will omit the mention of \mathcal{O} when $\mathcal{O} = \mathbb{R}^{n_x}$. As explained in the introduction, in the context of observer design, we are interested in estimating the state modulo its indistinguishable states. For that, we need the preimage of a map T to describe exactly the indistinguishable sets. This leads to the following definition.

Definition 3 (Characterizing indistinguishability) A map $T : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_z}$ is said to characterize the backward-distinguishable points in \mathcal{O} for (1) if, for each $(x_a, x_b) \in \mathcal{O} \times \mathcal{O}$, we have

$$T(x_a) = T(x_b) \iff x_a \sim x_b. \quad (9)$$

The aforementioned notions focus on backward-indistinguishable *points*. We will also exploit the link with the notion of indistinguishable *solutions*.

Definition 4 (Indistinguishable solutions) Two solutions $\phi_a : I_a \subset \mathbb{R} \rightarrow \mathbb{R}^{n_x}$ and $\phi_b : I_b \subset \mathbb{R} \rightarrow \mathbb{R}^{n_x}$ to (1) are indistinguishable if

$$h(\phi_a(t)) = h(\phi_b(t)) \quad \forall t \in I_a \cap I_b.$$

Note that when two points are backward-indistinguishable, then the two maximal backward solutions to (1) initialized at those points are indistinguishable. However, backward indistinguishability says nothing about what happens to those solutions in positive time. Conversely, two indistinguishable solutions verify

$$\phi_a(t) \sim \phi_b(t) \quad \forall t \in I_a \cap I_b \quad (10)$$

provided that they are *maximal* in backward time. Note though that, for analytic systems, the solutions are analytic in time, and therefore, equality of outputs on an arbitrarily small open subset of $I_a \cap I_b$, implies equality of outputs on $I_a \cap I_b$. Hence, when (1) is analytic, two solutions ϕ_a and ϕ_b , with $I_a \cap I_b$ nonempty, are indistinguishable if and only if there exists $t_o \in I_a \cap I_b$ such that $\phi_a(t_o) \sim \phi_b(t_o)$. Actually, in this case, (10) holds, and the forward and the backward indistinguishability of points are equivalent.

2.2 Set-valued KKL design

According to the KKL methodology, we are supposed to find a C^1 map $T : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_z}$ transforming the dynamics (1) into (4); namely, solution to (5). Without observability/distinguishability assumptions, we cannot hope to prove injectivity of T on \mathcal{X} . However, [2, Theorem 1] showed the existence of a map T solution to (5) that characterizes the backward-distinguishable states according to Definition 1.

Theorem 1 Assume the existence of an open bounded set \mathcal{O} , containing the compact set \mathcal{X} , that is backward invariant for (1); i.e, the backward solutions to (1) starting

from \mathcal{O} remain in \mathcal{O} . Denote $n_o := 2n_x + 1$ and $n_z := n_o n_y$. Then, there exists $\rho > 0$ such that, for almost any pair $(A_o, B_o) \in \mathbb{R}^{n_o \times n_o} \times \mathbb{R}^{n_o}$, with $A_o + \rho I_{n_o}$ Hurwitz, there exists $T : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_z}$ continuously differentiable, characterizing the backward-distinguishable states, and verifying (5) with $(A = I_{n_y} \otimes A_o, B = I_{n_y} \otimes B_o)$.

Remark 1 Note that it is always possible to make the set \mathcal{O} backward invariant by modifying f outside of \mathcal{O} . However, by doing so, (9) holds for a modified system. Hence, $T(x_a) = T(x_b)$ ensures equality of the outputs of the original system only as long as the backward solutions $t \mapsto (\phi_a(t), \phi_b(t))$ from (x_a, x_b) remain in the set where f has not been modified.

Remark 2 According to [10, Proposition 2.1], the solution T to (5) is unique on the backward-invariant set \mathcal{O} . So, if we find T verifying (5) everywhere on \mathcal{O} , for some $n_z \geq n_x$ and for (A, B) picked according to Theorem 1, then T characterizes the backward-distinguishable states.

Now, given $t \mapsto x(t)$ solution to (1) generating an output $t \mapsto y(t)$, we know, using (5) that, any solution $t \mapsto z(t)$ to (4), subject to output y , verifies

$$\lim_{t \rightarrow +\infty} |z(t) - T(x(t))| = 0. \quad (11)$$

Consider the set-valued preimage map $T^- : T(\mathcal{X}) \rightrightarrows \mathcal{X}$ defined by

$$T^-(z) := \{x \in \mathcal{X} : T(x) = z\}. \quad (12)$$

Since T characterizes the backward-indistinguishable states, in the sense that (3) holds, we know that

$$T^-(T(x)) = \mathcal{I}_{\mathcal{X}}(x) \quad \forall x \in \mathcal{X}. \quad (13)$$

Exploiting (11), it becomes tempting to use $T^-(z(t))$ as an estimate of the indistinguishable set $\mathcal{I}_{\mathcal{X}}(x(t))$ at time t . However, $z(t)$ is not guaranteed to be in the image set $T(\mathcal{X})$ where T^- is defined. Therefore, we need to find a set-valued extension $T^{\text{inv}} : \mathbb{R}^{n_z} \rightrightarrows \mathbb{R}^{n_x}$, having the same regularity as T^- , and verifying

$$T^{\text{inv}}(T(x)) = T^-(T(x)) \quad \forall x \in \mathcal{X}. \quad (14)$$

From there, one may compute online the set $T^{\text{inv}}(z(t))$, or at least a continuous selection $\hat{x}(t) \in T^{\text{inv}}(z(t))$, for instance through optimization schemes. In this paper, we study the following two questions:

- (1) Under which conditions does $t \mapsto T^{\text{inv}}(z(t))$ asymptotically converge to $t \mapsto \mathcal{I}_{\mathcal{X}}(x(t))$ in the Hausdorff sense ?
- (2) Can we ensure that a continuous selection $t \mapsto \hat{x}(t) \in T^{\text{inv}}(z(t))$ converges to a solution to (1) producing $t \mapsto y(t)$?

Regarding the first question, the answer is positive provided that T^{inv} is (Hausdorff) continuous, which means

upper and lower semicontinuous at the same time. In particular, it means that T^- must also be (Hausdorff) continuous. However, it is shown in [2] that the map T^- is, in general, only upper semicontinuous. Indeed, the following example shows a case where T^- is not lower semicontinuous and, therefore, it cannot be continuous. We refer the reader to Appendix A for the definitions of the set-valued regularity notions used in this paper.

Example 1 Consider the smooth map $T : \mathcal{X} := [-\pi, \pi] \rightarrow [-1, 1] \times [-1, 1]$ given by

$$T(x) := (\sin(2x), \sin(x)).$$

Note that $T^-(0, 0) = \{-\pi, 0, \pi\}$; whereas, for each $z \in T([-\pi, \pi]) \setminus \{0\}$, $T^-(z)$ contains only two elements. Hence, T^- is not lower semicontinuous at $z = 0$.

As a consequence, the continuity and convergence of $t \mapsto T^{\text{inv}}(z(t))$ to $t \mapsto \mathcal{I}_{\mathcal{X}}(x(t))$ is not guaranteed without extra assumptions that we investigate in the next section. Before that, we illustrate the aforementioned questions on the example studied in [2].

Example 2 ([2]) Consider system (1) with

$$f(x) := \begin{bmatrix} x_2 + x_1(1 - (x_1^2 + x_2^2)) \\ -x_1 + x_2(1 - (x_1^2 + x_2^2)) \end{bmatrix}, \quad (15)$$

$$h(x) := [x_1^2 - x_2^2 \quad 2x_1x_2]^\top.$$

For any $x \in \mathbb{R}^2$, $\mathcal{I}(x) = \{x, -x\}$. Next, using the KKL toolbox [12], a numerical solution to (5) defined on $\mathcal{X} := \mathcal{B}(0, 1.7)$ and for $n_z = (n_x + 1)n_y = 6$ (instead of $n_z = (2n_x + 1)n_y = 10$ as in Theorem 1) was obtained, after making some open set \mathcal{O} backward-invariant. Furthermore, it was numerically checked that T characterizes the backward-distinguishable states in the sense that

$$T^-(T(x)) = \{x, -x\} \quad \forall x \in \mathcal{X}.$$

Moreover, by online solving the optimization algorithm

$$\hat{x}(t) := \underset{x_s \in \mathcal{X}}{\operatorname{argmin}} |z(t) - T(x_s)|,$$

for z solution to (4) subject to an output y , we found that the obtained function $t \mapsto \hat{x}(t)$ converges asymptotically to the set-valued map $t \mapsto \{x(t), -x(t)\}$ formed by the two solutions to (1) generating y . However, this does not imply that $t \mapsto \hat{x}(t)$ tends to a solution to (1), unless continuity of $t \mapsto \hat{x}(t)$ is enforced in the optimization problem. In which case, $t \mapsto \hat{x}(t)$ either tends to $t \mapsto x(t)$ or to $t \mapsto -x(t)$; namely, to one of the two indistinguishable solutions. The goal of the paper is to prove these facts observed in simulation.

3 Lipschitzness of T^- and Lipschitz extension

In this section, we provide assumptions, under which, the set-valued map T^- is Lipschitz (and therefore con-

tinuous) and admits a Lipschitz extension T^{inv} . More precisely, we consider the following two assumptions.

Assumption 1 (Constant cardinality) There exists $p \in \mathbb{N}_{>0}$ such that, for each $z \in T(\mathcal{X})$, $\operatorname{card}(T^-(z)) = p$.

If T characterizes the backward-indistinguishable states as guaranteed in Theorem 1, namely, (3) holds, we know that (13) holds. Therefore, Assumption 1 holds if and only if

$$\operatorname{card}(\mathcal{I}_{\mathcal{X}}(x)) = p \quad \forall x \in \mathcal{X}.$$

In other words, any state in \mathcal{X} admits exactly $p-1$ other backward-indistinguishable states in \mathcal{X} for (1). For instance, in Example 2, Assumption 1 holds on any compact subset of $\mathbb{R}^2 \setminus \{0\}$ with $p = 2$, but it does not hold on \mathcal{X} since $0 \in \mathcal{X}$ and $\operatorname{card}(T^-(T(0))) = \operatorname{card}(\mathcal{I}_{\mathcal{X}}(0)) = 1$.

Assumption 2 (Full rank) For each $x \in \mathcal{X}$, the Jacobian matrix $\frac{\partial T}{\partial x}(x)$ is full rank.

Remark 3 When the map T solving (5) is known explicitly or learned numerically as in [12], the conditioning of $\frac{\partial T}{\partial x}(x)$ can be evaluated on a numerical grid. For instance, the matrix $\frac{\partial T}{\partial x}(x)$ obtained in [2], for the system in Example 2, can be checked to be full-rank on \mathcal{X} with a conditioning smaller than 10^3 ; see Figure 1.

The ‘‘physical’’ interpretation of Assumption 2 is less straight-forward. In Section 5.3, we exhibit a link between the rank of T and the rank of the differential observability map in (45), provided that the eigenvalues of A have a sufficiently large real part. Note also that a map T verifying Assumption 1 does not necessarily verify Assumption 2 (take, for instance, the map $T : \mathbb{R} \rightarrow \mathbb{R}$ with $T(x) := x^3$). The following result discusses a link between Assumptions 1 and 2.

Proposition 1 Let $T : \mathcal{O} \subset \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_z}$ be a C^1 map, and let $\mathcal{X} \subset \mathcal{O}$ be compact such that Assumption 2 holds. Then, for all $z \in T(\mathcal{X})$, $\operatorname{card}(T^-(z))$ is finite.

PROOF. Let $x_o \in \mathcal{X}$ and $z_o = T(x_o)$. Under Assumption 2, for any $x'_o \in T^-(z_o) \subseteq \mathcal{X}$, there exists a neighborhood $\mathcal{B}(x'_o, \varepsilon)$ such that $T : \mathcal{B}(x'_o, \varepsilon) \rightarrow \mathbb{R}^{n_z}$ is injective. Therefore, x'_o is the unique preimage of z_o in $\mathcal{B}(x'_o, \varepsilon)$, i.e., the preimages of z_o are isolated. Since \mathcal{X} is compact, $T^-(z_o)$ is finite. ■

We conclude from Proposition 1 that the contribution of Assumption 1 over 2 is not in the fact that $\operatorname{card}(T^-)$ is finite, but in the fact that it is constant. For instance, in Example 1, the map T verifies Assumption 2, and yet does not show constant cardinality of T^- . This is also the case of Example 2, where Assumption 2 holds and yet the cardinality drops at zero. This is because T is not a local homeomorphism on \mathcal{X} .

Under Assumptions 1 and 2, we establish Lipschitzness of T^- on $T(\mathcal{X})$. Furthermore, thanks to Almgren theory [3], we establish the existence of a set-valued map $T^{\text{inv}} : \mathbb{R}^{n_z} \rightrightarrows \mathbb{R}^{n_x}$ that is a Lipschitz extension of T^- to

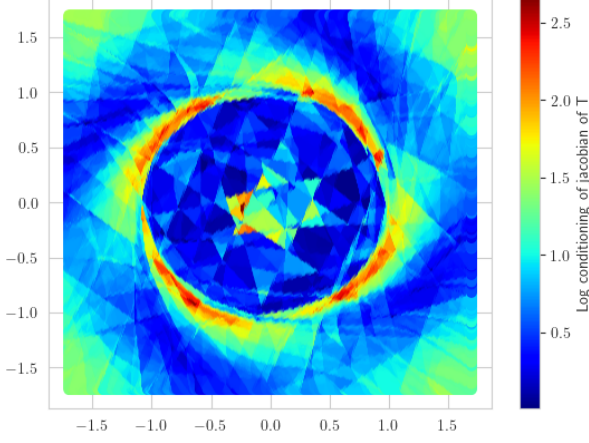


Fig. 1. Condition number of the Jacobian of T in Example 2

\mathbb{R}^{n_z} . The necessity of Assumptions 1 and 2 for the continuity/Lipschitzness of T^- is discussed in Section 5.1.

Theorem 2 Consider a C^1 map $T : \mathcal{O} \subset \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_z}$, with \mathcal{O} an open set containing the compact set \mathcal{X} and $n_z \geq n_x$, such that Assumptions 1 and 2 hold. Then, there exists a Lipschitz set-valued map $T^{\text{inv}} : \mathbb{R}^{n_z} \rightrightarrows \mathbb{R}^{n_x}$ such that

$$T^{\text{inv}}(z) = T^-(z) \quad \forall z \in T(\mathcal{X}) \quad (16)$$

and for any $z \in \mathbb{R}^{n_z}$, $T^{\text{inv}}(z)$ contains at most p elements.

PROOF. Consider $z_o \in T(\mathcal{X})$, and according to Assumption 1, there exist $(x_1, x_2, \dots, x_p) \in \mathcal{X}^p$ such that

$$T^-(z_o) = \{x_1, \dots, x_p\}.$$

From Assumption 2, the Jacobian of T is full-rank in a neighborhood of each x_i since the rank is lower-semicontinuous. Applying the constant rank theorem in Lemma 2, we deduce that, for each $i \in \{1, 2, \dots, p\}$, there exist $\varepsilon_i, \sigma_i > 0$ and two diffeomorphisms $\phi_i : \mathcal{B}(x_i, \varepsilon_i) \rightarrow \mathbb{R}^{n_x}$ and $\psi_i : \mathcal{B}(z_o, \sigma_i) \rightarrow \mathbb{R}^{n_z}$ such that

$$\psi_i \circ T(x) = (\phi_i(x), 0, \dots, 0) \quad \forall x \in \mathcal{B}(x_i, \varepsilon_i).$$

Now, denote by $\text{Proj} : \mathbb{R}^{n_z} \rightarrow \mathbb{R}^{n_x}$ the projection on the first n_x coordinates, we let the C^1 map $g_i := \phi_i^{-1} \circ \text{Proj} \circ \psi_i : \mathcal{B}(z_o, \sigma_i) \rightarrow \mathbb{R}^{n_x}$ verifying

$$g_i \circ T(x) = x \quad \forall x \in \mathcal{B}(x_i, \varepsilon_i).$$

In other words, g_i is a C^1 local left-inverse of T around x_i . We now prove that there exists $\sigma_o > 0$ such that

$$T^-(z) = \cup_{i=1}^p g_i(z) \quad \forall z \in \mathcal{B}(z_o, \sigma_o) \cap T(\mathcal{X}). \quad (17)$$

To do so, we pick a sequence $(z_k)_{k \in \mathbb{N}}$ in $T(\mathcal{X})$ converging to z_o . Since

$$T^-(z_o) := \{x_1, x_2, \dots, x_p\} \subset \cup_{i=1}^p \mathcal{B}(x_i, \varepsilon_i),$$

and since T^- is upper semicontinuous according to [2], it follows that there exists $k^* > 0$ such that, for each $k \geq k^*$,

$$T^-(z_k) \subset \cup_{i=1}^p \mathcal{B}(x_i, \varepsilon_i).$$

As a result, for each $k \geq k^*$ and each element $x_{kj} \in T^-(z_k)$, there exists $i_{kj} \in \{1, 2, \dots, p\}$ such that

$$z_k = T(x_{k,j}), \quad \text{and} \quad x_{k,j} \in \mathcal{B}(x_{i_{kj}}, \varepsilon_{i_{kj}}).$$

Thus, $x_{k,j} = g_{i_{kj}}(z_k)$, so that

$$T^-(z_k) \subseteq \cup_{i=1}^p g_i(z_k) \quad \forall k \geq k^*.$$

The latter establishes the existence of a neighborhood W_o of z_o such that

$$T^-(z) \subseteq \cup_{i=1}^p g_i(z) \quad \forall z \in W_o \cap T(\mathcal{X}).$$

However, by Assumption 1, the cardinal of $T^-(z)$ is p for $z \in T(\mathcal{X}) \cap W_o$, so that necessarily it is an equality and (17) follows. Then, we note that the set-valued map $g : W_o \rightrightarrows \mathbb{R}^{n_x}$ defined by $g(z) = \cup_{i=1}^p g_i(z)$ can be identified with an Almgren p -valued map \tilde{g} on $W_o \cap T(\mathcal{X})$. Thus, to g and T^- , we associate the Almgren p -valued maps \tilde{g} and \tilde{T}^- defined on W_o and $W_o \cap T(\mathcal{X})$, respectively.

Since each $g_i, i \in \{1, 2, \dots, p\}$, is C^1 , it is Lipschitz on W_o with a Lipschitz constant $L_i > 0$, and after (17), we conclude that, for each $z_a, z_b \in W_o \cap T(\mathcal{X})$, we have

$$\begin{aligned} \mathcal{G}(\tilde{T}^-(z_a), \tilde{T}^-(z_b)) &= \min_{\sigma \in \mathcal{P}_p} \max_{i \in \{1, \dots, p\}} |g_i(z_a) - g_{\sigma(i)}(z_b)| \\ &\leq \max_{i \in \{1, \dots, p\}} |g_i(z_a) - g_i(z_b)| \leq \max_{i \in \{1, \dots, p\}} L_i |z_a - z_b|. \end{aligned}$$

This being true on a neighborhood of z_o , for any $z_o \in T(\mathcal{X})$, we conclude that \tilde{T}^- is locally Lipschitz on $T(\mathcal{X})$. Now, since $T(\mathcal{X})$ is compact under the continuity of T , we show that \tilde{T}^- is (globally) Lipschitz on $T(\mathcal{X})$. Indeed, assume \tilde{T}^- is not Lipschitz on $T(\mathcal{X})$. Then, for every $k \in \mathbb{N}$, there exists $z_{a,k}, z_{b,k}$ in $T(\mathcal{X})$ such that $\mathcal{G}(\tilde{T}^-(z_{a,k}), \tilde{T}^-(z_{b,k})) \geq k|z_{a,k} - z_{b,k}|$. By compactness, we can assume that the sequence $\{z_{a,k}, z_{b,k}\}_{k=1}^\infty$ is convergent to $(z_a^*, z_b^*) \in T(\mathcal{X}) \times T(\mathcal{X})$. Since $\tilde{T}^-(z_{a,k})$ and $\tilde{T}^-(z_{b,k})$ are bounded, necessarily $|z_{a,k} - z_{b,k}|$ converges to 0, and, thus, $z_a^* = z_b^* := z^*$. Therefore, for sufficiently large k , $z_{a,k}, z_{b,k}$ must belong to a neighborhood of z^* where \tilde{T}^- is Lipschitz. This yields to a contradiction when k is sufficiently large.

Next, by using Almgren extension theorem; see Lemma 3 in the Appendix, we conclude the existence of a Lipschitz

Almgren p -valued map $\tilde{T}^{\text{inv}} : \mathbb{R}^{n_z} \rightarrow \mathcal{A}_p(\mathbb{R}^{n_x})$ (with multiplicity allowed) such that \tilde{T}^{inv} agrees with \tilde{T}^- on $T(\mathcal{X})$. In particular, there exists $L > 0$ such that

$$\mathcal{G}(\tilde{T}^{\text{inv}}(z_a), \tilde{T}^{\text{inv}}(z_b)) \leq L|z_a - z_b| \quad \forall (z_a, z_b) \in \mathbb{R}^{n_z} \times \mathbb{R}^{n_z}.$$

Then, forgetting about multiplicity and transforming the p -valued map into a set-valued one, we build $T^{\text{inv}} : \mathbb{R}^{n_z} \rightrightarrows \mathbb{R}^{n_x}$ such that

$$d_{\mathcal{H}}(T^{\text{inv}}(z_a), T^{\text{inv}}(z_b)) \leq L|z_a - z_b| \quad \forall (z_a, z_b) \in \mathbb{R}^{n_z} \times \mathbb{R}^{n_z}.$$

Indeed,

$$\begin{aligned} d_{\mathcal{H}}(T^{\text{inv}}(z_a), T^{\text{inv}}(z_b)) &= \max\{\max_i \min_j |\tilde{T}_i^{\text{inv}}(z_a) - \tilde{T}_j^{\text{inv}}(z_b)|, \\ &\quad \max_j \min_i |\tilde{T}_i^{\text{inv}}(z_a) - \tilde{T}_j^{\text{inv}}(z_b)|\} \end{aligned}$$

and for any permutation σ in \mathcal{P}_p ,

$$\begin{aligned} \max_i \min_j |\tilde{T}_i^{\text{inv}}(z_a) - \tilde{T}_j^{\text{inv}}(z_b)| &\leq \max_i |\tilde{T}_i^{\text{inv}}(z_a) - \tilde{T}_{\sigma(i)}^{\text{inv}}(z_b)| \\ \max_j \min_i |\tilde{T}_i^{\text{inv}}(z_a) - \tilde{T}_j^{\text{inv}}(z_b)| &\leq \max_j |\tilde{T}_{\sigma^{-1}(j)}^{\text{inv}}(z_a) - \tilde{T}_j^{\text{inv}}(z_b)|, \end{aligned}$$

which gives the result by an appropriate re-indexing. ■

Example 3 Consider the map T introduced in Example 1. We already showed that T^- is only upper semi-continuous. We can see that $\frac{\partial T}{\partial x}(x)$ is full rank for all $x \in \mathcal{X} := [-\pi, \pi]$. Hence, Assumption 2 is verified. However, Assumption 1 is not verified on $T(\mathcal{X}) \setminus \{0\}$. Thus, T^- is Lipschitz on any compact subset contained in $T(\mathcal{X}) \setminus \{0\}$.

Remark 4 From the proof of Theorem 2, the set-valued map $T^{\text{inv}} : \mathbb{R}^{n_z} \rightrightarrows \mathbb{R}^{n_x}$ is obtained from the p -valued map $\tilde{T}^{\text{inv}} : \mathbb{R}^{n_z} \rightarrow \mathcal{A}_p(\mathbb{R}^{n_x})$ (by ignoring multiplicities), which is a Lipschitz extension of $\tilde{T}^- : T(\mathcal{X}) \rightarrow \mathcal{A}_p(\mathbb{R}^{n_x})$ (T^- viewed as a p -valued map). The existence of \tilde{T}^{inv} is guaranteed by the Almgren Extension Theorem, whose proof in [19, Theorem 1.7] is constructive; hence, it allows to deduce a candidate \tilde{T}^{inv} , leading to a candidate T^{inv} .

Remark 5 Note that the conclusions of Theorem 2 remain valid if, instead of Assumption 1, we assume that the cardinality of T^- is constant only on the connected components of $T(\mathcal{X})$. The cardinality of T^{inv} will then not exceed the maximal cardinality of T^- . Such a relaxation will not help our study since, using (11), a solution z to (4), subject to the system's output y , converges to $T(\mathcal{X})$, and, since z is continuous, then it necessarily converges to a connected component in $T(\mathcal{X})$, on which, the cardinality of T^- is constant under both assumptions.

4 Convergence of the KKL Observer

4.1 Set-valued convergence to indistinguishable set

We start this section by establishing a direct consequence of Theorem 2 under the following assumption.

Assumption 3 The map $T : \mathcal{X} \rightarrow \mathbb{R}^{n_z}$ characterizes the backward-distinguishable points in \mathcal{X} for (1) according to Definition 3.

The existence of a map T verifying Assumption 3 is guaranteed by Theorem 1. Then, as noticed above, Assumption 1 holds if and only if any point in \mathcal{X} admits exactly $p - 1$ other backward indistinguishable states in \mathcal{X} .

Theorem 3 Consider a C^1 map $T : \mathcal{O} \subset \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_z}$, with \mathcal{O} open and $n_z \geq n_x$, verifying (5), for a given pair (A, B) and \mathcal{X} compact contained in \mathcal{O} , and satisfying Assumptions 1, 2, and 3. Let $T^{\text{inv}} : \mathbb{R}^{n_z} \rightarrow \mathbb{R}^{n_x}$ be the Lipschitz extension given in Theorem 2. Then, there exist $\rho, \lambda > 0$ such that, for any solution $x : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n_x}$ to (1) remaining in \mathcal{X} , and for any solution $z : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n_z}$ to (4) subject to $y = h(x)$, we have

$$\begin{aligned} d_{\mathcal{H}}(T^{\text{inv}}(z(t)), \mathcal{I}_{\mathcal{X}}(x(t))) &\leq \rho e^{-\lambda t} |z(0) - T(x(0))| \quad \forall t \geq 0. \end{aligned} \quad (18)$$

Therefore, if the system solution x eventually remains in \mathcal{X} , then the set-valued estimate $T^{\text{inv}}(z(t))$ asymptotically converges in the Hausdorff sense to the indistinguishable set $\mathcal{I}_{\mathcal{X}}(x(t))$.

PROOF. According to Theorem 2, T^{inv} is (globally) Lipschitz and verifies

$$T^{\text{inv}}(T(x)) = T^-(T(x)) \quad \forall x \in \mathcal{X}.$$

Furthermore, using Assumption 3, we conclude that

$$T^-(T(x)) = \mathcal{I}_{\mathcal{X}}(x) \quad \forall x \in \mathcal{X}.$$

Therefore, there exists $L > 0$ such that

$$d_{\mathcal{H}}(T^{\text{inv}}(z(t)), \mathcal{I}_{\mathcal{X}}(x(t))) \leq L|z(t) - T(x(t))|$$

for all $t \geq 0$ provided that $x(t) \in \mathcal{X}$. Hence, the proof is completed using the fact that the estimation error $\tilde{z}(t) := z(t) - T(x(t))$ in the z -coordinates verifies the exponentially stable dynamics $\dot{\tilde{z}} = A\tilde{z}$. ■

Remark 6 (ISS set-valued observer) If the available output y fed into (4) is noisy, i.e., $y = h(x) + \nu$, then, the error $\tilde{z}(t) := z(t) - T(x(t))$ in the z -coordinates evolves according to $\dot{\tilde{z}} = A\tilde{z} + B\nu$ instead of $\dot{\tilde{z}} = A\tilde{z}$ and there exist $\rho_1, \rho_2, \lambda > 0$ (depending on the pair (A, B))

only) such that

$$d_{\mathcal{H}}(T^{\text{inv}}(z(t)), \mathcal{I}_{\mathcal{X}}(x(t))) \leq L \left(\rho_1 e^{-\lambda t} |z(0) - T(x(0))| + \rho_2 \sup_{s \in [0, t]} |\nu(s)| \right) \quad (19)$$

for all $t \geq 0$ provided that $x(t) \in \mathcal{X}$. Besides, *since for any \hat{x}_0, x_0 in \mathcal{X} , $T(\hat{x}_0) = T(x_0)$ is equivalent to $T^{\text{inv}}(T(\hat{x}_0)) = \mathcal{I}_{\mathcal{X}}(\hat{x}_0) = \mathcal{I}_{\mathcal{X}}(x_0)$, there exists (by continuity) a class- \mathcal{K} function ρ such that for any \hat{x}_0, x_0 in \mathcal{X} ,*

$$|T(\hat{x}_0) - T(x_0)| \leq \rho(d_{\mathcal{H}}(\mathcal{I}_{\mathcal{X}}(\hat{x}_0), \mathcal{I}_{\mathcal{X}}(x_0))) .$$

Therefore, if $x(0) \in \mathcal{X}$ and $z(0)$ is chosen equal to $T(\hat{x}_0)$ for some initial guess $\hat{x}_0 \in \mathcal{X}$, then $|z(0) - T(x(0))| \leq \rho(d_{\mathcal{H}}(\mathcal{I}_{\mathcal{X}}(\hat{x}_0), \mathcal{I}_{\mathcal{X}}(x_0)))$ with $\mathcal{I}_{\mathcal{X}}(\hat{x}_0) = T^{\text{inv}}(T(\hat{x}_0))$. It follows from (19) that the KKL observer exhibits a set-valued ISS property with respect to measurement noise, in the Hausdorff sense.

Remark 7 In the case where $t \mapsto x(t)$ does not remain in \mathcal{X} , the map $t \mapsto T^{\text{inv}}(z(t))$ is guaranteed to approach $t \mapsto \mathcal{I}_{\mathcal{X}}(x(t))$ only during the time intervals, on which, the solution x is within the set \mathcal{X} .

4.2 Convergence of a continuous selection to a solution

In practice, one can compute a *continuous* selection $t \mapsto \hat{x}(t) \in T^{\text{inv}}(z(t))$, for instance, through an optimization algorithm. Hence, it is interesting to know whether \hat{x} converges to a solution generating y or not. To answer this question, we make the following assumption on the solutions generating a same output y .

Assumption 4 Given $y : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n_y}$, there exist $\bar{t} > 0$ and at least $p \in \mathbb{N}_{>0}$ solutions $\{x_i\}_{i=1}^p : [\bar{t}, +\infty) \rightarrow \mathcal{X}$ to (1) such that, for all $i \in \{1, \dots, p\}$, we have

- $h(x_i(t)) = y(t)$ for all $t \geq \bar{t}$,
- for all $j \in \{1, \dots, p\}$ with $j \neq i$, $x_i - x_j$ does not asymptotically converge to zero.

Given a solution z to (4) subject to y , we would like to relate the continuous selection $t \mapsto \hat{x}(t) \in T^{\text{inv}}(z(t))$ and the solutions x_i to (1) generating y . For that, we rely on the following *Splitting Lemma* first proved in [7], see also [11, Theorem 3.1.] for more details.

Lemma 1 (Splitting Lemma) Consider a continuous set-valued map $T^- : \mathcal{C} \subseteq \mathbb{R}^{n_z} \rightrightarrows \mathbb{R}^{n_x}$ such that \mathcal{C} is simply connected and there exists $p \in \mathbb{N}_{>0}$ such that, for each $z \in \mathcal{C}$, $\text{card}(T^-(z)) = p$. Then, there exist p continuous (single-valued) functions $\{g_i\}_{i=1}^p : \mathcal{C} \rightarrow \mathbb{R}^{n_x}$ such that

$$T^-(z) = \bigcup_{i=1}^p \{g_i(z)\} \quad \forall z \in \mathcal{C}. \quad (20)$$

The map T^- is then said to be split on \mathcal{C} .

Remark 8 Without any assumption on its domain \mathcal{C} (such as simple connectedness), a continuous p -valued map is not necessarily split in general when n_z and n_x are both greater than one [27]; see the following example.

Example 4 Let $\mathcal{C} \subset \mathbb{R}^2$ be the unit circle; namely,

$$\mathcal{C} := \{z \in \mathbb{R}^2 : |z| = 1\}.$$

We parameterize \mathcal{C} by the variable $t \in [0, 1)$, and we let the map $\phi : [0, 1) \rightarrow \mathcal{C}$ defined as

$$\phi(t) := (\sin(2\pi t), \cos(2\pi t)).$$

Note that ϕ is a diffeomorphism and

$$\phi^-(z) = \frac{\arg(z)}{2\pi} \quad \forall z \in \mathcal{C},$$

where $\arg : \mathcal{C} \rightarrow [0, 1)$ is the function associating each $z \in \mathcal{C}$ to its argument. Consider the set-valued map $H : [0, 1) \rightrightarrows \mathbb{R}^2$ given by

$$H(t) := \{f(t), g(t)\},$$

where $f, g : [0, 1) \rightarrow \mathbb{R}^2$ are continuous and defined as

$$\begin{aligned} f(t) &= (f_1(t), f_2(t)) := (2 - t, 4 - t), \\ g(t) &= (g_1(t), g_2(t)) := (1 + t, 3 + te^{t-1}). \end{aligned}$$

We start noting that $f(t) \neq g(t)$ for all $t \in [0, 1)$. Indeed, $f_1(t) = g_1(t)$ only when $t = 1/2$. However,

$$7/2 = f_2(1/2) \neq g_2(1/2) = 3 + 1/(2e^{1/2}).$$

Hence, H is a continuous set-valued map such that $\text{card}(H(t)) = 2$ for all $t \in [0, 1)$.

At this point, we define the set-valued map $T^- : \mathcal{C} \rightrightarrows \mathbb{R}^2$ given by

$$T^-(z) = H(\phi^-(z)) \quad \forall z \in \mathcal{C}.$$

Clearly, $\text{card}(T^-(z)) = 2$ for all $z \in \mathcal{C}$. Furthermore, since ϕ is a diffeomorphism and ϕ^- is continuous on $\mathcal{C} \setminus (1, 0)$, we conclude that T^- is continuous on $\mathcal{C} \setminus (1, 0)$. Actually, observing that $\lim_{t \rightarrow 1} f(t) = g(0)$ and $\lim_{t \rightarrow 1} g(t) = f(0)$, it follows that T^- is also continuous at $z = (1, 0)$. The latter shows that T^- is 2-valued and continuous on \mathcal{C} . But we next show that T^- is not split using contradiction. Indeed, assume that T^- is split, then we can find continuous functions $t_1^-, t_2^- : \mathcal{C} \rightarrow \mathbb{R}^2$ such that

$$T^-(z) = \{t_1^-(z), t_2^-(z)\} \quad z \in \mathcal{C}.$$

At the same time, by definition, we have that

$$T^-(z) = \{f(\phi^-(z)), g(\phi^-(z))\} \quad z \in \mathcal{C}.$$

Now, since $f \circ \phi^-$ and $g \circ \phi^-$ are continuous on $\mathcal{C} \setminus (1, 0)$ and never intersect, it follows that either

$$f(\phi^-(z)) = t_1^-(z) \quad \text{and} \quad g(\phi^-(z)) = t_2^-(z)$$

for all $z \in \mathcal{C} \setminus (1, 0)$ or

$$f(\phi^-(z)) = t_2^-(z) \quad \text{and} \quad g(\phi^-(z)) = t_1^-(z)$$

for all $z \in \mathcal{C} \setminus (1, 0)$. Both cases yield a contradiction since t_1^- and t_2^- are continuous whereas the functions $f \circ \phi^-$ and $g \circ \phi^-$ are discontinuous at $(1, 0)$. Therefore, T^- is not split.

In order to apply Lemma 1, given an output y to (1), we introduce the following assumption.

Assumption 5 *There exists a solution z_* to (4) subject to y and a simply connected subset $\mathcal{C} \subseteq T(\mathcal{X})$ such that $\lim_{t \rightarrow +\infty} |z_*(t)|_{\mathcal{C}} = 0$.*

Remark 9 *Given any output y generated by a solution x to (1), any solution z to (4) subject to y converges to $t \mapsto T(x(t)) \in T(\mathcal{X})$. Therefore, Assumption 5 holds in particular if x asymptotically converges to a point in \mathcal{X} or if $T(\mathcal{X})$ is simply connected. It also holds if the range of $t \mapsto T(x(t))$ is, at least after a certain time, simply connected in $T(\mathcal{X})$.*

Assumption 5 allows us to guarantee that T^- is split on \mathcal{C} . We then show that the same property holds for T^{inv} on a neighborhood of \mathcal{C} . This allows to show that any continuous selection $t \mapsto \hat{x}(t) \in T^{\text{inv}}(z(t))$, with z solution to (4) subject to y , converges to one of the solutions generating y given by Assumption 4, as stated in Theorem 4 below.

But, actually, we can prove that a continuous selection within $t \mapsto T^{\text{inv}}(z(t))$ converges to a solution even if T^{inv} is not split on a set containing the solutions after a certain time. So we propose the following alternative assumption when Assumption 5 does not hold.

Assumption 6 *There exists a solution z_* to (4) subject to y , for which, there exists $\tau_o > 0$ and $\delta > 0$ such that, for each $\tau \geq \tau_o$, $z^*([\tau, \tau + \delta])$ is simply connected.*

We can then state the following result.

Theorem 4 *Consider a C^1 map $T : \mathcal{O} \subset \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_z}$, with \mathcal{O} open and $n_z \geq n_x$, verifying (5), for a given pair (A, B) and \mathcal{X} compact contained in \mathcal{O} , and Assumptions 1 and 2. Consider an output $y : \mathbb{R}_{>0} \rightarrow \mathbb{R}^{n_y}$ to (1) such that Assumption 4 and either Assumption 5 or 6 hold. Then, for any continuous selection $t \mapsto \hat{x}(t) \in T^{\text{inv}}(z(t))$, where $z : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n_z}$ is a solution to (4) subject to y , there exists a solution \tilde{x} to (1) generating y such that*

$$\lim_{t \rightarrow +\infty} |\hat{x}(t) - \tilde{x}(t)| = 0. \quad (21)$$

PROOF. Consider an output $y : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n_y}$ to (1) and let $z : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n_z}$ be a solution to (4) subject to

the output y . We start by showing the result under Assumption 5. For that, let $\mathcal{U}_c \subset \mathbb{R}^{n_z}$ be a simply connected compact subset including \mathcal{C} in its interior. Next, we let T_u^{inv} be the restriction of T^{inv} to \mathcal{U}_c . Using Theorem 2, we know that T_u^{inv} is continuous and its images contain at most p elements. Since T^{inv} equals T^- on the compact set $\mathcal{U}_c \cap T(\mathcal{X})$, whose images contain exactly p elements from Assumption 1, it follows that, by choosing the boundary of \mathcal{U}_c sufficiently close to $T(\mathcal{X})$, we can say without loss of generality

$$\text{card}(T_u^{\text{inv}}(z)) = p \quad \forall z \in \mathcal{U}_c. \quad (22)$$

Now, since T_u^{inv} is continuous on \mathcal{U}_c simply connected and (22) holds, we conclude, using Lemma 1, the existence of a sequence of continuous functions $\{g_i\}_{i=1}^p : \mathcal{U}_c \rightarrow \mathcal{X}$ such that

$$T_u^{\text{inv}}(z) = \cup_{i=1}^p g_i(z) \quad \forall z \in \mathcal{U}_c.$$

According to (22), we have for all $z \in \mathcal{U}_c$ and for all $i \neq i'$, $g_i(z) \neq g_{i'}(z)$. By compactness of \mathcal{U}_c , continuity of the maps g_i and finiteness of pairs i, i' , there exists $\varepsilon > 0$ such that

$$|g_i(z) - g_{i'}(z)| \geq \varepsilon \quad \forall i \neq i', \quad \forall z \in \mathcal{U}_c. \quad (23)$$

On the other hand, knowing that the solution z_* in Assumption 5 converges to \mathcal{C} , we know that $z_*(t) \in \mathcal{U}_c$ at least after a certain time. And knowing that any solution z to (4) subject to y converges to z_* , we have by continuity that

$$\lim_{t \rightarrow \infty} d(\hat{x}(t), T_u^{\text{inv}}(z_*(t))) = 0. \quad (24)$$

Hence, since \hat{x} is continuous, (24) and (23) imply the existence of $i^* \in \{1, 2, \dots, p\}$ such that

$$\lim_{t \rightarrow \infty} |\hat{x}(t) - g_{i^*}(z_*(t))| = 0. \quad (25)$$

Now, we show that each $t \mapsto g_i(z_*(t))$, $i \in \{1, 2, \dots, p\}$, must converge to one of the solutions x_i generating y given by Assumption 4. Indeed, let $\{z_i\}_{i=1}^p : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n_z}$ such that

$$z_i(t) = T(x_i(t)) \quad \forall i \in \{1, 2, \dots, p\}, \quad \forall t \geq 0. \quad (26)$$

According to (5) and the first item of Assumption 4, all the z_i s and z_* satisfy (4) subject to y . Hence, they converge asymptotically to each other. Furthermore, according to (26) and since $x_i(t) \in \mathcal{X}$ for all $t \geq \bar{t}$, we have

$$x_i(t) \in T^-(z_i(t)) \quad \forall t \geq \bar{t}, \quad \forall i \in \{1, 2, \dots, p\}.$$

As above, using the continuity of T^{inv} given by Theorem 2, we deduce that

$$\lim_{t \rightarrow \infty} d(x_i(t), T_u^{\text{inv}}(z_*(t))) = 0 \quad \forall i \in \{1, \dots, p\}.$$

Now, according to the second item of Assumption 4 and the fact that $t \mapsto T_u^{\text{inv}}(z_*(t))$ describes p continuous functions of time not converging to each other according to (23), we conclude that, for each $i \in \{1, \dots, p\}$, there exists a unique $k_i \in \{1, 2, \dots, p\}$ such that x_i converges to $g_{k_i}(z_*)$ asymptotically. Similarly, for each $i \in \{1, \dots, p\}$, there exists a unique $l_i \in \{1, 2, \dots, p\}$ such that $g_i(z_*)$ converges to x_{l_i} asymptotically. From (25), we deduce the result.

We now suppose Assumption 6 holds instead of Assumption 5. Let $\mathcal{U} \subset \mathbb{R}^{n_z}$ be a compact subset including $T(\mathcal{X})$ in its interior and let T_u^{inv} be the restriction of T^{inv} to \mathcal{U} . By Theorem 2, T_u^{inv} is continuous and its images contain at most p elements. Besides, $T_u^{\text{inv}} = T^-$ on the compact set $T(\mathcal{X})$ where its images have exactly p distinct elements according to Assumption 1. It follows that, without loss of generality, when $\partial\mathcal{U}$ is sufficiently close to $T(\mathcal{X})$, then we may assume that

$$\text{card}(T_u^{\text{inv}}(z)) = p \quad \forall z \in \mathcal{U}. \quad (27)$$

[11, Proposition 4.1.] then shows the existence of $\bar{\epsilon} > 0$ such that

$$\min\{|x_a - x_b| : x_a \neq x_b \in T_u^{\text{inv}}(z), z \in \mathcal{U}\} \geq \bar{\epsilon}. \quad (28)$$

Furthermore, since y is generated by a solution to (1) that is eventually in \mathcal{X} according to Assumption 4, any solution to (4) fed with y converges to $T(\mathcal{X})$ so that $z_*(t) \in \mathcal{U}$ at least after some time, let's say τ_o given by Assumption 6 without loss of generality.

Finally, all solutions to (4) converge to each other, so in particular z converges to z_* and we have

$$\lim_{t \rightarrow \infty} d(\hat{x}(t), T_u^{\text{inv}}(z_*(t))) = 0. \quad (29)$$

Now, under Assumption 6, for each $\tau \geq \tau_o$, the set $z_*([\tau, \tau + \delta]) \subset \mathcal{U}$ is simply connected. Hence, using Lemma 1, we conclude the existence of a sequence $\{g_i^\tau : z_*([\tau, \tau + \delta]) \rightarrow \mathcal{X}\}$ such that

$$T_u^{\text{inv}}(z) = \cup_{i=1}^p g_i^\tau(z) \quad \forall z \in z_*([\tau, \tau + \delta]).$$

Then, (28) allows us to conclude that

$$|g_i^\tau(z) - g_{j \neq i}^\tau(z)| \geq \bar{\epsilon} \quad \forall z \in z_*([\tau, \tau + \delta]), \forall \tau > \tau_o. \quad (30)$$

Let $\{z_i\}_{i=1}^p : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n_z}$ such that

$$z_i(t) = T(x_i(t)) \quad \forall i \in \{1, 2, \dots, p\}, \quad \forall t \geq 0. \quad (31)$$

According to (5) and the first item of Assumption 4, all the z_i s and z_* satisfy (4) subject to y . Hence, they converge asymptotically to each other and are in \mathcal{U} after a certain time. T_u^{inv} being continuous on \mathcal{U} , and reasoning

as above, we conclude that, for each $\epsilon_o \in (0, \bar{\epsilon}/8]$, there exists $\tau_1 \geq \tau_o$ such that, for each $\tau \geq \tau_1$ the following two properties hold:

- For each $i \in \{1, 2, \dots, p\}$, there exists a unique $k_i \in \{1, 2, \dots, p\}$ such that

$$|g_{k_i}^\tau(z_*(t)) - x_i(t)| \leq \epsilon_o \leq \bar{\epsilon}/8 \quad \forall t \in [\tau, \tau + \delta].$$

- For each $i \in \{1, 2, \dots, p\}$, there exists a unique $l_i \in \{1, 2, \dots, p\}$ such that

$$|g_i^\tau(z_*(t)) - x_{l_i}(t)| \leq \epsilon_o \leq \bar{\epsilon}/8 \quad \forall t \in [\tau, \tau + \delta].$$

A consequence of the latter two items and (30) is that

$$|x_j(t) - x_i(t)| \geq 3\bar{\epsilon}/4 \quad \forall t \geq \tau_1, \forall i \neq j \in \{1, 2, \dots, p\}. \quad (32)$$

Now, using (29), we conclude the existence of $\tau_2 \geq \tau_1$ such that, for each $\tau \geq \tau_2$, there exists a unique $i \in \{1, 2, \dots, p\}$ such that

$$|\hat{x}(t) - g_i^\tau(z(t))| \leq \epsilon_o \quad \forall t \in [\tau, \tau + \delta].$$

Thus, for each $\tau \geq \tau_2$, there exists a unique $l \in \{1, 2, \dots, p\}$ such that

$$|x_l(t) - \hat{x}(t)| \leq 2\epsilon_o \leq \bar{\epsilon}/4 \quad \forall t \in [\tau, \tau + \delta].$$

Combining the latter inequality to (32) and using the continuity of solutions and the continuity of the selection \hat{x} , we conclude that l is invariant with τ , i.e., there exists a unique $l \in \{1, 2, \dots, p\}$ such that

$$|x_l(t) - \hat{x}(t)| \leq 2\epsilon_o \quad \forall t \in [\tau_2, +\infty). \quad \blacksquare$$

Example 5 In Example 2, if we pick $\tilde{\mathcal{X}}$ to be a compact subset of $\mathbb{R}^2 \setminus \{0\}$, Assumptions 1 and 2 are satisfied. Besides, if the circle \mathcal{C} centered at 0 and of radius 1 is contained in the interior of $\tilde{\mathcal{X}}$, then any solution x with $x(0) \neq 0$ converges to \mathcal{C} and is eventually in $\tilde{\mathcal{X}}$. Then, Assumption 4 holds since x and $-x$ are the only indistinguishable solutions producing the same output y and not converging to each other. On the other hand, Assumption 5 does not hold, as it can be seen in Figure 2, since the solutions $t \mapsto z(t)$ eventually circle around the “hole” in $T(\tilde{\mathcal{X}})$ obtained by removing a neighborhood of 0 in \mathcal{X} . Nevertheless, Assumption 6 holds because the time of revolution around the hole is lower bounded by a positive time δ^* , so that, even in the case of a periodic solution, $z([\tau, \tau + \delta])$ with $\delta < \delta^*$ is always simply connected. The convergence observed in [2] is thus justified by applying Theorem 4.

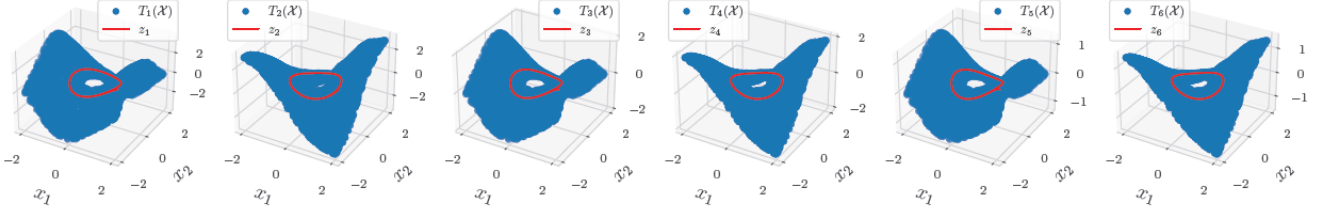


Fig. 2. Image sets $T_i(\tilde{\mathcal{X}})$ with $\tilde{\mathcal{X}} = [-2, 2]^2 \setminus \mathcal{B}(0, 0.5)$ and trajectories $t \mapsto z_i(t)$.

5 Discussion on Assumptions

5.1 Necessity of Assumptions 1 and 2 for Continuity and Lipschitzness of T^-

The next result shows that if $T_{\mathcal{O}}^- : T(\mathcal{O}) \rightrightarrows \mathcal{O}$ defined as

$$T_{\mathcal{O}}^-(z) := \{x \in \mathcal{O} : T(x) = z\} \quad (33)$$

takes finite values and is locally Lipschitz for some open neighborhood \mathcal{O} of \mathcal{X} , then Assumption 2 holds.

Proposition 2 *Let $T : \mathcal{O} \rightarrow \mathbb{R}^{n_z}$ be a C^1 map defined on an open set $\mathcal{O} \subseteq \mathbb{R}^{n_x}$ with $n_z \geq n_x$. Let $\mathcal{X} \subset \mathcal{O}$ such that any $x \in \mathcal{X}$ is isolated in $T_{\mathcal{O}}^-(T(x))$, with T^- defined in (33). Then, Assumption 2 is verified provided that $T_{\mathcal{O}}^-$ is locally Lipschitz.*

PROOF. Let $z \in T(\mathcal{X})$ and let $x \in \mathcal{X}$ such that $z = T(x)$. We will show that $\frac{\partial T}{\partial x}(x)$ is full-rank, namely

$$\frac{\partial T}{\partial x}(x)v = 0 \implies v = 0.$$

Since $T_{\mathcal{O}}^-$ is locally Lipschitz on $T(\mathcal{O})$, it follows that there exists $L > 0$ and a neighborhood U of z such that

$$d_{\mathcal{H}}(T_{\mathcal{O}}^-(z_o), T_{\mathcal{O}}^-(z)) \leq L|z_o - z| \quad \forall z_o \in U \cap T(\mathcal{O})$$

namely,

$$d_{\mathcal{H}}(T_{\mathcal{O}}^-(z_o), T_{\mathcal{O}}^-(T(x))) \leq L|z_o - T(x)| \quad \forall z_o \in U \cap T(\mathcal{O}).$$

Now, to find a contradiction, we assume the existence of $v \in \mathbb{R}^{n_x}$ such that $v \neq 0$ and $\frac{\partial T}{\partial x}(x)v = 0$. Note that, for all $h \in \mathbb{R}_{>0}$ sufficiently small, we have $x + hv \in \mathcal{O}$, and thus $T(x + hv) \in T(\mathcal{O}) \cap U$ by continuity of T . The latter implies that for all $h \in \mathbb{R}_{>0}$ sufficiently small,

$$d_{\mathcal{H}}(T_{\mathcal{O}}^-(T(x + hv)), T_{\mathcal{O}}^-(T(x))) \leq L|T(x + hv) - T(x)|.$$

Furthermore, since

$$\lim_{h \rightarrow 0} \frac{1}{h}|T(x + hv) - T(x)| = \frac{\partial T}{\partial x}(x)v = 0,$$

we conclude that

$$\lim_{h \rightarrow 0} \frac{1}{h} d_{\mathcal{H}}(T_{\mathcal{O}}^-(T(x + hv)), T_{\mathcal{O}}^-(T(x))) = 0.$$

By definition of the Hausdorff distance, we have

$$d_{\mathcal{H}}(T_{\mathcal{O}}^-(T(x + hv)), T_{\mathcal{O}}^-(T(x))) \geq d(y, T_{\mathcal{O}}^-(T(x))) \quad \forall y \in T_{\mathcal{O}}^-(T(x + hv)).$$

Since $x + hv \in T_{\mathcal{O}}^-(T(x + hv))$, we deduce that

$$\lim_{h \rightarrow 0} \frac{1}{h} d(x + hv, T_{\mathcal{O}}^-(T(x))) = 0. \quad (34)$$

Now, since $x \in T_{\mathcal{O}}^-(T(x))$ is isolated in $T_{\mathcal{O}}^-(T(x))$, we conclude that for h sufficiently small,

$$d(x + hv, T_{\mathcal{O}}^-(T(x))) = d(x + hv, x) = h|v|.$$

Hence, $\lim_{h \rightarrow 0} \frac{1}{h} d(x + hv, T_{\mathcal{O}}^-(T(x))) = |v| \neq 0$, which contradicts (34). \blacksquare

Remark 10 *Note that the “isolated-point” property is, in particular, verified if $T_{\mathcal{O}}^-$ takes finite values. On the other hand, we can show, in the same way, that Lipschitzness of T^- imposes from the Jacobian of T to be full-rank on the interior of \mathcal{X} provided that T^- takes finite values. But in order to show full-rank on the entire set \mathcal{X} , we propose to assume local Lipschitzness of an extension of T^- on an open neighborhood of \mathcal{X} ; hence, Proposition 2. Note that $T^-(x) = T_{\mathcal{O}}^-(x) \cap \mathcal{X}$ for all $x \in \mathcal{X}$, thus, T^- coincides with $T_{\mathcal{O}}^-$ on $T(\mathcal{X})$ if there are no extra preimages of points in $T(\mathcal{X})$ lying in \mathcal{O} . In that case, local Lipschitzness of $T_{\mathcal{O}}^-$ implies Lipschitzness of T^- .*

On the other hand, the next result shows that, under Assumption 2 and when $T(\mathcal{X})$ is connected, continuity of T^- on $T(\mathcal{X})$ implies that Assumption 1 holds.

Proposition 3 *Let $T : \mathcal{O} \rightarrow \mathbb{R}^{n_z}$ be a C^1 map defined on the open set $\mathcal{O} \subset \mathbb{R}^{n_x}$, let $\mathcal{X} \subset \mathcal{O}$ such that Assumption 2 holds and $T(\mathcal{X})$ is connected. Then, Assumption 1 holds provided that T^- defined in (12) is continuous on $T(\mathcal{X})$.*

PROOF. Assume T^- is continuous on $T(\mathcal{X})$. Let us pick $z_o \in T(\mathcal{X})$, and denote by $p \in \{1, 2, \dots\}$ the cardinal

of $T^-(z_o)$. Following the same first steps as in the proof of Theorem (exploiting Assumption 2, the constant rank theorem and the upper semicontinuity of T^-), we show the existence of a neighborhood W_o of z_o and p C^1 diffeomorphisms g_i defined on neighborhoods of each pre-image $x_i \in T^-(z_o)$ such that

$$T^-(z) \subseteq \cup_{i=1}^p g_i(z) \quad \forall z \in W_o \cap T(\mathcal{X}).$$

It follows that the cardinal of T^- is at most p in $W_o \cap T(\mathcal{X})$. Now, assume the existence of a sequence $\{z_k\}$ of points in $T(\mathcal{X})$ converging to z_o such that $\text{card}(T^-(z_k)) < p$ for all $k \in \{1, 2, \dots\}$. Let $\epsilon := \min\{|x_i - x_j| : \{x_i, x_j\} \subset T^-(z_o), i \neq j\}$ which is positive since $T^-(z_o)$ is finite. Then, by continuity of the g_i s, since $T^-(z_o) = \cup_{i=1}^p g_i(z_o)$ and $T^-(z_k)$ is a selection of less than p elements in $\cup_{i=1}^p g_i(z_k)$,

$$\liminf_{k \rightarrow \infty} \delta(T^-(z_o), T^-(z_k)) \geq \epsilon,$$

which contradicts lower semicontinuity of T^- . \blacksquare

5.2 Links Between Assumptions 3 and 4 Under Assumptions 1 and 2

Proposition 4 *Assume system (1) is analytic and let $T : \mathcal{O} \subset \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_z}$ be C^1 . Consider a compact set $\mathcal{X} \subset \mathcal{O}$ such that Assumptions 1, 2, and 3 hold. Let $y : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n_y}$ be an output to (1) generated by a solution x_1 that remains in \mathcal{X} after some $\bar{t} > 0$. Then, the following properties are true:*

- *There exist at most p solutions $\{x_i\}_{i=1}^p : [\bar{t}, \infty) \rightarrow \mathcal{X}$ to (1) generating y . Besides, such solutions cannot converge to each other.*
- *There are exactly p such solutions provided that one of the following holds:*
 - (1) *The set \mathcal{X} is forward invariant.*
 - (2) *There exists $\mathcal{U} \subset \mathbb{R}^{n_x}$ a neighborhood of \mathcal{X} such that*

$$\mathcal{I}_{\mathcal{U}}(x_1(t)) = \mathcal{I}_{\mathcal{X}}(x_1(t)) \quad \forall t \geq \bar{t}.$$

- (3) *The two following properties hold*

$$\begin{cases} \{f(x), -f(x)\} \cap T_{\mathcal{X}}(x) \neq \emptyset & \forall x \in T^-(x_1(\bar{t})), \\ \{-f(x), f(x)\} \cap T_{\partial\mathcal{X}}(x) = \emptyset & \forall x \in \partial\mathcal{X}. \end{cases}$$

PROOF. To prove the first item, given $\bar{t} > 0$, we consider two distinct solutions $x_1, x_2 : [\bar{t}, \infty) \rightarrow \mathcal{X}$ such that

$$h(x_1(t)) = h(x_2(t)) = y(t) \quad \forall t \in [\bar{t}, +\infty).$$

By uniqueness of solutions, $x_1(t) \neq x_2(t)$ for all $t \geq \bar{t}$. Also, by analyticity of (1), $x_1(t) \sim x_2(t)$ for all $t \geq \bar{t}$ and by Assumption 3, $T(x_1(t)) = T(x_2(t))$ for all $t \geq \bar{t}$. So for all $t \geq \bar{t}$, $x_1(t)$ and $x_2(t)$ are in $T^-(T(x_1(t)))$ and T^- has cardinal p by Assumption 1. It follows that there are

at most p such solutions. Now, let us show that any such solutions cannot converge to each other. Indeed, assume $x_1 - x_2$ converges to zero. Since \mathcal{X} is compact, there exists an increasing and diverging sequence $(t_n)_{n \in \mathbb{N}}$ and a limit point $x^* \in \mathcal{X}$ such that $\lim_{n \rightarrow +\infty} x_1(t_n) = x^*$. Then, also, $\lim_{n \rightarrow +\infty} x_2(t_n) = x^*$. Under Assumption 2, there exists a neighborhood of x^* where T is locally injective, which contradicts $x_2(t_n) \in T^-(T(x_1(t_n)))$ for sufficiently large n .

Now, to prove the second item, we assume that y is generated, on some interval $[\bar{t}, +\infty)$, by a solution $x_1 : [\bar{t}, +\infty) \rightarrow \mathcal{X}$. Under Assumption 1, we let

$$\{x_{1,o}, \dots, x_{p,o}\} := T^-(T(x_1(\bar{t}))) \text{ with } x_{1,o} = x_1(\bar{t}).$$

By Assumption 3, we conclude that

$$x_{i,o} \sim x_{j,o} \quad \forall i, j \in \{1, \dots, p\}.$$

Now, for each $i \in \{1, 2, \dots, p\}$, we let the (unique) maximal solution $x_i : I_i \rightarrow \mathbb{R}^{n_x}$ with $\bar{t} \in I_i$ such that $x_i(\bar{t}) =: x_{i,o}$. By definition of \sim , for each $i \in \{1, 2, \dots, p\}$, we have

$$h(x_i(t)) = h(x_1(t)) \quad \forall t \in I_i \cap I_1 \cap (-\infty, \bar{t}].$$

By analyticity, actually, we conclude that

$$h(x_i(t)) = h(x_1(t)) \quad \forall t \in I_i \cap I_1.$$

Denote $I := \cap_{i \in \{1, \dots, p\}} I_i$, which is an open set that contains \bar{t} . We; thus, have

$$x_i(t) \sim x_j(t) \quad \forall t \in I, \quad \forall i, j \in \{1, \dots, p\}.$$

To complete the proof, we need to show that

$$x_i(t) \in \mathcal{X} \quad \forall t \geq \bar{t}, \quad \forall i \in \{1, \dots, p\}. \quad (35)$$

Under item 1), when \mathcal{X} is forward invariant, (35) follows.

Furthermore, we show that (35) holds under item 2) using contradiction. That is, let $t_{\mathcal{X}} \geq \bar{t}$ be the maximal time, such that, all the x_i s remain in \mathcal{X} on $[\bar{t}, t_{\mathcal{X}}]$; namely,

$$t_{\mathcal{X}} := \max\{t \in I : x_i(s) \in \mathcal{X}, \forall s \in [\bar{t}, t], \forall i \in \{1, \dots, p\}\}.$$

Furthermore, let $k \in \{2, 3, \dots, p\}$ be such that the solution x_k leaves the set \mathcal{X} at $t_{\mathcal{X}}$. Hence, since x_k is continuous, we conclude the existence of $t_m > t_{\mathcal{X}}$ such that $x_k(t_m) \in \mathcal{U} \setminus \mathcal{X}$. By analyticity, we conclude that

$$h(x_k(t)) = y(t) \quad \forall t \in [\bar{t}, t_m].$$

The latter plus the fact that $x_k(\bar{t}) \in \mathcal{I}_{\mathcal{U}}(x_1(\bar{t}))$ imply that $x_k(t_m) \in \mathcal{I}_{\mathcal{U}}(x_1(t_m))$. However, using item 2), we know that $\mathcal{I}_{\mathcal{U}}(x_1(t_m)) = \mathcal{I}_{\mathcal{X}}(x_1(t_m)) \subset \mathcal{X}$, which yields a contradiction.

Suppose, now, that item 3) holds and let $t_{\mathcal{X}}$ be finite. By compactness of \mathcal{X} , there exists $\varepsilon > 0$ such that, for each $i, j \in \{1, \dots, p\}$ with $i \neq j$, we have

$$|x_i(t) - x_j(t)| \geq \varepsilon \quad \forall t \in [\bar{t}, t_{\mathcal{X}}], \quad (36)$$

and by Assumption 1, we conclude that

$$T^-(T(x_1(t))) = \{x_1(t), \dots, x_p(t)\} \in \mathcal{X} \quad \forall t \in [\bar{t}, t_{\mathcal{X}}].$$

Now, by definition of $t_{\mathcal{X}}$, we conclude the existence of $k \in \{2, 3, \dots, p\}$ such that $x_k(t_{\mathcal{X}}) \in \partial\mathcal{X}$ and x_k leaves \mathcal{X} right after $t_{\mathcal{X}}$. Hence, using Lemma 4 we conclude that $f(x_k(t_{\mathcal{X}})) \in T_{\mathbb{R}^{n_x} \setminus \mathcal{X}}(x_k(t_{\mathcal{X}}))$. Now, since $f(x_k(t_{\mathcal{X}})) \notin T_{\partial\mathcal{X}}(x_k(t_{\mathcal{X}}))$, we use Lemma 6 to conclude that $f(x_k(t_{\mathcal{X}})) \in \mathbb{R}^{n_x} \setminus T_{\mathcal{X}}(x_k(t_{\mathcal{X}}))$. Hence, using Lemma 5, we conclude that, for some $\delta > 0$ sufficiently small,

$$x_k(t) \notin \mathcal{X} \quad \forall t \in (t_{\mathcal{X}}, t_{\mathcal{X}} + \delta] \subset I. \quad (37)$$

Note that this k may not be unique; however, for simplicity we let it be unique in this proof, the exact same arguments apply in the general case. Now, we distinguish between two scenarios:

- When $t_{\mathcal{X}} > \bar{t}$, we conclude that $x_k([\bar{t}, t_{\mathcal{X}}]) \subset \mathcal{X}$. Hence, using Lemma 4, we conclude that

$$-f(x_k(t_{\mathcal{X}})) \in T_{\mathcal{X}}(x_k(t_{\mathcal{X}})).$$

However, since $-f(x_k(t_{\mathcal{X}})) \notin T_{\partial\mathcal{X}}(x_k(t_{\mathcal{X}}))$, we use Lemma 6 to conclude that $-f(x_k(t_{\mathcal{X}})) \in D_{\mathcal{X}}(x_k(t_{\mathcal{X}}))$. Hence, using Lemma 5, we conclude that for $\delta > 0$ sufficiently small, we have

$$x_k(t) \in \text{int}(\mathcal{X}) \quad \forall t \in [t_{\mathcal{X}} - \delta, t_{\mathcal{X}}) \subset I. \quad (38)$$

- When $t_{\mathcal{X}} = \bar{t}$, we use (37), to conclude that

$$f(x_k(\bar{t})) \in T_{\mathbb{R}^{n_x} \setminus \mathcal{X}}(x_k(\bar{t})).$$

Furthermore, since $f(x_k(\bar{t})) \notin T_{\partial\mathcal{X}}(x_k(\bar{t}))$; we conclude, using Lemma 6 that $f(x_k(\bar{t})) \notin T_{\mathcal{X}}(x_k(\bar{t}))$. As a result, according to item 3), we have

$$-f(x_k(\bar{t})) \in T_{\mathcal{X}}(x_k(\bar{t})) \setminus T_{\partial\mathcal{X}}(x_k(\bar{t})) = D_{\mathcal{X}}(x_k(\bar{t})).$$

Hence, using Lemma 5, (38) follows.

Now, using continuity of T^- and Assumption 1, we conclude that, for each $t \in (t_{\mathcal{X}}, t_{\mathcal{X}} + \delta]$, there exists $x'_{k,t} \in \mathcal{X}$ such that

$$T^-(T(x_1(t))) = \{x_1(t), \dots, x_{k-1}(t), x'_{k,t}, x_{k+1}(t), \dots, x_p(t)\},$$

and

$$\lim_{t \rightarrow t_{\mathcal{X}}^+} x'_{k,t} = x_k(t_{\mathcal{X}}).$$

Furthermore, for $\delta > 0$ even smaller, we conclude that all the elements of $T^-(T(x_1(t)))$, for all $t \in [t_{\mathcal{X}} - \delta, t_{\mathcal{X}} + \delta]$, are at a distance larger than $\varepsilon/2$ from each other. By continuity of solutions with respect to initial data, we conclude the existence $t_m \in (t_{\mathcal{X}}, t_{\mathcal{X}} + \delta]$ sufficiently close to $t_{\mathcal{X}}$ such that the solution x'_k starting from x'_{k,m,t_m} at time t_m verifies

$$\|x_k(t) - x'_k(t)\| < \varepsilon/2 \quad \forall [t_{\mathcal{X}} - \delta, t_{\mathcal{X}} + \delta]$$

and $x'_k(t_{\mathcal{X}} - \delta) \in \text{int}(\mathcal{X})$. But by Assumption 3, $x'_{k,t_m} \sim x_1(t_m)$ and therefore, by analyticity, we have

$$x_k(t_{\mathcal{X}} - \delta) \sim x'_k(t_{\mathcal{X}} - \delta).$$

It follows from (36) that $x_k(t_{\mathcal{X}} - \delta) = x'_k(t_{\mathcal{X}} - \delta)$ and therefore, $x_k = x'_k$ by uniqueness of solutions. This is impossible since $x_k(t_m) \notin \mathcal{X}$ while $x'_k(t_m) = x'_{k,t_m} \in \mathcal{X}$. ■

The previous proposition shows that, when system (1) is analytic and under some assumptions on the way solutions might exit or enter \mathcal{X} , Assumptions 1, 2, and 3 imply Assumption 4. In the following result, we investigate a converse statement. That is, for T solution to (5) satisfying Assumptions 1 and 2, we consider an output y satisfying the following assumption, which is slightly stronger than Assumption 4.

Assumption 7 *Given $y : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n_y}$, there exist $\bar{t} > 0$ and at least $p \in \mathbb{N}_{>0}$ solutions $\{x_i\}_{i=1}^p : [\bar{t}, +\infty) \rightarrow \mathcal{X}$ to (1) such that for all $i \in \{1, \dots, p\}$,*

- For each $t \geq \bar{t}$, $h(x_i(t)) = y(t)$.
- For each $i, j \in \{1, \dots, p\}$, we have

$$\liminf_{t \rightarrow \infty} |x_i(t) - x_{j \neq i}(t)| > 0. \quad (39)$$

As a result, we show that $T^-(T(x)) \subset \mathcal{I}_{\mathcal{X}}(x)$, for all x within the ω -limit set of the solutions in Assumption 7 generating y .

Proposition 5 *Consider a C^1 map $T : \mathcal{O} \subset \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_z}$ verifying (5) and Assumptions 1 and 2 on a compact set $\mathcal{X} \subset \mathcal{O}$. Let $y : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n_y}$ be an output to (1) such that Assumption 7 holds. Let x_1 be one of the solutions introduced in Assumption 7 that generates y . Then,*

$$T^-(T(x)) \subseteq \mathcal{I}_{\mathcal{X}}(x) \quad \forall x \in \omega(x_1(\bar{t})).$$

PROOF. Let the solutions $\{x_i\}_{i=1}^p : [\bar{t} \rightarrow \infty) \rightarrow \mathcal{X}$ to (1) generating y . Then, $x_{\text{ext}} := (x_1, x_2, \dots, x_p)$ is solution to a duplicated system with dynamics $f_{\text{ext}} = (f, f, \dots, f)$ and verifies

$$h(x_i(t)) = h(x_j(t)) \quad \forall t \geq \bar{t} \forall (i, j) \in \{1, \dots, p\}^2. \quad (40)$$

Let $\Omega := \omega(x_{\text{ext}}(\bar{t}))$, its ω -limit set. Since \mathcal{X} is compact and by uniqueness of solutions, Ω is compact, contained

in \mathcal{X} , and invariant by f_{ext} . Since h is continuous, we deduce from (40) that for all $(x_1^*, \dots, x_p^*) \in \Omega$,

$$h(x_i^*) = h(x_j^*) \quad \forall (i, j) \in \{1, \dots, p\}^2. \quad (41)$$

Now pick $x_{\text{ext}}^* := (x_1^*, \dots, x_p^*) \in \Omega$. By backward-invariance of Ω , the backward solution (ϕ_1, \dots, ϕ_p) by f_{ext} initialized at x_{ext}^* is defined on $\mathbb{R}_{\leq 0}$ and remains in Ω . It follows from (41) (which holds everywhere on Ω) that $h(\phi_i(t)) = h(\phi_j(t))$ for all $t \leq 0$ and for all $(i, j) \in \{1, \dots, p\}^2$.

Since for each $i \in \{1, \dots, p\}$, ϕ_i is the backward solution to (1) initialized at x_i^* by definition of f_{ext} , we deduce that $x_1^* \sim x_2^* \sim \dots \sim x_p^*$, and thus, since each $x_i^* \in \mathcal{X}$, $x_i^* \in \mathcal{I}_{\mathcal{X}}(x_1^*)$. Moreover, by uniqueness of the solutions to (5) on backward-invariant compact sets [10, Theorem 2], we know that, for each $i \in \{1, 2, \dots, p\}$, $T(x_i^*) = \int_{-\infty}^0 e^{-As} B h(\phi_i(s)) ds$, which implies that $T(x_1^*) = \dots = T(x_p^*)$. The latter allows us to conclude that

$$\{x_1^*, \dots, x_p^*\} \subseteq T^{-}(T(x_1^*)) \cap \mathcal{I}_{\mathcal{X}}(x_1^*). \quad (42)$$

Now, under the second item in Assumption 7, the x_i^* are distinct, which gives necessarily

$$\{x_1^*, \dots, x_p^*\} = T^{-}(T(x_1^*)) \subseteq \mathcal{I}_{\mathcal{X}}(y_1) \quad (43)$$

When, instead of (39), we assume that the solutions $\{x_i\}_{i=1}^p$ in Assumption 7 do not converge to each other (as in Assumption 4), we cannot conclude the equality in (43) based on (42). Indeed, it is possible to have, for example, the solutions x_1 and x_2 converge to a same ω -limit point $x_1^* = x_2^*$ along a same sequence of times, and they still do not converge to each other; see Example 6.

Example 6 Consider the two-dimensional system

$$\dot{x} = f(x) := \begin{bmatrix} (1 - |x|^2)x_1 + x_2 \\ (1 - |x|^2)x_2 - x_1 \end{bmatrix} \quad x \in \mathcal{X} := \{|x| \leq 2\}.$$

Note that this system admits a unique nontrivial limit cycle describing the unit circle $\{|x| = 1\}$. The latter attracts all the solutions except the one starting from the origin $\{x = 0\}$. Next, we propose to re-scale the system using the smooth function $\phi : \mathbb{R}_{\geq 0} \rightarrow [0, 1]$ given by

$$\phi(r) := \begin{cases} 0 & \text{if } r \geq 1 \\ (1 - r)^2 & \text{if } r < 1. \end{cases}$$

As a result, we introduce the system

$$\dot{x} = \phi(|x|^2)f(x) \quad x \in \mathcal{X}, \quad (44)$$

for which, every point within the set $\{|x| = 0\} \cup \{|x| \geq 1\}$ is a static equilibrium point, and the solutions starting

from the set $\{0 < |x| < 1\}$ converge to the set $\{|x| \geq 1\}$ by spiraling, i.e., without converging to any specific point in $\{|x| \geq 1\}$. Hence, by letting x_1 be a solution to (44) starting from $\{|x| = 1\}$ (which remains there, i.e., $x_1(t) = x_1(0)$ for all $t \geq 0$) and x_2 be a solution to (44) starting from $\{0 < |x| < 1\}$, we conclude that they do not converge to each other (since x_2 keeps spiraling), but still they share a common ω -limit point which is $x_1(0)$.

5.3 Link Between Assumption 2 and the Rank of a Differential Observability Map

In this section, we assume f and h to be smooth. Given a positive integer m , consider the map H_m defined by

$$H_m(x) := \left(h(x), L_f h(x), \dots, L_f^{m-1} h(x) \right), \quad (45)$$

containing the output map h and its $m - 1$ Lie derivatives. The injectivity of H_m characterizes the so-called *differential observability* of order m , meaning that the state is uniquely determined from the knowledge of the output and its first $m - 1$ time derivatives. In this paper where we handle non-observable systems, we will not make such an assumption. However, the next result shows that the rank of the KKL map T may be related to the rank of the map H_m , when the eigenvalues of A are picked sufficiently fast. This result allows to characterize the set of regular points where the previous convergence results hold by simply checking the rank of H_m .

Proposition 6 Consider a controllable pair $(A_o, B_o) \in \mathbb{R}^{m \times m} \times \mathbb{R}^m$ with A_o Hurwitz and $m \in \mathbb{N}$ such that the map H_m defined in (45) is full-rank on the compact set \mathcal{X} . Define $A = I_{n_y} \otimes A_o, B = I_{n_y} \otimes B_o$. There exists $k^* > 0$ such that for all $k > k^*$, there exists a C^1 map $T_k : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_z}$ which is full-rank on \mathcal{X} and verifies (5) with the pair (kA, B) .

PROOF. First, since in (5), the map f is only used on \mathcal{X} , we can consider an open bounded set \mathcal{O} and a modified vector field \check{f} such that $f = \check{f}$ on \mathcal{X} and \mathcal{O} is backward invariant for $\dot{x} = \check{f}(x)$. Then, as in Theorem 1, we show that the map defined by

$$T_k(x) := \int_{-\infty}^0 e^{-kAs} B h(\check{X}(x, s)) ds,$$

with $\check{X}(x, s)$ representing the flow operator for $\dot{x} = \check{f}(x)$, is C^1 on \mathcal{O} for k sufficiently large and verifies (5) for the pair (kA, B) . Given the structure of A and B , $T_k(x) = (T_{k,1}(x), \dots, T_{k,n_y}(x))$, where for all $i \in \{1, \dots, n_y\}$,

$$T_{k,i}(x) = \int_{-\infty}^0 e^{-kA_o s} B_o h_i(\check{X}(x, s)) ds.$$

Then, after m integration by parts, we see that

$$T_{k,i}(x) = A_o^{-m} C_o K \left(H_{m,i}(x) + \frac{1}{k^m} K^{-1} C_o^{-1} R_{k,i}(x) \right)$$

where C_o is a square controllability matrix associated to (A_o, B_o) (thus invertible), $K = \text{diag}(\frac{1}{k}, \dots, \frac{1}{k^m})$, $H_{m,i}$ is the observability map of order m associated to h_i and $R_{k,i} : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^m$ are the remainders defined as

$$R_{k,i}(x) := \int_{-\infty}^0 e^{-kA_o s} B_o L_f^m h_i(\check{X}(x, s)) ds.$$

Defining $R_k(x) = (R_{k,1}(x), \dots, R_{k,n_y}(x))$, and using similar arguments as in proving T_k is C^1 , for k sufficiently large, R_k is C^1 and

$$\frac{\partial T_k}{\partial x}(x) = M_1(k) \left(\frac{\partial H_m}{\partial x}(x) + M_2(k) \frac{\partial R_k}{\partial x}(x) \right),$$

where $M_1(k) := [I_{n_y} \otimes (A_o^{-m} C_o K)] M$, $M_2(k) := \frac{1}{k^m} M^{-1} [I_{n_y} \otimes (K^{-1} C_o^{-1})]$, and M is an invertible square matrix reordering the lines. Since H_m is full-rank on \mathcal{X} compact, there exists $\rho_H > 0$ such that for all $x \in \mathcal{X}$,

$$\frac{\partial H_m}{\partial x}(x)^\top \frac{\partial H_m}{\partial x}(x) \geq \rho_H I_{n_z}.$$

By invertibility of $A_o^{-m} C_o$ and M , and given the structure of k , there also exists $\rho_c > 0$ such that $M_1(k)^\top M_1(k) \geq \frac{\rho_c}{k^m} I_{n_z}$. It follows that for all $v \in \mathbb{R}^n$ and all $x \in \mathcal{X}$, $|\frac{\partial H_m}{\partial x}(x)v| \geq \rho_H |v|$, and

$$\left| \frac{\partial T_k}{\partial x}(x)v \right| \geq \frac{\rho_c}{k^m} \left(\rho_H |v| - \left| M_2(k) \frac{\partial R_k}{\partial x}(x)v \right| \right)$$

Upper-bounding $M_2(k)$ by $k^m c_c$ for some $c_c > 0$ independent from k , we thus get

$$\left| \frac{\partial T_k}{\partial x}(x)v \right| \geq \frac{\rho_c}{k^m} \left(\rho_H |v| - c_c \left| \frac{\partial R_k}{\partial x}(x)v \right| \right)$$

Let us now upper-bound the Jacobian of R_k ,

$$\left| \frac{\partial R_k}{\partial x}(x) \right| \leq \int_{-\infty}^0 |e^{-kAs}| \left| B \frac{\partial L_f^m h}{\partial x}(\check{X}(x, s)) \right| \left| \frac{\partial \check{X}}{\partial x}(x, s) \right|.$$

A being Hurwitz, there exists a positive definite matrix P and a positive scalar a such that $A^\top P + PA \leq -2aP$. It follows that for all s in $(-\infty, 0]$ $|e^{-kAs}| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} e^{kas}$, where λ_{\min} and λ_{\max} denote the minimal and maximal eigenvalues respectively. The bounded set \mathcal{O} being backward invariant under the modified dynamics, we can consider $c_0 = \max_{x \in \text{cl}(\mathcal{O})} \left| B \frac{\partial L_f^m h}{\partial x}(x) \right|$. Finally, we

show that the Jacobian of the flow operator $\frac{\partial \check{X}}{\partial x}(x, s)$ grows at most exponentially in time. To do so, we use the fact that $\psi(t) = \frac{\partial \check{X}}{\partial x}(x, t)$ satisfies the ODE : $\dot{\psi}(t) = \frac{\partial f}{\partial x}(\check{X}(x, t))\psi(t)$. Denoting $c_1 := \max_{x \in \text{cl}(\mathcal{O})} \left| \frac{\partial f}{\partial x}(x) \right|$ and bounding the previous differential inequality, we obtain that $\left| \frac{\partial \check{X}}{\partial x}(x, s) \right| \leq e^{-c_1 s}$ for all s in $(-\infty, 0]$. Therefore, for $k > \frac{c_1}{a}$, there exists a positive constant c_2 such that $\left| \frac{\partial R_k}{\partial x}(x) \right| \leq \frac{c_2}{ka - c_1}$. It follows that, for all $(x, v) \in \mathcal{X} \times \mathbb{R}^n$,

$$\left| \frac{\partial T_k}{\partial x}(x)v \right| \geq \frac{\rho_c}{k^{n_z}} \left(\rho_H - c_c \frac{c_2}{ka - c_1} \right) |v|,$$

and T_k is indeed full-rank on \mathcal{X} for k sufficiently large. ■

Remark 11 Note that if we add the backward invariance condition used in Theorem 1 and we ask from the PDE in (5) to hold on \mathcal{O} , the full-rank map T_k given by Proposition 6 is the unique solution to (5) (see Remark 2). However, we cannot directly prove that T_k characterizes the backward distinguishable points, i.e., verifies (9). According to Theorem 1 when $m \leq n_o$, (9) holds if (kA_o, B_o) is outside a zero-measure set.

Example 7 For the system in Example 2, we can see that $H_1(x) := (x_1^2 - x_2^2, x_1 x_2)$ is full-rank for all $x \neq (0, 0)$. On the other hand, H_m is never full-rank at $(0, 0)$ for any $m \in \mathbb{N}$. This means that, at least for $n_o \geq 1$ and $(A_o, B_o) \in \mathbb{R}^{n_o \times n_o} \times \mathbb{R}^{n_o}$, the solution to (5) with pair $(kI_{n_y} \otimes A_o, I_{n_y} \otimes B_o)$ should be full-rank on any compact set $\mathcal{X} \subset \mathbb{R}^2 \setminus \{(0, 0)\}$, provided that k is sufficiently large. Nothing can be said for compact sets that include $(0, 0)$.

6 Conclusion

This paper proposed a set-valued KKL observer for nonlinear systems disobeying the backward-distinguishability property. Provided that the transformation T is full rank on \mathcal{X} and its preimage has a constant cardinality, it is shown that the set-valued preimage is locally Lipschitz and admits a locally Lipschitz extension T^{inv} . When, additionally, a given output y is generated by a number of distinct solutions (that equals the cardinality of the preimage) not converging to each other, the designed observer is shown to asymptotically reconstructs each of such solutions. Such an approach applies to any nonlinear system, with no particular normal form, unlike high-gain or sliding-mode based methods in [22,23,21]. In the future, it would be interesting to explore other analysis tools to relax some of the considered assumptions, for example, without imposing the continuity/Lipschitzness of T^{inv} and study the convergence of (maybe not all) C^1 selections.

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A Useful definitions : continuity in set-valued and p -valued Maps

For regularity and convergence of set-valued maps, the space of subsets of \mathbb{R}^{n_x} is endowed with the Hausdorff distance as defined next.

Definition 5 (Hausdorff distance) Given two subsets \mathcal{X}_a and \mathcal{X}_b of \mathbb{R}^{n_x} , the Hausdorff distance is

$$d_{\mathcal{H}}(\mathcal{X}_a, \mathcal{X}_b) := \max \{ \delta(\mathcal{X}_a, \mathcal{X}_b), \delta(\mathcal{X}_b, \mathcal{X}_a) \}$$

where $\delta(\mathcal{X}_a, \mathcal{X}_b) := \sup_{x_a \in \mathcal{X}_a} d(x_a, \mathcal{X}_b) = \sup_{x_a \in \mathcal{X}_a} \inf_{x_b \in \mathcal{X}_b} d(x_a, x_b)$.

Remark 12 For closed sets, $\delta(\mathcal{X}_a, \mathcal{X}_b) = 0$ is equivalent to $\mathcal{X}_a \subseteq \mathcal{X}_b$, while $d_{\mathcal{H}}(\mathcal{X}_a, \mathcal{X}_b) = 0$ to $\mathcal{X}_a = \mathcal{X}_b$.

The continuity of a set-valued map $F : K \rightrightarrows \mathbb{R}^{n_x}$ then contains the following two properties ([6, Chapter 1]).

Definition 6 (Upper semicontinuity) A set-valued map $F : K \rightrightarrows \mathbb{R}^{n_x}$ is said to be upper semicontinuous at $z^* \in K$ if for any open neighbourhood V containing $F(z^*)$ there exists a neighbourhood W of z^* such that for all $z \in W$, $F(z) \subseteq V$. This is equivalent to

$$\lim_{z \rightarrow z^*} \delta(F(z), F(z^*)) = 0. \quad (\text{A.1})$$

Definition 7 (Lower semicontinuity) A set-valued map $F : K \rightrightarrows \mathbb{R}^{n_x}$ is said to be lower semicontinuous at $z^* \in K$ if for any $x \in F(z^*)$ and any neighborhood V of x , there exists a neighborhood W of z^* such that for all $z \in W$, $V \cap F(z) \neq \emptyset$. This is equivalent to

$$\lim_{z \rightarrow z^*} \delta(F(z^*), F(z)) = 0. \quad (\text{A.2})$$

In standard topology, we may then define stronger continuity properties such as Lipschitz continuity.

Definition 8 ((Local) Lipschitzness) The set-valued map $F : K \rightrightarrows \mathbb{R}^{n_x}$, with $K \subset \mathbb{R}^m$, is said to be locally Lipschitz if, for each $x \in K$, there exists a neighborhood U of x and a scalar $k > 0$ such that, for all $(x_a, x_b) \in (U \cap K) \times (U \cap K)$,

$$d_{\mathcal{H}}(F(x_a), F(x_b)) \leq k|x_a - x_b|. \quad (\text{A.3})$$

Besides, F is said to be Lipschitz if there exists $k > 0$ such that (A.3) holds for all $(x_a, x_b) \in K \times K$.

We also sometimes use the following topology of unordered p -tuples (counting multiplicity).

Definition 9 (Distance on $\mathcal{A}_p(\mathbb{R}^{n_x})$) For $S_a \in \mathcal{A}_p(\mathbb{R}^{n_x})$ and $S_b \in \mathcal{A}_p(\mathbb{R}^{n_x})$, we let the distance

$$\mathcal{G}(S_a, S_b) := \min_{\sigma \in \mathcal{P}_p} \max_{i \in \{1, \dots, p\}} |s_{a,i} - s_{b,\sigma(i)}|$$

where \mathcal{P}_p is the set of permutations of p elements, and S_a, S_b are in $\mathcal{A}_p(\mathbb{R}^{n_x})$ with elements $\{s_{a,i}\}_{i=1}^p$ and $\{s_{b,i}\}_{i=1}^p$, respectively.

The notions of continuity and (local) Lipschitzness of p -valued maps $\tilde{F} : \mathcal{Z} \subset \mathbb{R}^{n_z} \rightarrow \mathcal{A}_p(\mathbb{R}^{n_x})$ naturally follows from the usual definitions, while using the distance \mathcal{G} on $\mathcal{A}_p(\mathbb{R}^{n_x})$ instead of the Hausdorff distance.

Remark 13 In general, the distances $d_{\mathcal{H}}$ and \mathcal{G} are not equal nor equivalent. For instance, the Hausdorff distance between the unordered 3-tuples $\{1, 1, 2\}$ and $\{1, 2, 2\}$, seen as sets, is zero, while the \mathcal{G} distance between them is not.

B Some useful lemmas

Lemma 2 (Constant-rank theorem) Consider an open subset $\mathcal{O} \subset \mathbb{R}^{n_x}$, $x_o \in \mathcal{O}$ and a C^1 map $T : \mathcal{O} \rightarrow \mathbb{R}^{n_z}$ such that $\text{rank}(\frac{\partial T}{\partial x}(x)) = n_x$ for all $x \in \mathcal{O}$. Then, there exist a neighborhood of x_o denoted $U(x_o)$, a neighborhood of $T(x_o)$ denoted $U(T(x_o))$, and C^1 diffeomorphisms $\phi : U(x_o) \rightarrow \phi(U(x_o))$ and $\psi : U(T(x_o)) \rightarrow \psi(U(T(x_o)))$ such that

$$\psi \circ T(x) = (\phi(x), 0, 0, \dots, 0) \quad \forall x \in U(x_o).$$

Next, we recall the following extension theorem for Lipschitz maps [14, Theorem 1.1] and [19, Theorem 1.7].

Lemma 3 (Lipschitz extension theorem) Let $\mathcal{Z} \subset \mathbb{R}^{n_z}$ and $\tilde{F} : \mathcal{Z} \rightarrow \mathcal{A}_p(\mathbb{R}^{n_x})$ be Lipschitz. Then, there exists a Lipschitz extension $\tilde{F}_a : \mathbb{R}^{n_z} \rightarrow \mathcal{A}_p(\mathbb{R}^{n_x})$ of \tilde{F} .

Next, we recall from [5, Proposition 3.4.1] and [5, Theorem 4.3.4] some useful results on forward invariance of a set \mathcal{X} .

Lemma 4 Consider system (1) with $f : \mathcal{O} \rightrightarrows \mathbb{R}^n$ continuous. Consider a closed set $\mathcal{X} \subset \mathcal{O}$ and a solution x of $\dot{x} \in f(x)$ satisfying $\forall t > 0, \exists t \in (0, \bar{t}] : x(t) \in \mathcal{X}$. Then, $f(x(0)) \in T_{\mathcal{X}}(x(0))$.

Lemma 5 Consider system (1) with $f : \mathcal{O} \rightrightarrows \mathbb{R}^n$ continuous. Let $\mathcal{X} \subset \mathcal{O}$ be closed with nonempty interior and $x_o \in \partial \mathcal{X}$. If $f(x_o) \subset D_{\mathcal{X}}(x_o)$, then, for each solution x to $\dot{x} \in f(x)$ starting from x_o , there exists $\delta > 0$ such that $x((0, \delta]) \subset \text{int}(\mathcal{X})$.

The following result can be found in [5, Lemma 4.3.2 and Theorem 4.3.3].

Lemma 6 Given a closed set $\mathcal{X} \subset \mathbb{R}^{n_x}$, for each $x \in \partial \mathcal{X}$, we have $D_{\mathcal{X}}(x) = \mathbb{R}^{n_x} \setminus T_{\mathbb{R}^{n_x} \setminus \mathcal{X}}(x)$ and $T_{\partial \mathcal{X}}(x) = T_{\mathcal{X}}(x) \cap T_{\mathbb{R}^{n_x} \setminus \mathcal{X}}(x)$.

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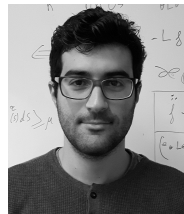
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