Observer Design for Hybrid Systems with Linear Maps and Known Jump Times

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Abstract This chapter unifies and develops recent developments in observer design for hybrid systems with linear dynamics and output maps, whose jump times are known. We define and analyze the (pre-)asymptotic detectability and uniform complete observability of this class of systems, then present two different routes for observer design. The first one relies on a synchronized Kalman-like observer that gathers observability from both flows and jumps. The second one consists in decomposing the state into parts with different observability properties and coupling observers estimating each of these parts, possibly exploiting an extra fictitious measurement coming from the combination of flows and jumps. These observers are based on a Linear Matrix Inequality (LMI) or the Kazantzis-Kravaris/Luenberger (KKL) paradigm. A comparison of these methods is presented in a table at the end.

1 Introduction

Consider a hybrid system with linear maps

$$\mathcal{H} \begin{cases} \dot{x} = Fx + u_c & (x, u_c) \in C \qquad y_c = H_c x \\ x^+ = Jx + u_d & (x, u_d) \in D \qquad y_d = H_d x \end{cases} \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state, C and D are the flow and jump sets, $y_c \in \mathbb{R}^{n_{y,c}}$ and $y_d \in \mathbb{R}^{n_{y,d}}$ are the outputs known during the flow intervals and at the jump times respectively, $u_c \in \mathbb{R}^{n_x}$ and $u_d \in \mathbb{R}^{n_x}$ are known exogenous input signals, as well

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as the dynamics matrices $F, J \in \mathbb{R}^{n_x \times n_x}$ and the output matrices $H_c \in \mathbb{R}^{n_{y,c} \times n_x}$, $H_d \in \mathbb{R}^{n_{y,d} \times n_x}$ which are all known and possibly time-varying. Models of the form (1) include not only hybrid systems with linear maps described in the setting of [23], but also *switched* and/or *impulsive* systems with linear maps where the active mode is seen as an exogenous signal making (F, J, H_c, H_d) time-varying (see [1, 17, 25] among many other ones), and continuous-time systems with *sporadic* or *multi-rate* sampled outputs where the "jumps" correspond to sampling events, $J = \mathrm{Id}$, $u_d = 0$, $y_c = 0$, and y_d the outputs available at the sampling event [21, 12, 24]. See [23, 28] for some examples of those classes of systems set in the framework of (1).

The goal of this chapter is to present in a unified and more complete way recent advances concerning the design of an asymptotic observer for system (1), assuming that its jump times are known or detected. In practice, we may be interested in estimating only certain trajectories of "physical interest", initialized in some set $\mathcal{X}_0 \subset \mathbb{R}^{n_x}$ and with exogenous terms $(F, J, H_c, H_d, u_c, u_d)$ in some set $\mathcal{U}$ of interest. We then denote $\mathcal{S}_{\mathcal{H}}(\mathcal{X}_0, \mathcal{U})$ as the set of those maximal solutions of interest. Because we look for an *asymptotic* observer, we assume maximal solutions are complete as stated next.

Assumption 1 Given $\mathcal{X}_0$ and $\mathcal{U}$, each maximal solution in $\mathcal{S}_{\mathcal{H}}(\mathcal{X}_0, \mathcal{U})$ is complete.

Since the jump times of system $\mathcal{H}$ in (1) are known, it is natural to strive for a *synchronized* asymptotic observer of the form

$$\hat{\mathcal{H}} \begin{cases} \dot{\hat{z}} = \Psi_c(\hat{z}, y_c, u_c) & \text{when } \mathcal{H} \text{ flows} \\ \hat{z}^+ = \Psi_d(\hat{z}, y_d, u_d) & \text{when } \mathcal{H} \text{ jumps} \\ \hat{x} = \Upsilon(\hat{z}, y_c, y_d, u_c, u_d) \end{cases} \tag{2}$$

where $\hat{z} \in \mathbb{R}^{n_z}$ is the observer state (with $n_z \geq n_x$ in general), $\Upsilon : \mathbb{R}^{n_z} \times \mathbb{R}^{n_{y,c}} \times \mathbb{R}^{n_{y,d}} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ is the observer output map, $\Psi_c : \mathbb{R}^{n_z} \times \mathbb{R}^{n_{y,c}} \times \mathbb{R}^{n_x} \to \mathbb{R}^{n_z}$ and $\Psi_d : \mathbb{R}^{n_z} \times \mathbb{R}^{n_{y,d}} \times \mathbb{R}^{n_x} \to \mathbb{R}^{n_z}$ are respectively the observer flow and jump maps designed such that each maximal solution $(x, \hat{z})$ to the cascade $\mathcal{H} - \hat{\mathcal{H}}$ initialized in $\mathcal{X}_0 \times \mathbb{R}^{n_z}$ with inputs in $\mathcal{U}$ is complete and verifies

$$\lim_{\substack{t+j \to +\infty \\ (t,j) \in \mathrm{dom}\, x}} |x(t, j) - \hat{x}(t, j)| = 0. \tag{3}$$

The knowledge of the jump times is not only used to trigger the observer jumps at the same time as those of the system, but it can also be used to design the observer maps Ψ_c and Ψ_d. The way this information is exploited varies depending on whether these maps rely on:

- Gains that are computed *offline* (for example via matrix inequalities), according to all possible lengths of flow intervals in between jumps, i.e., they depend on each individual flow length, not the particular sequence of them. This is the path taken by [1, 18, 12, 24, 22, 6, 29];
- Or, gains that are computed *online* along the time domain of each solution of interest, i.e., they depend on the *sequence* of flow lengths in each particular solution. This is the path taken by all Kalman-like approaches in [17, 26, 28].

In the former case, the design requires some information about the possible duration of flow intervals between successive jumps in each solution of interest (at least after a certain time) as defined next. In the context of switched (resp. sampled) systems, this corresponds to information about the possible switching (resp. sampling) rates.

Definition 1 (Set of flow lengths of a hybrid arc)

For a closed subset $\mathcal{I}$ of $[0, +\infty)$ and some $j_m \in \mathbb{N}$, we say that a hybrid arc $(t, j) \mapsto x(t, j)$ *has flow lengths within* $\mathcal{I}$ *after jump* j_m if

- $0 \leq t - t_j(x) \leq \sup \mathcal{I}$ for all $(t, j) \in \operatorname{dom} x$;
- $t_{j+1}(x) - t_j(x) \in \mathcal{I}$ holds for all $j \in \mathbb{N}$ with $j \geq j_m$ if $\sup \operatorname{dom}_j x = +\infty$, and for all $j \in \{j_m, j_m + 1, \ldots, \sup \operatorname{dom}_j x - 1\}$ otherwise.

In brief, $\mathcal{I}$ contains all the possible lengths of the flow intervals between successive jumps, at least after some time. The first item is to bound the length of the flow intervals not covered by the second item, namely possibly the first ones before j_m, and the last one, which is $\operatorname{dom}_t x \cap [t_{J(x)}, +\infty)$ where $t_{J(x)}$ is the time when the last jump happens (when defined). If $\mathcal{I}$ is unbounded, the system may admit (eventually) continuous solutions and the observer should correct the estimate at least during flows, while $0 \in \mathcal{I}$ means the hybrid arc can jump more than once at the same time instance or have flow lengths going to zero (including (eventually) discrete and Zeno solutions) and the observer should reduce the estimation error at least at jumps.

From there, one may design either:

- A *flow-based* observer with an innovation term during flows only, exploiting the observability of the full state during flows from y_c when $0 \notin \mathcal{I}$ [1, 6];
- A *jump-based* observer with an innovation term at jumps only, exploiting the detectability of the full state via the combination of flows and jumps from y_d available at the jumps only when $\mathcal{I}$ is bounded [6, 17, 12, 24, 26];
- An observer with innovation terms during both flows and jumps, exploiting the observability from both y_c and y_d and the combination of flows and jumps: this is done via a hybrid Kalman-like approach in [28], or via an observability decomposition in [29], or Lyapunov-based LMIs in [22, 6].

This chapter deals with the third case, where the full state is not necessarily instantaneously observable during flows and not observable from the jump output only. It unifies and extends the work of [28, 29]. More precisely, in Section 2, we start with an observability analysis allowing us to exhibit necessary conditions and sufficient conditions for observer design: first through hybrid Gramian conditions, and then via an observability decomposition. The latter decomposes the state in two parts: the first one is instantaneously observable through the flow output y_c, while the second one must be detectable from an extended jump output featuring the *available* jump output y_d and an additional *fictitious* one, describing how the non-observable states impact the observable ones at jumps and become *visible* through y_c. While the Gramian-based analysis has led in [28] to a *systematic* hybrid Kalman-like design, we show how the observability decomposition lets us design observers made of:

- A *high-gain* flow-based observer of the state components that are instantaneously observable from y_c;

- A jump-based observer for the remaining components, derived from a *discrete* LMI-based (resp. KKL-based) observer in Section 3 (resp. Section 4), thus revisiting and extending [28].

Notations: Let $\mathbb{R}$ (resp. $\mathbb{N}$) denote the set of real numbers (resp. natural numbers, i.e., $\{0, 1, 2, \ldots\}$) and $\mathbb{N}_{>0} = \mathbb{N} \setminus \{0\}$. We denote $\mathbb{R}^{m\times n}$ (resp. $\mathcal{S}^n_{>0}$) as the set of real $(m \times n)$- (resp. symmetric positive definite $(n \times n)$-) dimensional matrices. Given a set S, $\mathrm{int}(S)$ denotes its interior. Let $\mathfrak{R}(z)$ and $\mathfrak{I}(z)$ be the real and imaginary parts of the complex variable z, respectively. Denote Id as the identity matrix of appropriate dimension. Let $|\cdot|$ be the Euclidean norm and $\|\cdot\|$ the induced matrix norm. Let $A^\perp$ be the orthogonal complement of matrix A satisfying $AA^\perp = 0$ and such that $\begin{pmatrix} A^\top & A^\perp \end{pmatrix}$ is invertible, and $A^\dagger$ be the Moore-Penrose inverse of A [20]. For a solution $(t, j) \mapsto x(t, j)$ of a hybrid system, we denote $\mathrm{dom}\, x$ its time domain [13], $\mathrm{dom}_t\, x$ (resp. $\mathrm{dom}_j\, x$) the domain's projection on the ordinary time (resp. jump) component, and for $j \in \mathrm{dom}_j\, x$, $t_j(x)$ the unique time such that $(t_j(x), j) \in \mathrm{dom}\, x$ and $(t_j(x), j-1) \in \mathrm{dom}\, x$, and $\mathcal{T}_j(x) := \{t \in \mathrm{dom}_t\, x : (t, j) \in \mathrm{dom}\, x\}$ (for hybrid systems with inputs, see [23]). The mention of x is omitted when no confusion is possible. A solution x to a hybrid system is *complete* if $\mathrm{dom}\, x$ is unbounded and *Zeno* if it is complete and $\sup \mathrm{dom}_t\, x < +\infty$. Let $\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ be the diagonal matrix with entries λ_i, $i = 1, 2, \ldots, n$. Occasionally, $\star$ denotes the symmetric part, i.e., $\star^\top P = P^\top P$ or sometimes $\star^\top QP = P^\top QP$. Last, let sat_s be a saturation function with level s, i.e., $\mathrm{sat}_s(M) = M$ if $\|M\| \le s$ and sat_s is bounded otherwise.

2 Detectability and observability analysis

The existence of an *asymptotic* and *synchronized* observer (2) for system (1) requires system (1) to be *asymptotically detectable*, in the following sense.

Definition 2 ((Pre-)asymptotic detectability with known jump times)

System (1) with known jump times, initialized in $\mathcal{X}_0$, and with inputs in $\mathcal{U}$ is *pre-asymptotically detectable* if any complete solutions x_a and x_b in $\mathcal{S}_\mathcal{H}(\mathcal{X}_0, \mathcal{U})$ with the same inputs $(F, J, H_c, H_d, u_c, u_d)$, such that $\mathrm{dom}\, x_a = \mathrm{dom}\, x_b$, and whose flow outputs $y_{a,c}$, $y_{b,c}$ and jump outputs $y_{a,d}$, $y_{b,d}$ satisfy

$$y_{a,c}(t, j) = y_{b,c}(t, j), \qquad \forall t \in \mathrm{int}(\mathcal{T}_j(x_a)), \forall j \in \mathrm{dom}_j\, x_a, \tag{4a}$$

$$y_{a,d}(t_j, j-1) = y_{b,d}(t_j, j-1), \qquad \forall j \in \mathrm{dom}_j\, x_a, j \ge 1, \tag{4b}$$

verify

$$\lim_{\substack{t+j\to+\infty \\ (t,j)\in\mathrm{dom}\, x_a}} |x_a(t, j) - x_b(t, j)| = 0. \tag{5}$$

If in addition, all solutions in $\mathcal{S}_\mathcal{H}(\mathcal{X}_0, \mathcal{U})$ are complete, then we have *asymptotic detectability*. The set $\mathcal{X}_0$ (resp. $\mathcal{U}$) may be omitted if the property holds for any initial condition in $\mathcal{X}_0 = C \cup D$ (resp. any input).

Remark 1 In Definition 2, when all the flow intervals have non-empty interior (i.e., $\mathrm{int}(\mathcal{T}_j(x_a)) \neq \emptyset$ for all $j \in \mathrm{dom}_j\, x_a$), condition (4a) is equivalent to

$$y_{a,c}(t,j) = y_{b,c}(t,j), \quad \forall (t,j) \in \mathrm{dom}\, x_a, \tag{6}$$

by the continuity of $t \mapsto y_c(t,j)$ during flows for all $j \in \mathrm{dom}_j\, x_a$. On the contrary, when a solution admits consecutive jumps, condition (4a) is required only on the flow intervals with a non-empty interior since it holds vacuously on the other ones. In other words, the equality of y_c is only required when the system is *flowing*.

The *necessity* of asymptotic detectability is typically obtained as follows. First, by definition of a synchronized asymptotic observer initialized in $\mathcal{X}_0$ with inputs in $\mathcal{U}$, all solutions in $\mathcal{S}_{\mathcal{H}}(\mathcal{X}_0, \mathcal{U})$ must be complete. Then, pick a pair of solutions (x_a, x_b) as in Definition 2 verifying (4). A solution $\hat{x}$ produced by observer (2) fed with outputs $(y_{a,c}, y_{a,d})$ shares the same time domain and must converge asymptotically to x_a; but, $\hat{x}$ is also a solution to observer (2) fed with outputs $(y_{b,c}, y_{b,d})$ according to (4), so that it must also converge asymptotically to x_b. It follows that x_a and x_b necessarily converge asymptotically to each other, thus giving us asymptotic detectability. Note that compared to [5], Definition 2 is restricted to pairs of complete solutions with the *same* time domain because the knowledge of the jump times is used to trigger the jumps of the observer so that only complete solutions with the same time domain and the same outputs are required to converge asymptotically to each other.

For observer design, one typically requires stronger *observability* assumptions depending on the class of observers and the required observer properties [3]. Observability typically means that the equality of the outputs in (4), possibly over a large enough time window, implies that the solutions x_a and x_b are actually the same or said differently, there does not exist any pair of different solutions with the same time domain and the same outputs in the sense of (4). Actually, in the context of observer design, a more relevant property is the ability to determine uniquely the current state from the knowledge of the past outputs over a certain time window $\Delta > 0$, which is typically called *backward distinguishability* [27] or *constructibility* [15]. In other words, for all (t,j) in the domain such that $t + j \geq \Delta$, the equality of the outputs (y_c, y_d) along x_a and x_b at all past times (t', j') in the domain such that $0 \leq (t+j) - (t'+j') \leq \Delta$ implies that $x_a(t,j) = x_b(t,j)$ (see later in Section 2.1). For continuous-time systems, this property is equivalent to observability over a time window because of the uniqueness of solutions in forward and backward time. However, they cease to be equivalent in discrete-time or hybrid systems when the jump maps are not invertible: a system could be constructible without being observable (see [15, Section 2.3.3] for a detailed discussion on those notions).

2.1 Hybrid observability Gramian

Consider a pair of solutions x_a and x_b in $\mathcal{S}_{\mathcal{H}}(\mathcal{X}_0, \mathcal{U})$ with the same inputs $(F, J, H_c, H_d, u_c, u_d)$ such that $\mathrm{dom}\, x_a = \mathrm{dom}\, x_b := \mathbb{D}$. Then, for all hybrid times

$((t', j'), (t, j)) \in \mathbb{D} \times \mathbb{D}$, we have

$$x_a(t, j) - x_b(t, j) = \Phi_{F,J}((t, j), (t', j'))(x_a(t', j') - x_b(t', j')), \tag{7}$$

where $\Phi_{F,J}$ is a *hybrid transition matrix* defined as

$$\Phi_{F,J}((t, j), (t', j')) = \phi_F(t, t_{j+1}) \left(\prod_{k=j}^{j'+1} \phi_F(t_{k+1}, t_k) J(t_k, k-1) \right) \phi_F(t_{j'+1}, t'), \tag{8}$$

if $t \geq t'$ and $j \geq j'$, and, if the jump matrix J is invertible at the jump times,

$$\Phi_{F,J}((t, j), (t', j')) = \phi_F(t, t_j) \left(\prod_{k=j+1}^{j'} \phi_F(t_{k-1}, t_k) J^{-1}(t_k, k-1) \right) \phi_F(t_{j'}, t'), \tag{9}$$

if $t \leq t'$ and $j \leq j'$, with the domain of F and J inherited from $\mathbb{D}$, where ϕ_F denotes the continuous-time transition matrix associated with F, i.e., describing solutions to $\dot{x} = Fx$. By summing and integrating squares, it follows that the equality of the outputs (y_c, y_d) along x_a and x_b between time $(t', j') \in \mathbb{D}$ and a later time $(t, j) \in \mathbb{D}$ is equivalent to

$$(x_a(t', j') - x_b(t', j'))^\top \mathcal{G}_{(F,J,H_c,H_d)}((t', j'), (t, j))(x_a(t', j') - x_b(t', j')) = 0, \tag{10}$$

or, assuming the invertibility of J at the jump times,

$$(x_a(t, j) - x_b(t, j))^\top \mathcal{G}^{bw}_{(F,J,H_c,H_d)}((t', j'), (t, j))(x_a(t, j) - x_b(t, j)) = 0, \tag{11}$$

where $\mathcal{G}_{(F,J,H_c,H_d)}((t', j'), (t, j))$ (resp. $\mathcal{G}^{bw}_{(F,J,H_c,H_d)}((t', j'), (t, j))$) is the observability Gramian (resp. *backward observability Gramian*) between those times as defined next.

Definition 3 (Observability Gramians)

The *observability Gramian* and *backward observability Gramian* of a quadruple (F, J, H_c, H_d) defined on a hybrid time domain $\mathbb{D}$, between time $(t', j') \in \mathbb{D}$ and a later time $(t, j) \in \mathbb{D}$, are defined as

$$\begin{aligned}
&\mathcal{G}_{(F,J,H_c,H_d)}((t', j'), (t, j)) = \\
&\quad \int_{t'}^{t_{j'+1}} \star^\top \Psi_c((s, j'), (t', j'))ds + \sum_{k=j'+1}^{j-1} \int_{t_k}^{t_{k+1}} \star^\top \Psi_c((s, k), (t', j'))ds \\
&\qquad + \sum_{k=j'}^{j-1} \star^\top \Psi_d((t_{k+1}, k), (t', j')) + \int_{t_j}^{t} \star^\top \Psi_c((s, j), (t', j'))ds,
\end{aligned} \tag{12}$$

and, when J is invertible at the jump times,

$$\mathcal{G}^{bw}_{(F,J,H_c,H_d)}((t',j'),(t,j)) = \int_{t'}^{t_{j'+1}} \star^\top \Psi_c((s,j'),(t,j))ds + \sum_{k=j'+1}^{j-1} \int_{t_k}^{t_{k+1}} \star^\top \Psi_c((s,k),(t,j))ds + \sum_{k=j'}^{j-1} \star^\top \Psi_d((t_{k+1},k),(t,j)) + \int_{t_j}^{t} \star^\top \Psi_c((s,j),(t,j))ds, \quad (13)$$

respectively, where

$$\Psi_c((s,k),(t,j)) = H_c(s,k)\Phi_{F,J}((s,k),(t,j)), \quad (14a)$$

$$\Psi_d((t_{k+1},k),(t,j)) = H_d(t_{k+1},k)\Phi_{F,J}((t_{k+1},k),(t,j)), \quad (14b)$$

with all the jump times determined from $\mathbb{D}$.

According to (10), we deduce that the observability between times (t',j') and (t,j), namely the ability to reconstruct the initial state $x(t',j')$ from the knowledge of the *future* output until (t,j), is equivalent to the positive definiteness of the observability Gramian over this period. On the other hand, when J is invertible at the jump times, the ability to reconstruct the current state $x(t,j)$ from the knowledge of the *past* output until (t',j'), i.e. the backward distinguishability or constructibility, is characterized by the positive definiteness of the backward observability Gramian over this period according to (11). Actually, in that case, both notions are actually equivalent but the backward observability Gramian tends to appear more naturally in the analysis of observers.

Remark 2 Note that unlike for purely continuous- or discrete-time linear systems, the inputs (u_c, u_d) involved in the hybrid dynamics (1) may impact the observability properties since they may change the domain of the solutions and thus the Gramian.

For observer design, the invertibility of the Gramian is typically assumed to be *uniform*, leading to the following hybrid *uniform complete observability*, extending the classical UCO condition of the Kalman and Bucy's filter [16].

Definition 4 (Uniform complete observability (UCO))

The quadruple (F,J,H_c,H_d) defined on a hybrid time domain $\mathbb{D}$ is *uniformly completely observable* with data (Δ,μ) if there exists $\Delta > 0$ and $\mu > 0$ such that for all $((t',j'),(t,j)) \in \mathbb{D}\times\mathbb{D}$ verifying $(t-t')+(j-j') \geq \Delta$, we have

$$\mathcal{G}_{(F,J,H_c,H_d)}((t',j'),(t,j)) \geq \mu\,\mathrm{Id}\,. \quad (15)$$

In [28], this condition is stated with $\mathcal{G}^{bw}$ replacing $\mathcal{G}$ because the former appears directly in the analysis. They are actually equivalent, assuming the uniform invertibility of J at the jump times and the boundedness of F and J. In [28], this UCO condition is exploited to design a *systematic* hybrid Kalman-like observer of the form

$$\begin{cases} \left.\begin{aligned} \dot{\hat{x}} &= F\hat{x} + u_c + PH_c^\top R_c^{-1}(y_c - H_c\hat{x}) \\ \dot{P} &= \lambda P + FP + PF^\top - PH_c^\top R_c^{-1} H_c P \end{aligned}\right\} \text{when } \mathcal{H} \text{ flows} \\ \left.\begin{aligned} \hat{x}^+ &= J\hat{x} + u_d + JK(y_d - H_d\hat{x}) \\ P^+ &= \gamma^{-1} J(I - KH_d)PJ^\top \end{aligned}\right\} \text{when } \mathcal{H} \text{ jumps} \end{cases} \tag{16a}$$

with

$$K = PH_d^\top (H_d P H_d^\top + R_d)^{-1}, \tag{16b}$$

where $\lambda \geq 0$ and $\gamma \in (0, 1]$ are design parameters, $R_c \in \mathcal{S}_{>0}^{n_{y,c}}$ and $R_d \in \mathcal{S}_{>0}^{n_{y,d}}$ are (possibly time-varying) weighting matrices that are positive definite and are uniformly upper- and lower-bounded. In [28], it is shown that the estimation error:

- Converges asymptotically to zero under UCO and the boundedness of the system matrices *along each of the considered solutions* in $\mathcal{S}_{\mathcal{H}}(\mathcal{X}_0, \mathcal{U})$;
- Is exponentially stable with an arbitrarily fast rate and robustly stable (as defined in [2] but for hybrid systems) after a certain time, under UCO and the boundedness of the system matrices *uniformly across the considered solutions* in $\mathcal{S}_{\mathcal{H}}(\mathcal{X}_0, \mathcal{U})$, when additionally λ is sufficiently large and γ is sufficiently small.

Example 1 Consider a pendulum equipped with an IMU and bouncing on a vertical wall, with angular position $\boldsymbol{\theta} \in \mathbb{R}^3$. The IMU contains a gyroscope measuring its angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ and an accelerometer measuring its proper acceleration (linear acceleration minus gravity) $\boldsymbol{y_a} \in \mathbb{R}^3$ in the IMU frame, modulo an unknown constant bias $\boldsymbol{b_a} \in \mathbb{R}^3$. We assume its tilt $\boldsymbol{t} \in \mathbb{R}^3$ is measured at the jump times (when the mass impacts the wall) and that the linear velocity $\boldsymbol{v} \in \mathbb{R}^3$ in the sensor frame can be deduced from the gyroscope measurement via kinematics [31]. We also assume the velocity magnitude is reduced by an unknown constant restitution coefficient $c \in (0, 1]$ at each impact. With these in mind, we model the system in hybrid form with state $x = (\boldsymbol{t}, \boldsymbol{v}, \boldsymbol{b_a}, c)$ as (see [31] for the flow dynamics)

$$\begin{cases} \dot{\boldsymbol{t}} = -[\boldsymbol{\omega}]_\times \boldsymbol{t} \\ \dot{\boldsymbol{v}} = -[\boldsymbol{\omega}]_\times \boldsymbol{v} + \boldsymbol{y_a} - \boldsymbol{b_a} + g\boldsymbol{t} \\ \dot{\boldsymbol{b}}_{\boldsymbol{a}} = 0 \\ \dot{c} = 0 \end{cases} \quad y_c = \boldsymbol{v}, \qquad \begin{cases} \boldsymbol{t}^+ = \boldsymbol{t} \\ \boldsymbol{v}^+ = -c\boldsymbol{v} \\ \boldsymbol{b}_{\boldsymbol{a}}^+ = \boldsymbol{b_a} \\ c^+ = c \end{cases} \quad y_d = (\boldsymbol{v}, \boldsymbol{t}), \tag{17}$$

where $[\boldsymbol{\omega}]_\times = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$ and g is the gravitational acceleration, with the flow and jump sets depending on the wall configuration. This system takes the form (1) where $F = \begin{pmatrix} -[\boldsymbol{\omega}]_\times & 0 & 0 & 0 \\ g\,\mathrm{Id} & -[\boldsymbol{\omega}]_\times & -\mathrm{Id} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $J = \begin{pmatrix} \mathrm{Id} & 0 & 0 & 0 \\ 0 & 0 & 0 & -y_{d,\boldsymbol{v}} \\ 0 & 0 & \mathrm{Id} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$[1], $u_c = \begin{pmatrix} 0 \\ \boldsymbol{y_a} \\ 0 \\ 0 \end{pmatrix}$, and $u_d = 0$,

[1] Note that J is not invertible at jumps, but it can be made invertible by considering an alternative jump map $\boldsymbol{v}^+ = \boldsymbol{v} + c\,y_{d,\boldsymbol{v}} - y_{d,\boldsymbol{v}}$ and seeing $-y_{d,\boldsymbol{v}}$ as part of u_d. Simulations have shown that this invertibility may not be necessary for the Kalman-like observer.

with $y_{d,\boldsymbol{v}}$ the $\boldsymbol{v}$-component of y_d. We would like to estimate both $\boldsymbol{b_a}$ and c. Assuming the system is persistently not at rest (with external excitation if $c < 1$), it is observable over each period of time containing at least one jump and a flow interval, because

- $\boldsymbol{t}$ is available at jumps and its dynamics are independent of the other state components, making it observable, independently of the input signal $[\boldsymbol{\omega}]_\times$;
- $\boldsymbol{v}$ and $\boldsymbol{b_a}$ are both instantaneously observable during flows from y_c once $\boldsymbol{t}$ is known, also independently of $[\boldsymbol{\omega}]_\times$;
- c is observable at jumps by seeing $\boldsymbol{v}$ as a known input, because $\boldsymbol{v}$ is measured.

It follows that if the pendulum velocity is uniformly lower-bounded (thanks to an appropriate input in the mechanical system, whose effects are in fact contained in $\boldsymbol{y_a}$ in the IMU frame), the observability Gramian computed over a time window Δ larger than the maximal length of flow intervals would be uniformly positive definite. Then, assuming the boundedness of F and J, a Kalman-like observer (16) fed with $(y_c, y_d, \boldsymbol{\omega}, \boldsymbol{y_a})$ can be designed for system (17). Now, consider the case where there is an unknown constant bias $\boldsymbol{b_g} \in \mathbb{R}^3$ in the gyroscope measurement. As a result, $[\boldsymbol{\omega}]_\times$ in system (17) is replaced by $[\boldsymbol{\omega_m} - \boldsymbol{b_g}]_\times$, where $\boldsymbol{\omega_m}$ is the biased measurement. Assume an estimate $\hat{\boldsymbol{b}}_{\boldsymbol{g}}$ of $\boldsymbol{b_g}$ is available such that $\hat{\boldsymbol{b}}_{\boldsymbol{g}} - \boldsymbol{b_g}$ asymptotically vanishes. Then, the dynamics (17) may be re-written as

$$\dot{x} = \hat{F}x + (F - \hat{F})x, \tag{18}$$

where $[\boldsymbol{\omega_m} - \hat{\boldsymbol{b}}_{\boldsymbol{g}}]_\times$ replaces $[\boldsymbol{\omega_m} - \boldsymbol{b_g}]_\times$ in $\hat{F}$. Consider the previous Kalman-like observer, but designed with the known $\hat{F}$ instead of F. According to our observability analysis above, the observability of the quadruple (F, J, H_c, H_d) does not depend on F and therefore still holds for $(\hat{F}, J, H_c, H_d)$. The *robust stability* of the Kalman-like observer [28, Theorem 3] then ensures that the error converges asymptotically to zero because: (i) The UCO condition holds with $\hat{F}$ replacing F, and (ii) The "disturbance" $(F - \hat{F})x$ vanishes asymptotically thanks to $\hat{\boldsymbol{b}}_{\boldsymbol{g}}$ converging to $\boldsymbol{b_g}$ and the boundedness of x. In other words, we only need to find an asymptotic estimate of $\boldsymbol{b_g}$ and feed it to the hybrid Kalman-like observer of x. For that, notice that the pendulum position $\boldsymbol{\theta}$ verifies the hybrid dynamics

$$\begin{cases} \dot{\boldsymbol{\theta}} = \boldsymbol{\omega_m} - \boldsymbol{b_g} \\ \dot{\boldsymbol{b}}_{\boldsymbol{g}} = 0 \end{cases} \qquad y_c' = 0, \qquad \begin{cases} \boldsymbol{\theta}^+ = \boldsymbol{\theta} \\ \boldsymbol{b}_{\boldsymbol{g}}^+ = \boldsymbol{b_g} \end{cases} \qquad y_d' = \boldsymbol{\theta}, \tag{19}$$

with the flow and jump sets depending on the wall configuration. In other words, $\boldsymbol{\theta}$ has continuous-time dynamics, but with sampled measurement at each impact obtained via the impact condition. Because system (19) has linear maps and only the jump output, we can design a *jump-based observer* with a constant gain using LMIs on the equivalent *discrete*-time system sampled at the jumps (see [6, Corollary 5.2]). All in all, still if the pendulum velocity is uniformly lower-bounded away from zero and (F, J) upper-bounded, an observer of x is obtained by the cascade of a jump-based observer of $\boldsymbol{b_g}$ and a Kalman-like one for system (17) fed with $[\boldsymbol{\omega_m} - \hat{\boldsymbol{b}}_{\boldsymbol{g}}]_\times$ instead of $[\boldsymbol{\omega}]_\times$.

In what follows, we attempt to analyze more precisely the observability/detectability of the system by decomposing the state according to the different sources of observability. Beyond a finer comprehension, this allows for the design of observers when UCO is not satisfied, for instance under mere detectability properties, or to design observers of smaller dimensions through decoupling (see Table 1 for a comparison).

2.2 Observability decomposition

In the case where the full state is *instantaneously observable* during flows via the flow output y_c and the system admits an average dwell time, a high-gain flow-based observer (using only y_c) can be designed (see [6, Section 4]); and when the full state is observable from the jump output y_d only, a jump-based observer based on an equivalent discrete-time system can be designed if the jumps are persistent (see [6, Section 5]). We are thus interested here in the case where observability rather comes from the combination of flows and jumps and/or the combination of (y_c, y_d). The idea of the decomposition is thus to isolate state components that are instantaneously observable during flows from y_c, from other ones that become visible thanks to y_d or the combination of flows and jumps. It follows that both the flow and jump outputs may need to be fully exploited to reconstruct the state and that neither (eventually) continuous nor discrete/Zeno trajectories are allowed: both flows and jumps need to be persistent at least after a certain time, unlike in Section 2.1, as assumed next.

Assumption 2 There exists $j_m \in \mathbb{N}$ such that solutions have flow lengths within a compact set $\mathcal{I} \subseteq [\tau_m, \tau_M]$ where $\tau_m > 0$ after jump j_m.

Assumption 2 means that, for all solutions $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0, \mathcal{U})$, the hybrid arc $(t, j) \mapsto (x(t, j), t - t_j)$ is solution after some time to the hybrid system

$$\mathcal{H}^\tau \begin{cases} \left.\begin{aligned} \dot{x} &= Fx + u_c \\ \dot{\tau} &= 1 \end{aligned}\right\} (x, \tau) \in C^\tau & y_c = H_c x \\ \left.\begin{aligned} x^+ &= Jx + u_d \\ \tau^+ &= 0 \end{aligned}\right\} (x, \tau) \in D^\tau & y_d = H_d x \end{cases} \tag{20a}$$

with the flow and jump sets

$$C^\tau = \mathbb{R}^{n_x} \times [0, \tau_M], \qquad D^\tau = \mathbb{R}^{n_x} \times \mathcal{I}, \tag{20b}$$

where $\tau \in \mathbb{R}$ is a timer keeping track of the time elapsed since the previous jump. Note that $\mathcal{H}^\tau$ admits (after the first j_m jumps) a larger set of solutions than $\mathcal{S}_{\mathcal{H}}(\mathcal{X}_0, \mathcal{U})$ since the information of the flow and jump sets are replaced by the knowledge of flow lengths in $\mathcal{I}$ only (as long as the inputs $(F, J, H_c, H_d, u_c, u_d)$ are defined along the time domains of those extra solutions). But, as discussed in Section 1, when the observer contains gains that are computed offline, based on the knowledge of the possible flow lengths only, it is actually designed for $\mathcal{H}^\tau$ instead of $\mathcal{H}$ and it is thus the detectability/observability of $\mathcal{H}^\tau$ that is relevant. In that case, the

design depends only implicitly on the sets $\mathcal{X}_0$, $\mathcal{U}$, C, and D through the choice of $\mathcal{I}$ satisfying Assumption 2.

In view of observer design and motivated by [10], we start by proposing a change of variables decomposing the state x of $\mathcal{H}^\tau$ into components associated with different types of observability. In order to guarantee the existence of the decomposition, we assume in the next section that the flow pair (F, H_c) is constant. However, the subsequent results of this chapter still hold with (F, H_c) varying, as long as the transformation into the decomposition form exists and is invertible uniformly in time as explained in Remark 3.

2.3 Observability from y_c during flows

Assume (F, H_c) is constant. Let the (flow) observability matrix be

$$\mathcal{O} := \operatorname{row}(H_c, H_c F, \dots, H_c F^{n_x - 1}), \tag{21}$$

and assume it is of rank $n_o := \dim \operatorname{Im} \mathcal{O} < n_x$. Consider a basis $(v_i)_{1 \le i \le n_x}$ of $\mathbb{R}^{n_x}$ such that $(v_i)_{1 \le i \le n_o}$ is a basis of the observable subspace and $(v_i)_{n_o+1 \le i \le n_x}$ is a basis of the non-observable subspace $\ker \mathcal{O}$. Then, we define the invertible matrix $\mathcal{D} := \begin{pmatrix} \mathcal{D}_o & \mathcal{D}_{no} \end{pmatrix}$ where

$$\mathcal{D}_o := \begin{pmatrix} v_1 & \dots & v_{n_o} \end{pmatrix} \in \mathbb{R}^{n_x \times n_o}, \tag{22a}$$

$$\mathcal{D}_{no} := \begin{pmatrix} v_{n_o+1} & \dots & v_{n_x} \end{pmatrix} \in \mathbb{R}^{n_x \times n_{no}}, \tag{22b}$$

which by definition satisfies for all $\tau \ge 0$,

$$\mathcal{O} \mathcal{D}_{no} = 0, \qquad H_c e^{F\tau} \mathcal{D}_{no} = 0. \tag{23}$$

We denote $\mathcal{V} := \mathcal{D}^{-1}$ which we decompose consistently into two parts $\mathcal{V} =: \begin{pmatrix} \mathcal{V}_o \\ \mathcal{V}_{no} \end{pmatrix}$, so that $\mathcal{V}_o x$ represents the part of the state that is instantaneously observable during flows (see [9, Theorem 6.O6]).

A first idea could be to stop the decomposition here and design a sufficiently fast high-gain observer for $\mathcal{V}_o x$ while estimating the rest of the state $\mathcal{V}_{no} x$ through y_d and detectability. However, as noticed in [10, Proposition 6], the fact that $\mathcal{V}_o x$ and $\mathcal{V}_{no} x$ possibly interact with each other during flows prevents achieving stability by further pushing the high gain. The case of such a decomposition where $\mathcal{V}_o x$ and $\mathcal{V}_{no} x$ evolve independently during flows is exploited in a more general context in [30]. Here, because the maps are linear, we can go further and solve this possible coupling by more efficiently decoupling the state components as follows.

Indeed, the estimation of any state that is not instantaneously observable during flows needs to take into account the combination of flows and jumps. That is why it is relevant to exhibit explicitly this combination via the change of coordinates

$$x \mapsto z = \begin{pmatrix} z_o \\ z_{no} \end{pmatrix} = \mathcal{V} e^{-F\tau} x = \begin{pmatrix} \mathcal{V}_o \\ \mathcal{V}_{no} \end{pmatrix} e^{-F\tau} x, \tag{24}$$

whose inverse transformation is

$$x = e^{F\tau} \mathcal{D} z = e^{F\tau} (\mathcal{D}_o z_o + \mathcal{D}_{no} z_{no}), \tag{25}$$

and which, according to (23), transforms $\mathcal{H}^\tau$ into

$$\begin{cases} \dot{z}_o = G_o(\tau) u_c \\ \dot{z}_{no} = G_{no}(\tau) u_c \\ \dot{\tau} = 1 \\ \\ z_o^+ = J_o(\tau) z_o + J_{ono}(\tau) z_{no} + \mathcal{V}_o u_d \\ z_{no}^+ = J_{noo}(\tau) z_o + J_{no}(\tau) z_{no} + \mathcal{V}_{no} u_d \\ \tau^+ = 0, \end{cases} \tag{26a}$$

with the flow and jump sets

$$\mathbb{R}^{n_o} \times \mathbb{R}^{n_{no}} \times [0, \tau_M], \qquad \mathbb{R}^{n_o} \times \mathbb{R}^{n_{no}} \times \mathcal{I}, \tag{26b}$$

and the measurements

$$y_c = H_{c,o}(\tau) z_o, \qquad y_d = H_{d,o}(\tau) z_o + H_{d,no}(\tau) z_{no}, \tag{26c}$$

where $G_o(\tau) = \mathcal{V}_o e^{-F\tau}$, $G_{no}(\tau) = \mathcal{V}_{no} e^{-F\tau}$, $J_o(\tau) = \mathcal{V}_o J e^{F\tau} \mathcal{D}_o$, $J_{ono}(\tau) = \mathcal{V}_o J e^{F\tau} \mathcal{D}_{no}$, $J_{noo}(\tau) = \mathcal{V}_{no} J e^{F\tau} \mathcal{D}_o$, $J_{no}(\tau) = \mathcal{V}_{no} J e^{F\tau} \mathcal{D}_{no}$, $H_{c,o}(\tau) = H_c e^{F\tau} \mathcal{D}_o$, $H_{d,o}(\tau) = H_d e^{F\tau} \mathcal{D}_o$, and $H_{d,no}(\tau) = H_d e^{F\tau} \mathcal{D}_{no}$. This idea of bringing at the jumps the whole combination of flows and jumps is similar to the so-called *equivalent discrete-time system* exhibited in [6] for jump-based observer designs. Notice that by definition and thanks to linearity, the observability decomposition through $\mathcal{V}$ ensures that the flow dynamics of z_o and y_c are totally independent of z_{no}, which only impacts z_o at jumps. In other words, the whole dependence of the observable part on the non-observable part via flows and jumps has been gathered at the jumps. Besides, z_o is by definition instantaneously observable from y_c. More precisely, for any $\delta > 0$, there exists $\alpha > 0$ such that the observability Gramian of the continuous-time pair $(0, H_{c,o}(\tau))$ satisfies

$$\int_t^{t+\delta} H_{c,o}^\top(s) H_{c,o}(s) ds = \int_t^{t+\delta} \mathcal{D}_o^\top e^{F^\top s} H_c^\top H_c e^{Fs} \mathcal{D}_o ds \geq \alpha \, \mathrm{Id}, \qquad \forall t \geq 0. \tag{27}$$

Indeed, this Gramian corresponds to the observability Gramian of the pair (F, H_c) projected onto the observable subspace. This condition is thus related to the *uniform complete observability* of the continuous-time pair $(0, H_{c,o}(\tau))$ in the Kalman literature [16] (continuous-time version of the one in Definition 4), but here with an arbitrarily small window δ. Since z_o is observable via y_c, we propose to estimate z_o sufficiently fast during flows to compensate for the interaction with z_{no} at jumps.

Then, intuitively from system (26), information about z_{no} may be drawn from two sources: the jump output y_d and the part of z_{no} impacting z_o at jumps, namely $J_{ono}(\tau)z_{no}$, which may become "visible" in z_o, via y_c during the following flow interval. This is illustrated in Example 2 below. Actually, we show next in Section 2.4 that the detectability of z_{no} comes from these two sources of information only.

Example 2 Consider a hybrid system of form (1) with state $x = (x_1, x_2, x_3, x_4)$, $u_c = 0$, $u_d = 0$, and the matrices

$$F = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \quad H_c = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad H_d = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}, \quad (28)$$

with random flow lengths varying in some compact set $\mathcal{I} \subset \left(0, \frac{\pi}{2}\right)$. It can be seen that only x_1 and x_2 are instantaneously observable during flows from y_c, but x_3 impacts x_1 at jumps (or y_d) and x_4 impacts x_3 during flows. Therefore, we may hope to estimate the full state. In order to decouple those various impacts and analyze detectability more easily, we proceed with the change of variables (24). We obtain

$$z_o = \begin{pmatrix} \cos(\tau) & \sin(\tau) & 0 & 0 \\ -\sin(\tau) & \cos(\tau) & 0 & 0 \end{pmatrix} x, \qquad z_{no} = \begin{pmatrix} 0 & 0 & \cos(2\tau) & \sin(2\tau) \\ 0 & 0 & -\sin(2\tau) & \cos(2\tau) \end{pmatrix} x, \qquad (29)$$

and system (26) with the matrices $J_o(\tau) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $J_{ono}(\tau) = \begin{pmatrix} \cos(2\tau) & -\sin(2\tau) \\ 0 & 0 \end{pmatrix}$, $J_{noo}(\tau) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $J_{no}(\tau) = \begin{pmatrix} \cos(2\tau) & -\sin(2\tau) \\ \sin(2\tau) & \cos(2\tau) \end{pmatrix}$, $H_{c,o}(\tau) = \begin{pmatrix} \cos(\tau) & -\sin(\tau) \end{pmatrix}$, $H_{d,o}(\tau) = \begin{pmatrix} 0 & 0 \end{pmatrix}$, and $H_{d,no}(\tau) = \begin{pmatrix} \cos(2\tau) & -\sin(2\tau) \end{pmatrix}$. It can be seen that in this case, the terms $J_{ono}(\tau)z_{no}$ and $H_{d,no}(\tau)z_{no}$ contain the same information on z_{no} and both should be able to let us estimate this part.

Remark 3 In what follows, a varying pair (F, H_c) can be considered as long as the transformation into the form (26) satisfying (27) exists and is invertible uniformly in time. This can be done with the transition matrix of F replacing the exponential form, if the observable subspace remains the same at all times. In that case, the jump matrices J_o, J_{ono}, J_{noo}, J_{no}, $H_{c,o}$, $H_{d,o}$, and $H_{d,no}$ are (discrete) known inputs that are no longer functions of τ only, but of the (discrete) jump index, which is not considered in this chapter. Similarly, J could vary at each jump as long as every related condition in the rest of this chapter holds uniformly in u_d.

2.4 Detectability analysis

We first provide a more specific characterization of the (pre-)asymptotic detectability of system (26) in the case of zero inputs (u_c, u_d). Indeed, we will see in Theorem 2 that this detectability is relevant to characterize that of the initial system $\mathcal{H}$.

Lemma 1 *System* (26) *with known jump times and zero inputs* (u_c, u_d) *is pre-asymptotically detectable if and only if any of its complete solutions* (z, τ) *with zero inputs* (u_c, u_d) *and flow and jump outputs satisfying*

$$y_c(t, j) = 0, \qquad \forall t \in \text{int}(\mathcal{T}_j(z)), \forall j \in \text{dom}_j\, z, \tag{30a}$$

$$y_d(t_j, j-1) = 0, \qquad \forall j \in \text{dom}_j\, z, j \geq 1, \tag{30b}$$

verifies

$$\lim_{\substack{t+j\to+\infty \\ (t,j)\in\text{dom}\, z}} z(t, j) = 0. \tag{31}$$

Proof First, assume system (26) is pre-asymptotically detectable. Let (z, τ) be a complete solution to system (26) with zero inputs (u_c, u_d) and with outputs satisfying (30). Notice that the hybrid arc (z', τ), with $\text{dom}\, z' = \text{dom}\, z$ and z' constantly zero, is also solution to system (26) (thanks to linearity in the maps and the inputs (u_c, u_d) being zero). It can be seen that this solution is complete and also satisfies (30). By the pre-asymptotic detectability of system (26), we have $\lim_{\substack{t+j\to+\infty \\ (t,j)\in\text{dom}\, z}} |z(t, j) - z'(t, j)| = 0$, which implies that $\lim_{\substack{t+j\to+\infty \\ (t,j)\in\text{dom}\, z}} z(t, j) = 0$. Second, let us prove the converse. For that, assume that any complete solution to system (26) with zero inputs (u_c, u_d) and with outputs verifying (30) is such that z converges to zero. Consider two complete solutions (z_a, τ_a) and (z_b, τ_b) to system (26) with $\text{dom}(z_a, \tau_a) = \text{dom}(z_b, \tau_b)$, with zero inputs (u_c, u_d), and with outputs satisfying (4). By definition of the flow set, z_a and z_b jump at least once. Because the timers are both reset to zero at jumps, we have $\tau_a(t, j) = \tau_b(t, j) := \tau(t, j)$ for all $(t, j) \in \text{dom}\, z_a$ such that $t \geq t_1$ and $j \geq 1$. By removing $[0, t_1] \times \{0\}$ from the time domain, we see that $(z_a - z_b, \tau)$ is a complete solution to system (26) with outputs verifying (30) (thanks to system (26) having linear maps in z and the flow and jump sets being independent of z). Therefore, by assumption, we have $\lim_{\substack{t+j\to+\infty \\ (t,j)\in\text{dom}\, z_a}} (z_a - z_b)(t, j) = 0$, which implies the (pre-)asymptotic detectability of system (26). □

Note that the equivalence of the *incremental* detectability as in Definition 2 with the *zero* detectability as in Lemma 1 is classical for linear continuous- or discrete-time systems, but it is not automatic for hybrid systems with linear maps due to the flow/jump conditions. Here, it holds only because:

- The flow and jump conditions in system (26) do not depend on z but only on τ;
- τ is determined uniquely after the first jump by the time domain of solutions;
- The inputs (u_c, u_d) are removed, thus avoiding a restriction of solutions to system (26) due to a mismatch of time domains.

Theorem 1 *Assume that* $0 \notin \mathcal{I}$ *and* $\mathcal{I}$ *is compact. Then, the following three statements are equivalent:*

1. *The hybrid system* (20) *with zero inputs* (u_c, u_d) *and known jump times is asymptotically detectable;*

2. *The hybrid system* (26) *with zero inputs* (u_c, u_d) *and known jump times is asymptotically detectable;*
3. *The discrete-time system defined as*

$$z_{no,k+1} = J_{no}(\tau_k) z_{no,k}, \qquad y_k = H_{d,\text{ext}}(\tau_k) z_{no,k}, \tag{32}$$

where $H_{d,\text{ext}}(\tau_k) = \begin{pmatrix} H_{d,no}(\tau_k) \\ J_{ono}(\tau_k) \end{pmatrix}$*, with* $\tau_k \in \mathcal{I}$ *for all* $k \in \mathbb{N}$*, is asymptotically detectable.*

Proof First, notice that all maximal solutions to systems (20) and (26) are complete because their dynamics maps are linear and the flow and jump conditions do not depend on the state but only on the timer. Notice also that since $0 \notin \mathcal{I}$, consecutive jumps cannot happen, so that condition (4a) is equivalent to (6) following Remark 1. Because systems (20) and (26) are the same system modulo a uniformly invertible change of variables, 1. and 2. are equivalent. Then, let us prove that 2. implies 3. So assume 2. holds and consider a solution $(z_{no,k})_{k\in\mathbb{N}}$ to system (32) with input $(\tau_k)_{k\in\mathbb{N}}$ in $\mathcal{I}$, such that $y_k = 0$ for all $k \in \mathbb{N}$. We want to show that $(z_{no,k})_{k\in\mathbb{N}}$ asymptotically goes to zero. For that, we build and analyze the complete solution $z = (z_o, z_{no})$ to system (26) initialized as $z_o(0,0) = 0$, $z_{no}(0,0) = z_{no,0}$, and $\tau(0,0) = 0$ with jumps verifying $\tau(t_j, j-1) = \tau_{j-1} \in \mathcal{I}$ for all $j \geq 1$ and zero inputs (u_c, u_d). It follows[2] from the fact that (i) z_o and z_{no} are constant during flows, (ii) y_c is independent of z_{no}, and (iii) $y_k = 0$ for all $k \in \mathbb{N}$, that for all $j \in \mathbb{N}$,

$$\begin{aligned}
z_o(t,j) &= 0, && \forall t \in [t_j, t_{j+1}], \\
z_{no}(t,j) &= z_{no,j}, && \forall t \in [t_j, t_{j+1}], \\
y_c(t,j) &= 0, && \forall t \in [t_j, t_{j+1}], \\
z_o(t_{j+1}, j+1) &= J_{ono}(\tau_j) z_{no,j} = 0, \\
z_{no}(t_{j+1}, j+1) &= J_{no}(\tau_j) z_{no,j} = z_{no,j+1}, \\
y_d(t_{j+1}, j) &= H_{d,no}(\tau_j) z_{no,j} = 0.
\end{aligned}$$

By 2. and Lemma 1, this implies $\lim_{\substack{t+j\to+\infty \\ (t,j)\in\text{dom}\, z}} z_{no}(t,j) = 0$. Because this solution z_{no} coincides with $(z_{no,k})_{k\in\mathbb{N}}$ at the jumps, we deduce $\lim_{\substack{k\to+\infty \\ k\in\mathbb{N}}} z_{no,k} = 0$, implying 3. Finally, let us prove that 3. implies 2. Consider a complete solution $(z, \tau) = (z_o, z_{no}, \tau)$ to system (26) with zero inputs (u_c, u_d) and such that (30) holds. By definition of the jump set, for all $j \in \text{dom}_j\, z$, $\tau_j := \tau(t_{j+1}, j) \in \mathcal{I}$. Since $0 \notin \mathcal{I}$, the solution admits a dwell time and, because we look for an asymptotic property, we may assume without any loss of generality that the solution starts with a flow, possibly overlooking the first part of the domain (with $j = 0$) in case of a jump at time 0. Since z_o is instantaneously observable during flows according to (27), $y_c = 0$ implies that z_o is zero during each flow interval. Next, as $0 \notin \mathcal{I}$, there is no more than one jump at each jump time so that $z_o(t,j) = 0$ for all $(t,j) \in \text{dom}\, z$. Besides, since $\mathcal{I}$ is compact,

[2] Note that $0 \notin \mathcal{I}$ and the compactness of $\mathcal{I}$ are not needed to prove this direction.

$\mathrm{dom}_j\, z = \mathbb{N}$ and from (26a), we then have $J_{ono}(\tau(t_j, j-1))z_{no}(t_j, j-1) = 0$ for all $j \in \mathrm{dom}_j\, z$ with $j \geq 1$. Therefore, since z_{no} is constant during flows and $y_d(t_j, j-1) = 0$, for all $j \in \mathrm{dom}_j\, z$, $j \geq 1$, we have for all $j \in \mathbb{N}$,

$$\begin{aligned} z_{no}(t_{j+1}, j+1) = J_{no}(\tau(t_{j+1}, j))z_{no}(t_{j+1}, j) = J_{no}(\tau_j)z_{no}(t_j, j), \\ H_{d,no}(\tau(t_{j+1}, j))z_{no}(t_{j+1}, j) = H_{d,no}(\tau_j)z_{no}(t_j, j) = 0, \\ J_{ono}(\tau(t_{j+1}, j))z_{no}(t_{j+1}, j) = J_{ono}(\tau_j)z_{no}(t_j, j) = 0. \end{aligned}$$

By considering the sequence $(z_{no}(t_j, j))_{j\in\mathbb{N}}$ solution to system (32) with input $(\tau_j)_{j\in\mathbb{N}}$ in $\mathcal{I}$ and applying 3., we obtain $\lim_{\substack{j\to+\infty \\ j\in\mathbb{N}}} z_{no}(t_j, j) = 0$. Since z_{no} is constant during flows, we have $\lim_{\substack{t+j\to+\infty \\ (t,j)\in\mathrm{dom}\, z}} = 0$, implying 2. according to Lemma 1. □

We thus conclude that (at least when the inputs (u_c, u_d) are zero) the asymptotic detectability of system (20) requires z_{no} to be asymptotically detectable through the output made of the measured output y_d and the *fictitious* one $J_{ono}(\tau)z_{no}$, which describes how z_{no} impacts z_o at jumps. We insist also that the detectability of z_{no} comes from the combination of flows and jumps and not due to jumps alone since the useful information contained in the flow dynamics and output is gathered at the jumps via the transformation (24).

It follows from this analysis that the design of an asymptotic observer for $\mathcal{H}$ with gains computed offline from the knowledge of $\mathcal{I}$ only (without any special consideration of $\mathcal{X}_0, \mathcal{U}, C$, or D), namely an observer for system (20), without considering the possible restrictions of time domains by the inputs (u_c, u_d), requires the asymptotic detectability of the discrete-time system (32). This is because any time domain with flow lengths in $\mathcal{I}$ is a priori possible. This will be done in Section 3 with an LMI-based design. On the other hand, if we consider the more precise problem of observer design with time domains restricted to that of solutions in $\mathcal{S}_{\mathcal{H}}(\mathcal{X}_0, \mathcal{U})$, we end up with the following sufficient condition for asymptotic detectability of $\mathcal{H}$.

Theorem 2 *Suppose Assumptions 1 and 2 hold. Then, $\mathcal{H}$ initialized in $\mathcal{X}_0$ with inputs in $\mathcal{U}$ is asymptotically detectable if for each $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0, \mathcal{U})$, the discrete-time system* (32) *with input $(\tau_k)_{k\in\mathbb{N}}$ defined as $\tau_k = t_{k+1}(x) - t_k(x)$ for all $k \in \mathbb{N}$ is asymptotically detectable.*

Proof Pick solutions x_a and x_b in $\mathcal{S}_{\mathcal{H}}(\mathcal{X}_0, \mathcal{U})$ with the same inputs $(F, J, H_c, H_d, u_c, u_d)$, such that $\mathrm{dom}\, x_a = \mathrm{dom}\, x_b$, and with outputs $y_{a,c}, y_{b,c}, y_{a,d}, y_{b,d}$ satisfying (4). By Assumptions 1 and 2, these solutions are complete with $\mathrm{dom}_j\, x_a = \mathrm{dom}_j\, x_b = \mathbb{N}$. By Assumption 2, for all $j \geq j_m$, $\tau_j := t_{j+1}(x_a) - t_j(x_a) = t_{j+1}(x_b) - t_j(x_b) \in \mathcal{I}$. Since $0 \notin \mathcal{I}$, the solutions admit a dwell time after the first j_m jumps. Since we look for an asymptotic property, we may discard the first part of the solutions with $j < j_m$, and assume without any loss of generality that they start with a flow interval and have a dwell time. Then, condition (4a) is equivalent to (6) following Remark 1. Consider the hybrid signals (z_a, z_b, τ) defined for all $(t, j) \in \mathrm{dom}\, x_a$ as

$$\begin{aligned}\tau(t,j) &= t - t_j,\\ \begin{pmatrix} z_{a,o}(t,j)\\ z_{a,no}(t,j)\end{pmatrix} &= \begin{pmatrix}\mathcal{V}_o\\ \mathcal{V}_{no}\end{pmatrix} e^{-F\tau(t,j)} x_a(t,j),\\ \begin{pmatrix} z_{b,o}(t,j)\\ z_{b,no}(t,j)\end{pmatrix} &= \begin{pmatrix}\mathcal{V}_o\\ \mathcal{V}_{no}\end{pmatrix} e^{-F\tau(t,j)} x_b(t,j).\end{aligned}$$

We see that both (z_a, τ) and (z_b, τ) are solutions to system (26) with the same τ-dependent matrices $(G_o, G_{no}, J_o, J_{ono}, J_{noo}, J_{no}, H_{c,o}, H_{d,o}, H_{d,no})$ and the same inputs (u_c, u_d). Since $z_{a,o}$ and $z_{b,o}$ are instantaneously observable during flows according to (27), $y_{a,c} = y_{b,c}$ implies that $z_{a,o} = z_{b,o}$ during each flow interval. Next, as $0 \notin \mathcal{I}$, there is no more than one jump at each jump time so that $z_{a,o}(t,j) = z_{b,o}(t,j)$ for all $(t,j) \in \operatorname{dom} x_a$. Besides, since u_d is the same for both solutions, from (26a), we have $J_{ono}(\tau(t_j, j-1)) z_{a,no}(t_j, j-1) = J_{ono}(\tau(t_j, j-1)) z_{b,no}(t_j, j-1)$ for all $j \geq 1$. Therefore, since $z_{a,no}$ and $z_{b,no}$ evolve in the same way during flows (with the same u_c) and $y_{a,d}(t_j, j-1) = y_{b,d}(t_j, j-1)$, for all $j \geq 1$, by defining $\tilde{z}_{no} = z_{a,no} - z_{b,no}$, we have that $\tilde{z}_{no}$ is constant during flows and for all $j \in \mathbb{N}$,

$$\begin{aligned}\tilde{z}_{no}(t_{j+1}, j+1) &= J_{no}(\tau(t_{j+1}, j))\tilde{z}_{no}(t_{j+1}, j) = J_{no}(\tau_j)\tilde{z}_{no}(t_j, j),\\ H_{d,no}(\tau(t_{j+1}, j))\tilde{z}_{no}(t_{j+1}, j) &= H_{d,no}(\tau_j)\tilde{z}_{no}(t_j, j) = 0,\\ J_{ono}(\tau(t_{j+1}, j))\tilde{z}_{no}(t_{j+1}, j) &= J_{ono}(\tau_j)\tilde{z}_{no}(t_j, j) = 0.\end{aligned}$$

By considering the sequence $(\tilde{z}_{no}(t_j, j))_{j \in \mathbb{N}}$ solution to system (32) with input $(\tau_j)_{j \in \mathbb{N}}$ in $\mathcal{I}$ and using the asymptotic detectability of system (32) for the particular sequence $(\tau_k)_{k \in \mathbb{N}}$ generated by x_a and x_b, we obtain $\lim_{\substack{j \to +\infty \\ j \in \mathbb{N}}} \tilde{z}_{no}(t_j, j) = 0$. Since $\tilde{z}_{no}$ is constant during flows, we have $\lim_{\substack{t+j \to +\infty \\ (t,j) \in \operatorname{dom} x_a}} \tilde{z}_{no}(t,j) = 0$, which means that $\lim_{\substack{t+j \to +\infty \\ (t,j) \in \operatorname{dom} x_a}} (z_{a,no}(t,j) - z_{b,no}(t,j)) = 0$, implying that $\mathcal{H}$ initialized in $\mathcal{X}_0$ with inputs in $\mathcal{U}$ is asymptotically detectable according to Lemma 1. □

Unlike Theorem 1, Theorem 2 does not give a *necessary* condition for detectability (and thus observer design). The reason is that the flow and jump conditions of $\mathcal{H}$ are not taken into account in system (32). But it still suggests us to build observers for $\mathcal{H}$ under observability/detectability conditions on system (32) for the flow length sequences $(\tau_k)_{k \in \mathbb{N}}$ appearing in $\mathcal{S}_{\mathcal{H}}(\mathcal{X}_0, \mathcal{U})$. This will be done in Section 3 and 4 through LMI- or KKL-based design.

Remark 4 Compared to the preliminary work [29], the observer designs in this chapter do not require an extra (constant) transformation decoupling the part of z_{no} detectable from y_d and the part detectable from the fictitious output $J_{ono}(\tau) z_{no}$. We instead consider an extended output in system (32) and the decomposition (26) proposed in this chapter is thus less conservative. For instance, Example 2 can be cast into the decomposition (26), but fails to fall into the scope of the decomposition of [29] unless $\mathcal{I} = \left\{\frac{\pi}{2}\right\}$.

Example 3 Consider the system in Example 2. It is possible to check, by computing the (time-varying) observability matrix of the pair $(J_{no}(\tau_k), H_{d,no}(\tau_k))$, that system (32) is observable for any sequence $(\tau_k)_{k\in\mathbb{N}}$ as long as $\sin(\tau_k + \tau_{k+1}) \neq 0$ at some $k \in \mathbb{N}$, which is the case for $\tau_k \in \mathcal{I}$ since $\mathcal{I} \subset \left(0, \frac{\pi}{2}\right)$. This implies in particular that system (32) is detectable. Actually, even if $H_d = 0$, i.e., no output is available at jumps, the pair $(J_{no}(\tau_k), J_{ono}(\tau_k))$ is also observable using the same arguments. This means that z_{no} is actually observable through the fictitious measurement of z_o at jumps. We see from this example that thanks to the flow-jump coupling, by using z_o as a fictitious measurement, state components that are not observable during flows from the flow output may become observable via jumps even without any additional measurements at jumps (hidden dynamics), only through the way they impact the observable ones.

3 LMI-based observer design from discrete quadratic detectability in observability decomposition

Inspired by the detectability analysis of Theorem 1, we propose a first observer design under the following detectability assumption.

Assumption 3 Given τ_M and $\mathcal{I}$ defined in Assumption 2, there exist $Q_{no} \in \mathcal{S}^{n_{no}}_{>0}$, $L_{d,no} : [0, \tau_M] \to \mathbb{R}^{n_{no}\times n_{y,d}}$ bounded on $[0, \tau_M]$ and continuous on $\mathcal{I}$, and $K_{no} \in \mathbb{R}^{n_{no}\times n_o}$ such that

$$\star^\top Q_{no} \left(J_{no}(\tau) - \begin{pmatrix} L_{d,no}(\tau) & K_{no} \end{pmatrix} \begin{pmatrix} H_{d,no}(\tau) \\ J_{ono}(\tau) \end{pmatrix} \right) - Q_{no} < 0, \qquad \forall \tau \in \mathcal{I}. \tag{33}$$

We refer the reader to Remark 5 for constructive methods to solve (33), where it is shown that the solvability of (33) in Q_{no} and $\begin{pmatrix} L_{d,no}(\tau) & K_{no} \end{pmatrix}$ is equivalent to that of a reduced LMI involving Q_{no} only. Consistently with Theorem 1, Assumption 3 requires the detectability of the pair $\left(J_{no}(\tau), \begin{pmatrix} H_{d,no}(\tau) \\ J_{ono}(\tau) \end{pmatrix} \right)$ for each frozen $\tau \in \mathcal{I}$. But it is actually stronger because it further requires Q_{no} and K_{no} to be independent of τ. It corresponds to a stronger version of the *quadratic detectability* of system (32) defined in [32]. Actually, the detectability of Assumption 3 allows us to build an observer for any sequence of flow lengths $(\tau_k)_{k\in\mathbb{N}} \in \mathcal{I}$ and thus requires the detectability of the discrete-time pair for any such sequences, which is still consistent with the result of Theorem 1. Note that the reason why K_{no} is required to be independent of τ is that it is used to carry out another change of variables in the proof of Theorem 3 below, allowing us to exhibit the *fictitious* output in the analysis.

3.1 LMI-based observer design in the z-coordinates

Because the flow output matrix $H_{c,o}(\cdot)$ varies and satisfies the observability condition (27), we design a flow-based *Kalman-like* observer of z_o during flows using y_c [7]. Its advantage over a Kalman observer is that it admits a *strict* Lyapunov function, allowing for direct robust Lyapunov analysis. Besides, it provides a direct relationship between the Lyapunov matrix and the observability Gramian. Then, as suggested by the detectability analysis, z_{no} should be estimated thanks to both y_d and its interaction with z_o at jumps via a fictitious output. The latter is not available for injection in the observer, but it becomes visible through z_o after the jump, and thus through y_c during flows. This justifies correcting the estimate of z_{no} during flows with y_c, via the gain K_{no}. The dynamics of the observer are then given by

$$\begin{cases} \dot{\hat{z}}_o = G_o(\tau)u_c + P^{-1}H_{c,o}^\top(\tau)R^{-1}(\tau)(y_c - H_{c,o}(\tau)\hat{z}_o) \\ \dot{\hat{z}}_{no} = G_{no}(\tau)u_c + K_{no}P^{-1}H_{c,o}^\top(\tau)R^{-1}(\tau)(y_c - H_{c,o}(\tau)\hat{z}_o) \\ \dot{P} = -\lambda P + H_{c,o}^\top(\tau)R^{-1}(\tau)H_{c,o}(\tau) \\ \dot{\tau} = 1 \\ \\ \hat{z}_o^+ = J_o(\tau)\hat{z}_o + J_{ono}(\tau)\hat{z}_{no} + \mathcal{V}_o u_d \\ \hat{z}_{no}^+ = J_{noo}(\tau)\hat{z}_o + J_{no}(\tau)\hat{z}_{no} + \mathcal{V}_{no}u_d \\ \qquad + L_{d,no}(\tau)(y_d - H_{d,o}(\tau)\hat{z}_o - H_{d,no}(\tau)\hat{z}_{no}) \\ P^+ = P_0 \\ \tau^+ = 0, \end{cases} \tag{34}$$

with jumps triggered at the same time as $\mathcal{H}$ in the same way as observer (2), $P_0 \in \mathcal{S}_{>0}^{n_o}$, K_{no} and $L_{d,no}$ given by Assumption 3, and where $\tau \mapsto R(\tau) \in \mathcal{S}_{>0}^{n_{y,c}}$ is a positive definite weighting matrix that is defined and is continuous on $[0, \tau_M]$ to be chosen arbitrarily. The estimate is then recovered by using (25) on $\hat{z}$ with the global exponential stability (GES) of the estimation error as stated next.

Theorem 3 *Under Assumptions 1, 2, and 3, given any $P_0 \in \mathcal{S}_{>0}^{n_o}$, there exists $\lambda^\star > 0$ such that for any $\lambda > \lambda^\star$, there exist $\rho_1 > 0$ and $\lambda_1 > 0$ such that for any solution $x \in \mathcal{S}_\mathcal{H}(\mathcal{X}_0, \mathcal{U})$ and any solution $(\hat{z}, P, \tau)$ to observer (34) with $P(0,0) = P_0$, $\tau(0,0) = 0$, and jumps triggered at the same time as in x, $(\hat{z}, P, \tau)$ is complete and we have*

$$|x(t,j) - \hat{x}(t,j)| \leq \rho_1 |x(0,0) - \hat{x}(0,0)| e^{-\lambda_1(t+j)}, \qquad \forall (t,j) \in \operatorname{dom} x, \tag{35}$$

with $\hat{x}$ obtained by $\hat{x} = e^{F\tau}\mathcal{D}\hat{z}$ with $\mathcal{D}$ defined in (25)*.*

Proof Consider a solution $x \in \mathcal{S}_\mathcal{H}(\mathcal{X}_0, \mathcal{U})$ and a solution $(\hat{z}, P, \tau)$ to observer (34) with $P(0,0) = P_0$ and $\tau(0,0) = 0$, and jumps triggered at the same time as in x. By Assumption 1, it is complete and so is $(\hat{z}, P, \tau)$. Following (24), define

$$z(t,j) := \mathcal{V} e^{-F\tau(t,j)} x(t,j), \qquad \forall (t,j) \in \operatorname{dom} x, \tag{36}$$

and consider the error $\tilde{z} = (\tilde{z}_o, \tilde{z}_{no}) = (z_o - \hat{z}_o, z_{no} - \hat{z}_{no})$. Because the flow lengths of x are in $\mathcal{I}$ by Assumption 2 after the first j_m jumps only, the proof consists of two parts: first, we use Lemma 2 to show the exponential convergence of $\tilde{z}$ starting at hybrid time (t_{j_m}, j_m) by putting the error dynamics into the appropriate form, and then, we analyze the behavior of the error before (t_{j_m}, j_m) using Lemma 3. Consider first the solution $(\tilde{z}, P, \tau)$ starting from (t_{j_m}, j_m). According to Assumption 2 and since the observer's jumps are synchronized with those of $\mathcal{H}$, $(\tilde{z}, P, \tau)$ is solution to

$$\begin{cases} \dot{\tilde{z}}_o = -P^{-1}H_{c,o}^\top(\tau)R^{-1}(\tau)H_{c,o}(\tau)\tilde{z}_o \\ \dot{\tilde{z}}_{no} = -K_{no}P^{-1}H_{c,o}^\top(\tau)R^{-1}(\tau)H_{c,o}(\tau)\tilde{z}_o \\ \dot{P} = -\lambda P + H_{c,o}^\top(\tau)R^{-1}(\tau)H_{c,o}(\tau) \\ \dot{\tau} = 1 \\ \\ \tilde{z}_o^+ = J_o(\tau)\tilde{z}_o + J_{ono}(\tau)\tilde{z}_{no} \\ \tilde{z}_{no}^+ = (J_{noo}(\tau) - L_{d,no}(\tau)H_{d,o}(\tau))\tilde{z}_o + (J_{no}(\tau) - L_{d,no}(\tau)H_{d,no}(\tau))\tilde{z}_{no} \\ P^+ = P_0 \\ \tau^+ = 0, \end{cases} \tag{37a}$$

with the flow and jump sets

$$\mathbb{R}^{n_o} \times \mathbb{R}^{n_{no}} \times \mathbb{R}^{n_o \times n_o} \times [0, \tau_M], \qquad \mathbb{R}^{n_o} \times \mathbb{R}^{n_{no}} \times \mathbb{R}^{n_o \times n_o} \times \mathcal{I}. \tag{37b}$$

We next perform the change of variables

$$\tilde{\eta} = \tilde{z}_{no} - K_{no}\tilde{z}_o, \tag{38}$$

which transforms the error system (37) into

$$\begin{cases} \dot{\tilde{z}}_o = -P^{-1}H_{c,o}^\top(\tau)R^{-1}(\tau)H_{c,o}(\tau)\tilde{z}_o \\ \dot{\tilde{\eta}} = 0 \\ \dot{P} = -\lambda P + H_{c,o}^\top(\tau)R^{-1}(\tau)H_{c,o}(\tau) \\ \dot{\tau} = 1 \\ \\ \tilde{z}_o^+ = \overline{J}_o(\tau)\tilde{z}_o + J_{ono}(\tau)\tilde{\eta} \\ \tilde{\eta}^+ = (\overline{J}_{noo}(\tau) - L_{d,no}(\tau)\overline{H}_{d,o}(\tau))\tilde{z}_o + J_\eta(\tau)\tilde{\eta} \\ P^+ = 0 \\ \tau^+ = 0, \end{cases} \tag{39}$$

with the same flow and jump sets where $\overline{J}_o(\tau) = J_o(\tau) + J_{ono}(\tau)K_{no}$, $\overline{J}_{noo}(\tau) = J_{noo}(\tau) + J_{no}(\tau)K_{no} - K_{no}J_o(\tau) - K_{no}J_{ono}(\tau)K_{no}$, $\overline{H}_{d,o}(\tau) = H_{d,o}(\tau) + H_{d,no}(\tau)K_{no}$, and $J_\eta(\tau) = J_{no}(\tau) - L_{d,no}(\tau)H_{d,no}(\tau) - K_{no}J_{ono}(\tau)$, with the flow set $\mathbb{R}^{n_o} \times \mathbb{R}^{n_{no}} \times [0, \tau_M]$ and the jump set $\mathbb{R}^{n_o} \times \mathbb{R}^{n_{no}} \times \mathcal{I}$. From Assumption 3, $J_\eta(\tau)$ is Schur for all $\tau \in \mathcal{I}$, and more precisely, there exists $Q_\eta \in \mathcal{S}_{>0}^{n_{no}}$ such that

$$J_\eta^\top(\tau)Q_\eta J_\eta(\tau) - Q_\eta < 0, \qquad \forall \tau \in \mathcal{I}. \tag{40}$$

Using Lemma 2, we proceed to prove the GES of the error $(\tilde{z}_o, \tilde{\eta})$ with respect to the value $(\tilde{z}_o, \tilde{\eta})(t_{j_m}, j_m)$. Then the GES with respect to $(\tilde{z}_o, \tilde{\eta})(0,0)$ is proven using Lemma 3. Last, because the transformations (24) and (38) are linear with $\tau \mapsto e^{F\tau}$ bounded with a strictly positive lower bound on the compact set $[0, \tau_M]$, we obtain (35) observing that $(\tilde{z}_o, \tilde{\eta}) = \begin{pmatrix} \mathcal{V}_o \\ \mathcal{V}_{no} - K_{no}\mathcal{V}_o \end{pmatrix} e^{-F\tau}(x - \hat{x})$. □

Remark 5 Applying Schur's lemma and then the elimination lemma [8] to (33), we see that $Q_{no} \in \mathcal{S}^{n_{no}}_{>0}$ exists if and only if there exists a solution to the LMI

$$\begin{pmatrix} \begin{pmatrix} H_{d,no}(\tau) \\ J_{ono}(\tau) \end{pmatrix}^{\perp\top} Q_{no} \begin{pmatrix} H_{d,no}(\tau) \\ J_{ono}(\tau) \end{pmatrix}^{\perp} & \star \\ Q_{no}J_{no}(\tau) \begin{pmatrix} H_{d,no}(\tau) \\ J_{ono}(\tau) \end{pmatrix}^{\perp} & Q_{no} \end{pmatrix} > 0, \qquad \forall \tau \in \mathcal{I}. \tag{41}$$

If such a Q_{no} is obtained, the gains $L_{d,no}(\cdot)$ and K_{no} are then found by using (33) with Q_{no} known. If $\mathcal{I}$ is infinite, then there is an infinite number of LMIs to solve. Actually, it is worth noting that the exponential term $e^{F\tau}$ contained in all the τ-dependent matrices in (41) can be expanded using residue matrices [12], as

$$\begin{aligned} e^{F\tau} = &\sum_{i=1}^{\sigma_r} \sum_{j=1}^{m_i^r} R_{ij} e^{\lambda_i \tau} \frac{\tau^{j-1}}{(j-1)!} \\ &+ \sum_{i=1}^{\sigma_c} \sum_{j=1}^{m_i^c} 2e^{\Re(\lambda_i)\tau} (\Re(R_{ij}) \cos(\Im(\lambda_i)\tau) - \Im(R_{ij}) \sin(\Im(\lambda_i)\tau)) \frac{\tau^{j-1}}{(j-1)!}, \end{aligned} \tag{42}$$

where σ_r and σ_c are the numbers of distinct real eigenvalues and complex conjugate eigenvalue pairs; m_i^r and m_i^c are the multiplicity of the real eigenvalue λ_i and of the complex conjugate eigenvalue pair λ_i, λ_i^* in the minimal polynomial of F; $R_{ij} \in \mathbb{R}^{n_x \times n_x}$ are matrices corresponding to the residues associated to the partial fraction expansion of $(sI - F)^{-1}$. This in turn allows $e^{F\tau}$ to be written as a finite sum of matrices affine in N scalar functions $\beta_{ij} = e^{\lambda_i \tau} \tau^{j-1}$, $\gamma_{ij} = e^{\Re(\lambda_i)\tau} \cos(\Im(\lambda_i)\tau) \tau^{j-1}$, and $\gamma^*_{ij} = e^{\Re(\lambda_i)\tau} \sin(\Im(\lambda_i)\tau) \tau^{j-1}$. It then implies that (41) can be solved in a polytopic approach, i.e., the LMIs are satisfied for all $\tau \in \mathcal{I}$ compact if they are satisfied at the finite number 2^N of vertices of the polytope formed by these scalar functions when τ varies in $\mathcal{I}$. Alternatively, the LMIs can be solved in a grid-based approach followed by post-analysis of the solution's stability as in [32], possibly with a theoretical proof extended from [24].

3.2 LMI-based observer design in the x-coordinates

In this section, we show that the observer can equivalently be implemented directly in the original x-coordinates, with dynamics given by

$$\begin{cases} \dot{\hat{x}} = F\hat{x} + u_c + \mathcal{P}H_c^\top R^{-1}(\tau)(y_c - H_c\hat{x}) \\ \dot{\mathcal{P}} = \lambda\mathcal{P} + F\mathcal{P} + \mathcal{P}F^\top - \mathcal{P}H_c^\top R^{-1}(\tau)H_c\mathcal{P} \\ \dot{\tau} = 1 \\ \\ \hat{x}^+ = J\hat{x} + u_d + \mathcal{D}_{no}L_{d,no}(\tau)(y_d - H_d\hat{x}) \\ \mathcal{P}^+ = (\mathcal{D}_o + \mathcal{D}_{no}K_{no})P_0^{-1}(\mathcal{D}_o + \mathcal{D}_{no}K_{no})^\top \\ \tau^+ = 0, \end{cases} \tag{43}$$

with jumps still triggered at the same time as $\mathcal{H}$. The GES of the error is proven in Theorem 4.

Theorem 4 *Under Assumptions 1, 2, and 3, given any $P_0 \in \mathcal{S}_{>0}^{n_o}$, there exists $\lambda^\star > 0$ such that for any $\lambda > \lambda^\star$, there exist $\rho_1 > 0$ and $\lambda_1 > 0$ such that for any solution $x \in \mathcal{S}_\mathcal{H}(\mathcal{X}_0, \mathcal{U})$ and for any solution $(\hat{x}, \mathcal{P}, \tau)$ to observer (43) with $\mathcal{P}(0,0) = (\mathcal{D}_o + \mathcal{D}_{no}K_{no})P_0^{-1}(\mathcal{D}_o + \mathcal{D}_{no}K_{no})^\top$, $\tau(0,0) = 0$, and jumps triggered at the same time as in x, $(\hat{x}, \mathcal{P}, \tau)$ is complete and we have*

$$|x(t,j) - \hat{x}(t,j)| \leq \rho_1|x(0,0) - \hat{x}(0,0)|e^{-\lambda_1(t+j)}, \qquad \forall (t,j) \in \operatorname{dom} x. \tag{44}$$

Proof Pick a solution $x \in \mathcal{S}_\mathcal{H}(\mathcal{X}_0, \mathcal{U})$ and a solution $(\hat{\zeta}, \mathcal{P}, \tau)$ to observer (43) with $\mathcal{P}(0,0) = (\mathcal{D}_o + \mathcal{D}_{no}K_{no})P_0^{-1}(\mathcal{D}_o + \mathcal{D}_{no}K_{no})^\top$, and $\tau(0,0) = 0$, with jumps triggered at the same time as in x. Consider $(\hat{z}, P, \underline{\tau})$ solution to observer (34), with $\hat{z}(0,0) = \mathcal{V}\hat{x}(0,0)$, $P(0,0) = P_0$, and $\underline{\tau}(0,0) = 0$, with jumps triggered at the same time as in x. First, notice from its dynamics that $\underline{\tau} = \tau$. Then, applying Theorem 3, we get

$$|x(t,j) - \underline{\hat{x}}(t,j)| \leq \rho_1|x(0,0) - \underline{\hat{x}}(0,0)|e^{-\lambda_1(t+j)}, \qquad \forall (t,j) \in \operatorname{dom} x,$$

where $\underline{\hat{x}} = e^{F\tau}\mathcal{D}\hat{z}$. The proof consists in showing that $\underline{\hat{x}} = \hat{x}$, thus obtaining (44). Observe that

$$\dot{\underline{\hat{x}}} = F\underline{\hat{x}} + u_c + L_c(P,\tau)(y_c - H_c\underline{\hat{x}}), \tag{45a}$$

during flows and

$$\underline{\hat{x}}^+ = J\underline{\hat{x}} + u_d + L_d(\tau)(y_d - H_d\underline{\hat{x}}), \tag{45b}$$

at jumps where

$$\begin{aligned} L_c(P,\tau) &= e^{F\tau}\mathcal{D}\begin{pmatrix} P^{-1}H_{c,o}^\top(\tau)R^{-1}(\tau) \\ K_{no}P^{-1}H_{c,o}^\top(\tau)R^{-1}(\tau) \end{pmatrix} \\ &= e^{F\tau}(\mathcal{D}_o + \mathcal{D}_{no}K_{no})P^{-1}\mathcal{D}_o^\top e^{F^\top\tau}H_c^\top R^{-1}(\tau), \end{aligned} \tag{46a}$$

and

$$L_d(\tau) = \mathcal{D}\begin{pmatrix} 0 \\ L_{d,no}(\tau) \end{pmatrix} = \mathcal{D}_{no} L_{d,no}(\tau). \tag{46b}$$

From (23), $L_c(P,\tau)$ can be re-written as $L_c(P,\tau) = \underline{\mathcal{P}} H_c^\top R^{-1}(\tau)$ where

$$\underline{\mathcal{P}} = e^{F\tau}(\mathcal{D}_o + \mathcal{D}_{no}K_{no})P^{-1}(\mathcal{D}_o + \mathcal{D}_{no}K_{no})^\top e^{F^\top \tau}. \tag{47}$$

Calculating $\dot{\underline{\mathcal{P}}}$ while noting that the time derivative of P^{-1} is $-P^{-1}\dot{P}P^{-1}$, we obtain the same flow/jump dynamics as $\mathcal{P}$ in observer (43). Besides, $\underline{\mathcal{P}}(0,0) = \mathcal{P}(0,0)$, so $\underline{\mathcal{P}} = \mathcal{P}$ thanks to the uniqueness of solutions. We deduce that $\underline{\hat{x}}$ follows the same dynamics as $\hat{x}$, and since

$$\underline{\hat{x}}(0,0) = \mathcal{D}'\hat{z}(0,0) = \hat{x}(0,0),$$

we have $\underline{\hat{x}} = \hat{x}$, which concludes the proof. □

It is interesting to see that the observability provided at jumps by the fictitious output in the non-observable subspace $\mathcal{D}_{no}$ is stored into $\mathcal{P}$ at jumps. This allows the use of y_c to correct the estimate in the non-observable subspace during flows, while the Riccati dynamics of $\mathcal{P}$ instead excites only the observable directions provided by (27). In terms of implementation, observer (43) is of a larger dimension than that in observer (34) but it allows us to avoid the online inversion of the change of variables. Actually, observer (43) has the same dimension and the same flow dynamics as the Kalman-like observer proposed in [28]. The difference lies in (i) the jump dynamics, which here contains *a priori* gains K_{no} and $L_{d,no}$ instead of dynamic gains computed online via $\mathcal{P}$, and (ii) the quadratic *detectability* assumption (3) which replaces the UCO in [28].

Remark 6 Denote $L_d(\tau) = \mathcal{D}_{no}L_{d,no}(\tau)$ and observe that $L_{d,no}(\tau) = \mathcal{V}_{no}L_d(\tau)$. Similarly, denote $K'_{no} = \mathcal{D}_{no}K_{no}$ and observe that $K_{no} = \mathcal{V}_{no}K'_{no}$. The conditions in Assumption 3 for the design of $L_{d,no}(\cdot)$ and K_{no} are equivalent to solving for $L_d(\cdot)$ and K'_{no} directly in the x-coordinates and for all $\tau \in \mathcal{I}$,

$$\star^\top Q_{no}\mathcal{V}_{no}(J - L_d(\tau)H_d - K'_{no}\mathcal{V}_o J)e^{F\tau}\mathcal{D}_{no} - Q_{no} < 0, \tag{48a}$$

$$\mathcal{V}_o L_d(\tau) = 0, \tag{48b}$$

$$\mathcal{V}_o K'_{no} = 0. \tag{48c}$$

Actually, these are projections of more conservative LMIs (where variables have the full dimensions corresponding to the plant) onto the observable subspaces.

Example 4 The spiking behavior of a cortical neuron may be modeled with state $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$ as

$$\begin{cases} \dot{\eta} = \begin{pmatrix} 0.04\eta_1^2 + 5\eta_1 + 140 - \eta_2 + I_{\text{ext}} \\ a(b\eta_1 - \eta_2) \end{pmatrix} & \text{when } \eta_1 \le v_m \\ \eta^+ = \begin{pmatrix} c \\ \eta_2 + d \end{pmatrix} & \text{when } \eta_1 = v_m \end{cases} \tag{49}$$

where η_1 is the membrane potential, η_2 is the recovery variable, and I_{ext} represents the (constant) synaptic current or injected DC current [14]. We pick here the parameters as $I_{\text{ext}} = 10$, $a = 0.02$, $b = 0.2$, $c = -55$, $d = 4$, and $v_m = 30$ (all in appropriate units), thus characterizing the neuron type and its firing pattern [14]. The solutions of this system are known to have a dwell time with flow lengths remaining in a compact set $\mathcal{I} = [\tau_m, \tau_M]$ where $\tau_m > 0$, and the jump times can be detected from the discontinuities of the measured output $y_c = \eta_1$. Since $y_c = \eta_1$ is known during flows, we treat $0.04\eta_1^2 + 140 + I_{\text{ext}}$ as a known term that can be compensated using output injection with $u_c = (0.04y_c^2 + 140 + I_{\text{ext}}, 0)$. On the other hand, in the jump map of (49), we assume c and d are unknown and include them in the state to be estimated along with (η_1, η_2). We show here that, using the decomposition in this paper, we can design a flow-based observer using the knowledge of $y_c = \eta_1$ during flows only, although the flow dynamics are not observable. In other words, we take $y_d = 0$, which comes back to not using any output injection at jumps. We re-model system (49) extended with (c, d) into the form (1) with $x = (x_1, x_2, x_3, x_4) = (\eta_1, \eta_2, c, d) \in \mathbb{R}^4$ and the matrices

$$F = \begin{pmatrix} 5 & -1 & 0 & 0 \\ ab & -a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad H_c = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad H_d = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}. \quad (50)$$

We see that x_1 and x_2 are instantaneously observable from y_c and following (24), we get $z_o = \begin{pmatrix} \mu_1(\tau) & \mu_2(\tau) & 0 & 0 \\ \mu_3(\tau) & \mu_4(\tau) & 0 & 0 \end{pmatrix} x \in \mathbb{R}^2$, $z_{no} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x = (x_3, x_4) \in \mathbb{R}^2$, and the form (26) with matrices $J_o(\tau) = \begin{pmatrix} 0 & 0 \\ \mu_5(\tau) & \mu_6(\tau) \end{pmatrix}$, $J_{ono}(\tau) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $J_{noo}(\tau) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $J_{no}(\tau) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $H_{c,o}(\tau) = \begin{pmatrix} \mu_7(\tau) & \mu_8(\tau) \end{pmatrix}$, $H_{d,o}(\tau) = \begin{pmatrix} 0 & 0 \end{pmatrix}$, and $H_{d,no}(\tau) = \begin{pmatrix} 0 & 0 \end{pmatrix}$, where μ_i, $i = 1, 2, \ldots, 8$ are known exponential functions of τ. We see that for any $\tau \in \mathcal{I}$, z_{no} cannot be seen from y_d because $H_{d,no}(\tau) = 0$, but it can be accessed through z_o via $J_{ono}(\tau)$ (*hidden* dynamics). Solving (33), we obtain $K_{no} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ with any $Q_{no} \in \mathcal{S}^1_{>0}$ and any $L_{d,no}(\cdot)$. Let us then take $L_{d,no} = 0$. In this application, we see that (x_3, x_4) is estimated thanks to its interaction with z_o at jumps, the latter being estimated during flows, namely we exploit the *hidden* observability analyzed in Theorem 1. Then, an LMI-based observer as in observer (34) or (43), with the mentioned gains K_{no} and $L_{d,no}$, a weighting matrix taken as $R = \text{Id}$, and a large enough λ, can be designed for this system.

4 KKL-based observer design from discrete uniform backward distinguishability in observability decomposition

The idea of this section is to replace the LMI-based design of the observer for the z_{no} part with a *systematic* KKL-based one. To do that, following [27], we exploit a discrete-time KKL observer design for the discrete-time linear time-varying system (32), for which observability is assumed as suggested by Theorem 2. The main reasons for relying on this discrete-time design, as opposed to a discrete-time Kalman(-like) design, for instance, are two-fold:

- Compared to a Kalman design [11], it admits a *strict* Input-to-State Stable (ISS) Lyapunov function, allowing for an interconnection with the high-gain flow-based observer of z_o;
- Compared to a Kalman-like design [34], its gain with respect to the fictitious output in system (32) is *constant*, allowing us to re-produce in the analysis a similar change of variables as (38) in the LMI-based design of Theorem 3.

For this method, we make the following assumption.

Assumption 4 Given a subset $\mathcal{I} \subset [0, +\infty)$, the matrix $J_{no}(\tau)$ is uniformly invertible for all $\tau \in \mathcal{I}$, i.e., there exists $s_J > 0$ such that $\|J_{no}^{-1}(\tau)\| \leq s_J$ for all $\tau \in \mathcal{I}$.

Remark 7 Contrary to discrete-time systems, the jump map of a hybrid system has little chance of being invertible since it is not a discretization of some continuous-time dynamics. However, here thanks to the transformation (24), the flow dynamics are somehow merged with the jumps and thus it is reasonable to expect $J_{no}(\tau)$ to be invertible for all $\tau \in \mathcal{I}$. On the other hand, the ability to deal with the non-invertibility of the dynamics has been studied in [19] (in the linear context). With hybrid systems, it can even be coped with using the non-uniqueness of system representation (see for instance [28, Example 2]). Note that while simulations suggest that the invertibility of the jump dynamics is typically not necessary in the Kalman-like design of [28], and may only be for theoretical analysis, it is needed here to implement observer (60) below, namely to compute T^+ correctly.

4.1 Discrete-time KKL observer design for system (32)

Consider the discrete-time system (32) with $\tau_k \in \mathcal{I}$ for all $k \in \mathbb{N}$. Following the KKL spirit, we look for a transformation $(T_k)_{k\in\mathbb{N}}$ such that in the new coordinates $\eta_k := T_k z_{no,k}$, system (32) follows the dynamics

$$\eta_{k+1} = \gamma A \eta_k + B y_k, \tag{51}$$

where $A \in \mathbb{R}^{n_\eta \times n_\eta}$ is Schur, $B \in \mathbb{R}^{n_\eta \times n_{d,\text{ext}}}$ where $n_{d,\text{ext}} := n_{y,d} + n_o$ such that the discrete-time pair (A, B) is controllable, and $\gamma \in (0, 1]$ is a design parameter. It then

follows that the transformation $(T_k)_{k\in\mathbb{N}}$ must be such that for every $(\tau_k)_{k\in\mathbb{N}}$ with $\tau_k \in \mathcal{I}$ for all $k \in \mathbb{N}$,

$$T_{k+1}J_{no}(\tau_k) = \gamma A T_k + B H_{d,\text{ext}}(\tau_k), \tag{52}$$

with $H_{d,\text{ext}}(\cdot)$ defined in system (32) in Theorem 1. The interest of this form is that an observer for system (32) in the η-coordinates is a simple filter of the output

$$\hat{\eta}_{k+1} = \gamma A \hat{\eta}_k + B y_k, \tag{53}$$

making the error $\tilde{\eta}_k = \eta_k - \hat{\eta}_k$ verify

$$\tilde{\eta}_{k+1} = \gamma A \tilde{\eta}_k, \tag{54}$$

and thus is exponentially stable. Then, if $(T_k)_{k\in\mathbb{N}}$ is uniformly left-invertible, the estimate defined by $\hat{z}_{no,k} = T_k^* \hat{\eta}$, where $(T_k^*)_{k\in\mathbb{N}}$ is a bounded sequence of left inverses of $(T_k)_{k\in\mathbb{N}}$ verifying $T_k^* T_k = \text{Id}$ for all $k \geq k^\star$ for some $k^\star \in \mathbb{N}$, is such that for any $(\tau_k)_{k\in\mathbb{N}}$ with $\tau_k \in \mathcal{I}$ for all $k \in \mathbb{N}$, there exist $c_1 > 0$ and $c_2 \in (0,1)$ such that for any initial conditions $z_{no,0}$ and $\hat{z}_{no,0}$ and for all $k \in \mathbb{N}$,

$$|z_{no,k} - \hat{z}_{no,k}| \leq c_1 c_2^k |z_{no,0} - \hat{z}_{no,0}|. \tag{55}$$

From [27, Corollary 1], we know that this is possible under the uniform backward distinguishability of system (32) as defined next.

Definition 5 (Uniform backward distinguishability of system (32)**)**

Given $(\tau_k)_{k\in\mathbb{N}}$ with $\tau_k \in \mathcal{I}$ for all $k \in \mathbb{N}$, system (32) is *uniformly backward distinguishable* if for each $i \in \{1, 2, \ldots, n_{d,\text{ext}}\}$ its output dimension, there exists $m_i \in \mathbb{N}_{>0}$ such that there exists $\alpha_m > 0$ such that for all $k \geq \max_i m_i$, the backward distinguishability matrix sequence $(\mathbb{O}_k^{bw})_{k\in\mathbb{N}}$ defined as

$$\mathbb{O}_k^{bw} = (\mathbb{O}_{1,k}^{bw}, \mathbb{O}_{2,k}^{bw}, \ldots, \mathbb{O}_{n_{d,\text{ext}},k}^{bw}) \in \mathbb{R}^{(\sum_{i=1}^{n_{d,\text{ext}}} m_i)\times n_{no}}, \tag{56a}$$

where

$$\mathbb{O}_{i,k}^{bw} := \begin{pmatrix} H_{d,\text{ext},i}(\tau_{k-1})J_{no}^{-1}(\tau_{k-1}) \\ H_{d,\text{ext},i}(\tau_{k-2})J_{no}^{-1}(\tau_{k-2})J_{no}^{-1}(\tau_{k-1}) \\ \ldots \\ H_{d,\text{ext},i}(\tau_{k-(m_i-1)})J_{no}^{-1}(\tau_{k-(m_i-1)})\ldots J_{no}^{-1}(\tau_{k-1}) \\ H_{d,\text{ext},i}(\tau_{k-m_i})J_{no}^{-1}(\tau_{k-m_i})J_{no}^{-1}(\tau_{k-(m_i-1)})\ldots J_{no}^{-1}(\tau_{k-1}) \end{pmatrix}, \tag{56b}$$

where $H_{d,\text{ext},i}(\cdot)$ denotes the i^{th} row of the extended output matrix $H_{d,\text{ext}}(\cdot)$ of system (32), has full rank and satisfies $\mathbb{O}_k^{bw\top}\mathbb{O}_k^{bw} \geq \alpha_m \,\text{Id} > 0$.

Note that the forward version $(\mathbb{O}_k^{fw})_{k\in\mathbb{N}}$ of $(\mathbb{O}_k^{bw})_{k\in\mathbb{N}}$, which is much easier to compute, can be considered as in [27, Remark 3], under the additional assumptions that the m_i are the same for $i \in \{1, 2, \ldots, n_{d,\text{ext}}\}$ and J_{no} is uniformly invertible,

namely there exists $c_J > 0$ such that $(J_{no}^{-1}(\tau_k))^\top J_{no}^{-1}(\tau_k) \geq c_J \,\mathrm{Id}$ for all $k \in \mathbb{N}$. Theorem 5, which is a particular case of [27, Theorems 2 and 3], states the existence and uniform left-invertibility of $(T_k)_{k\in\mathbb{N}}$ solving (52).

Theorem 5 *Consider $n_\eta \in \mathbb{N}_{>0}$, $\gamma > 0$, and a pair $(A, B) \in \mathbb{R}^{n_\eta \times n_\eta} \times \mathbb{R}^{n_\eta \times n_{d,\mathrm{ext}}}$. Under Assumption 4, for any $(\tau_k)_{k\in\mathbb{N}}$ with $\tau_k \in \mathcal{I}$ for all $k \in \mathbb{N}$ and any $T_0 \in \mathbb{R}^{n_\eta \times n_{no}}$, the transformation $(T_k)_{k\in\mathbb{N}}$ initialized as T_0 and satisfying* (52) *uniquely exists under the following closed form*

$$T_k = (\gamma A)^k T_0 \prod_{p=0}^{k-1} J_{no}^{-1}(\tau_p) + \sum_{p=0}^{k-1} (\gamma A)^{k-p-1} B H_{d,\mathrm{ext}}(\tau_p) \prod_{r=p}^{k-1} J_{no}^{-1}(\tau_r). \tag{57}$$

Moreover, for any $(\tau_k)_{k\in\mathbb{N}}$ with $\tau_k \in \mathcal{I}$ for all $k \in \mathbb{N}$ such that system (32) *is uniformly backward distinguishable for some $m_i \in \mathbb{N}_{>0}$, $i \in \{1, 2, \ldots, n_{d,\mathrm{ext}}\}$ and any controllable pairs $(\tilde{A}_i, \tilde{B}_i) \in \mathbb{R}^{m_i \times m_i} \times \mathbb{R}^{m_i}$ with $\tilde{A}_i$ Schur, $i \in \{1, 2, \ldots, n_{d,\mathrm{ext}}\}$, there exists $0 < \gamma^\star \leq 1$ such that for any $0 < \gamma < \gamma^\star$, there exist $k^\star \in \mathbb{N}_{>0}$, $\underline{c}_T > 0$, and $\overline{c}_T > 0$ such that $(T_k)_{k\in\mathbb{N}}$ in* (57) *with*

$$A = \mathrm{diag}(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_{n_{d,\mathrm{ext}}}) \in \mathbb{R}^{n_\eta \times n_\eta}, \tag{58a}$$

$$B = \mathrm{diag}(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_{n_{d,\mathrm{ext}})} \in \mathbb{R}^{n_\eta \times n_{d,\mathrm{ext}}}, \tag{58b}$$

where $n_\eta := \sum_{i=1}^{n_{d,\mathrm{ext}}} m_i$, verifies $T_k^\top T_k \geq \underline{c}_T \,\mathrm{Id}$ for all $k \geq k^\star$ and $\|T_k\| \leq \overline{c}_T$ for all $k \in \mathbb{N}$.

In other words, for γ sufficiently small, $(T_k)_{k\in\mathbb{N}}$ is uniformly left-invertible and upper-bounded after $k^\star$. Note that the dependence of $\gamma^\star$, $\underline{c}_T$, $\overline{c}_T$, and $k^\star$ on $(\tau_k)_{k\in\mathbb{N}}$ and T_0 is only through α_m and the m_i coming from the uniform backward distinguishability and the upper bounds of T_0 and $J_{no}^{-1}(\cdot)$. Note also that the (non-uniform) injectivity of T can be obtained from non-uniform distinguishability conditions, as seen in [27, Example 1], which may suffice in some cases to ensure the convergence of the KKL observer.

Proof These results are the particular case of [27, Theorems 2 and 3]. Note that by continuity, $\tau \mapsto J_{no}(\tau)$ is uniformly invertible on $\mathcal{I}$ because $\mathcal{I}$ is compact, and $\tau \mapsto H_{d,no}(\tau)$ is uniformly bounded on the compact set $[0, \tau_M]$. □

Remark 8 Interestingly, Definition 5, whose nonlinear version is defined in [27, Definition 1], coincides with the *uniform complete observability* (UCO) condition required by the discrete-time Kalman(-like) filter (see [11, Condition (13)], [34, Assumption 3], or [33, Definition 3]), on the pair $\left(J_{no}(\tau), \begin{pmatrix} H_{d,no}(\tau) \\ J_{ono}(\tau) \end{pmatrix}\right)$, which is the discrete-time version of that in Definition 4 above, i.e., there exist $m_i \in \mathbb{N}_{>0}$ and $c_o > 0$ such that for all $(\tau_k)_{k\in\mathbb{N}}$ with $\tau_k \in \mathcal{I}$ for all $k \in \mathbb{N}$, we have for all $k \geq \max_i m_i$,

$$\sum_{i=1}^{n_{d,\mathrm{ext}}} \sum_{p=k-m_i}^{k-1} \star^\top H_{d,\mathrm{ext},i}(\tau_p) J_{no}^{-1}(\tau_p) \ldots J_{no}^{-1}(\tau_{k-2}) J_{no}^{-1}(\tau_{k-1}) \geq c_o \,\mathrm{Id} > 0. \tag{59}$$

It is thus interesting to compare these two discrete-time observers. In terms of dimensions, the complexity of the Kalman filter is $\frac{n_{no}(n_{no}+1)}{2} + n_{no}$, while that of the KKL observer is $\left(\sum_{i=1}^{n_{d,\text{ext}}} m_i\right) n_{no} + \sum_{i=1}^{n_{d,\text{ext}}} m_i$. Therefore, the Kalman filter is advantageous in dimension compared to the KKL observer. However, the advantage of the latter (besides being applicable in the nonlinear context) is that there exists a strict ISS Lyapunov function of quadratic form that allows us to prove exponential ISS, unlike the discrete-time Kalman filter [11] whose Lyapunov function is not strict. This advantage is then exploited in the next part, where we design a KKL-based observer to estimate the z_{no} part in the hybrid system (26) while the error in z_o is seen as a disturbance. Note that a discrete-time Kalman-like observer [34] could seem like a possible alternative to the KKL-based one since it also exhibits a strict Lyapunov function. However, the gain multiplied with the fictitious output in the observer must be *constant* during flows for us to perform the analysis (see K_{no} in (33)), which is not the case in a Kalman-like observer (unless the pair $(J_{no}(\tau), H_{d,no}(\tau))$ at the jump times is UCO, so without the need for the fictitious output). This is ensured in KKL design since it relies on a transformation into a linear time-invariant form (see below in the proof of Theorem 6).

Next, in Section 4.2, we exploit this section's results for the hybrid system (26).

4.2 KKL-based observer design for system (26)

The KKL-based observer we propose for system (26) has the form

$$\begin{cases} \dot{\hat{z}}_o = G_o(\tau)u_c + P^{-1}H_{c,o}^\top(\tau)R^{-1}(\tau)(y_c - H_{c,o}(\tau)\hat{z}_o) \\ \dot{\hat{\eta}} = (TG_{no}(\tau) - B_{ono}G_o(\tau))u_c \\ \dot{P} = -\lambda P + H_{c,o}^\top(\tau)R^{-1}(\tau)H_{c,o}(\tau) \\ \dot{T} = 0 \\ \dot{\tau} = 1 \\ \\ \hat{z}_o^+ = J_o(\tau)\hat{z}_o + J_{ono}(\tau)\hat{z}_{no} + \mathcal{V}_o u_d \\ \hat{\eta}^+ = (T^+J_{noo}(\tau) + \gamma AB_{ono} - B_{ono}J_o(\tau))\hat{z}_o \\ \qquad + \gamma A\hat{\eta} + (T^+\mathcal{V}_{no} - B_{ono}\mathcal{V}_o)u_d + B_{d,no}(y_d - H_{d,o}(\tau)\hat{z}_o) \\ P^+ = P_0 \\ T^+ = (\gamma AT + B_{d,no}H_{d,no}(\tau) + B_{ono}J_{ono}(\tau))\,\text{sat}_{s_J}(J_{no}^\dagger(\tau)) \\ \tau^+ = 0, \end{cases} \tag{60a}$$

with

$$\hat{z}_{no} = \text{sat}_{s_T}(T^\dagger)(\hat{\eta} + B_{ono}\hat{z}_o), \tag{60b}$$

with jumps triggered at the same time as $\mathcal{H}$ in the same way as observer (2), $P_0 \in \mathcal{S}_{>0}^{n_o}$, $\tau \mapsto R(\tau) \in \mathcal{S}_{>0}^{n_{y,c}}$ is a positive definite weighting matrix that is defined and is continuous on $[0, \tau_M]$ to be chosen only for design purpose, $\gamma \in (0, 1]$, and $(A, B_{d,no}, B_{ono})$ are design parameters to be chosen. Following similar reasoning

as in Section 3.1, those dynamics are picked so that T coincides with the $(T_k)_{k\in\mathbb{N}}$ in Section 4.1 at jumps and so that the corresponding discrete-time KKL error dynamics (54) appear after a certain change of coordinates, modulo some errors on z_o (see (66) below). The difficulty comes here from the fact that the discrete-time output y_k in the discrete-time KKL dynamics (51) is not fully available at jumps since it contains the fictitious output.

Assumption 5 Given j_m defined in Assumption 2, there exist $m_i \in \mathbb{N}_{>0}$ for each $i = 1, 2, \ldots, n_{d,\text{ext}}$ and $\alpha_m > 0$ such that for every complete solution $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0, \mathcal{U})$, the sequence of flow lengths $(\tau_j)_{j\in\mathbb{N}, j\geq j_m}$ where $\tau_j = t_{j+1} - t_j$ is such that system (32), scheduled with that $(\tau_j)_{j\in\mathbb{N}, j\geq j_m}$, is uniformly backward distinguishable with the parameters m_i and α_m following Definition 5.

Theorem 6 *Suppose Assumptions 1, 2, 4, and 5 hold. Define* $n_\eta := \sum_{i=1}^{n_{d,\text{ext}}} m_i$ *and consider for each* $i \in \{1, 2, \ldots, n_{d,\text{ext}}\}$ *a controllable pair* $(\tilde{A}_i, \tilde{B}_i) \in \mathbb{R}^{m_i\times m_i} \times \mathbb{R}^{m_i}$ *with* $\tilde{A}_i$ *Schur. Define then*

$$A = \text{diag}(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_{n_{d,\text{ext}}}) \in \mathbb{R}^{n_\eta\times n_\eta}, \tag{61a}$$

$$B_{d,no} = \text{diag}(\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_{n_{y,d}}) \in \mathbb{R}^{n_\eta\times n_{y,d}}, \tag{61b}$$

$$B_{ono} = \text{diag}(\tilde{B}_{n_{y,d}+1}, \tilde{B}_{n_{y,d}+2}, \ldots, \tilde{B}_{n_{d,\text{ext}}}) \in \mathbb{R}^{n_\eta\times n_o}. \tag{61c}$$

Given any $\lambda_1 > 0$*, any* $P_0 \in \mathcal{S}^{n_o}_{>0}$*, and any* $T_0 \in \mathbb{R}^{n_\eta\times n_{no}}$*, there exists* $0 < \gamma^\star \leq 1$ *such that there exists* $\lambda^\star > 0$ *such that for any* $0 < \gamma < \gamma^\star$ *and any* $\lambda > \lambda^\star$*, there exist* $\bar{j} \in \mathbb{N}_{>0}$*, saturation levels* $s_T > 0$*,* $s_J > 0$*, and scalar* $\rho_1 > 0$ *such that for any solution* $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0, \mathcal{U})$ *and any solution* $(\hat{z}_o, \hat{\eta}, P, T, \tau)$ *to observer* (60) *with* $P(0,0) = P_0$*,* $T(0,0) = T_0$*,* $\tau(0,0) = 0$*, the chosen* $(A, B_{d,no}, B_{ono})$*,* sat_{s_T} *at level* s_T*,* sat_{s_J} *at level* s_J*, and jumps triggered at the same time as in* x*,* $(\hat{z}_o, \hat{\eta}, P, T, \tau)$ *is complete and we have*

$$|x(t,j) - \hat{x}(t,j)| \leq \rho_1 |x(t_{\bar{j}}, \bar{j}) - \hat{x}(t_{\bar{j}}, \bar{j})| e^{-\lambda_1(t+j)}, \qquad \forall (t,j) \in \text{dom}\, x, j \geq \bar{j}, \tag{62}$$

with $\hat{x}$ *obtained by* $\hat{x} = e^{F\tau}\mathcal{D}\hat{z}$ *with* $\mathcal{D}$ *defined in* (25)*.*

The parameter $\bar{j}$ is related to j_m in Assumption 4 and to the number of jumps needed to get the uniform left-injectivity of $(T_k)_{k\in\mathbb{N}}$ in Theorem 5.

Proof First, according to Assumption 2, the flow lengths of solutions in $\mathcal{S}_{\mathcal{H}}(\mathcal{X}_0, \mathcal{U})$ are in the compact set $[0, \tau_M]$, so there exists $\overline{c}_{T,m} > 0$ such that for every solution $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0, \mathcal{U})$ and any solution $(\hat{z}_o, \hat{\eta}, P, T, \tau)$ to observer (60) with $P(0,0) = P_0$, $T(0,0) = T_0$, $\tau(0,0) = 0$, and any $\gamma \in (0, 1]$, with jumps triggered at the same time as in x, we have $\|T(t,j)\| \leq \overline{c}_{T,m}$ for all $(t,j) \in \text{dom}\, x$ such that $j \leq j_m$. Then, from Assumptions 2, 4, and 5, and according to Theorem 5 starting from jump j_m, there exists $0 < \gamma_0^\star \leq 1$ such that for all $0 < \gamma < \gamma_0^\star$, there exist $j^\star \in \mathbb{N}_{>0}$, $\underline{c}_T > 0$, and $\overline{c}_T > 0$ such that for every solution $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0, \mathcal{U})$, the solution $(T_j)_{j\in\mathbb{N}, j\geq j_m}$ to (52) with $\tau_j = t_{j+1} - t_j$, initialized at any T_{j_m} verifying $\|T_{j_m}\| \leq \overline{c}_{T,m}$, is uniformly left-invertible for all $j \geq j_m + j^\star$ and uniformly bounded for all $j \geq j_m$, i.e.,

$$T_j^\top T_j \geq \underline{c}_T \,\mathrm{Id}, \qquad \forall j \geq j_m + j^\star, \qquad \|T_j\| \leq \overline{c}_T, \qquad \forall j \geq j_m.$$

It follows that $(T_j^\dagger)^\top T_j^\dagger \leq \frac{1}{\underline{c}_T}\,\mathrm{Id}$, for all $j \geq j_m + j^\star$, and there exists a saturation level $s_T > 0$ such that $\mathrm{sat}_{s_T}(T_j^\dagger) = T_j^\dagger$ for all $j \geq j_m + j^\star$. Pick $0 < \gamma < \gamma_0^\star$ and consider a solution $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0, \mathcal{U})$ and a solution $(\hat{z}_o, \hat{\eta}, P, T, \tau)$ to observer (60) with $P(0,0) = P_0$, $T(0,0) = T_0$, $\tau(0,0) = 0$, the chosen $(A, B_{d,no}, B_{ono})$, the saturation level s_T, and jumps triggered at the same time as in x. Following (24), define

$$z(t,j) := \mathcal{V} e^{-F\tau(t,j)} x(t,j), \qquad \forall (t,j) \in \mathrm{dom}\, x,$$

and consider the error $\tilde{z} = (\tilde{z}_o, \tilde{z}_{no}) = (z_o - \hat{z}_o, z_{no} - \hat{z}_{no})$. As justified above, $\|T(t_{j_m}, j_m)\| \leq \overline{c}_{T,m}$. Since $\dot{T} = 0$ during flows, the sequence $(T(t_j, j))_{j\in\mathbb{N}, j\geq j_m}$ coincides with the sequence $(T_j)_{j\in\mathbb{N}, j\geq j_m}$ solution to (52) with $\tau_j = t_{j+1} - t_j$ for all $j \in \mathbb{N}$ with $j \geq j_m$. Therefore,

$$T^\top(t,j) T(t,j) \geq \underline{c}_T \,\mathrm{Id}, \qquad \forall (t,j) \in \mathrm{dom}\, x,\ j \geq j_m + j^\star. \tag{63}$$

We now use Corollary 1 to show exponential convergence of $\tilde{z}$ starting at hybrid time $(t_{j_m+j^\star}, j_m + j^\star)$ by putting the error dynamics into the appropriate form. In order to exploit the KKL design, we define

$$\eta(t,j) = T(t,j) z_{no}(t,j) - B_{ono} z_o(t,j), \qquad \forall (t,j) \in \mathrm{dom}\, x. \tag{64}$$

Notice that η verifies $\dot{\eta} = (TG_{no}(\tau) - B_{ono}G_o(\tau))u_c$ during flows. From Assumptions 2 and 4, after hybrid time (t_{j_m}, j_m) we get $\mathrm{sat}_{s_J}(J_{no}^\dagger(\tau)) = J_{no}^\dagger(\tau)$ and $J_{no}^\dagger(\tau) J_{no}(\tau) = \mathrm{Id}$ so that at jumps,

$$\begin{aligned}\eta^+ = (T^+ J_{noo}(\tau) + \gamma A B_{ono} - B_{ono} J_o(\tau)) z_o \\ + \gamma A \eta + (T^+ \mathcal{V}_{no} - B_{ono}\mathcal{V}_o) u_d + B_{d,no} H_{d,no}(\tau) z_{no}.\end{aligned} \tag{65}$$

Given the dynamics of $\hat{\eta}$ in observer (60), the error $\tilde{\eta} := \eta - \hat{\eta}$ verifies $\dot{\tilde{\eta}} = 0$ during flows and at jumps (after hybrid time (t_{j_m}, j_m)),

$$\tilde{\eta}^+ = (T^+ J_{noo}(\tau) + \gamma A B_{ono} - B_{ono} J_o(\tau) - B_{d,no} H_{d,no}(\tau)) \tilde{z}_o + \gamma A \tilde{\eta}, \tag{66}$$

which is a contracting dynamics in $\tilde{\eta}$. After $(t_{j_m+j^\star}, j_m + j^\star)$, we have (i) $T^\dagger T = \mathrm{Id}$ so that $z_{no} = T^\dagger \eta + T^\dagger B_{ono} z_o$, and (ii) $\mathrm{sat}_{s_T}(T^\dagger) = T^\dagger$ so that $\hat{z}_{no} = T^\dagger \hat{\eta} + T^\dagger B_{ono} \hat{z}_o$. Therefore, after $(t_{j_m+j^\star}, j_m + j^\star)$, $(\tilde{z}_o, \tilde{\eta}, P, \tau)$ is solution to

$$\begin{cases} \dot{\tilde{z}}_o = -P^{-1}H_{c,o}^\top(\tau)R^{-1}(\tau)H_{c,o}(\tau)\tilde{z}_o \\ \dot{\tilde{\eta}} = 0 \\ \dot{P} = -\lambda P + H_{c,o}^\top(\tau)R^{-1}(\tau)H_{c,o}(\tau) \\ \dot{\tau} = 1 \\ \\ \tilde{z}_o^+ = \overline{J}_o(\tau)\tilde{z}_o + J_{ono}(\tau)T^\dagger\tilde{\eta} \\ \tilde{\eta}^+ = (T^+J_{noo}(\tau) + \gamma AB_{ono} - B_{ono}J_o(\tau) - B_{d,no}H_{d,no}(\tau))\tilde{z}_o + \gamma A\tilde{\eta} \\ P^+ = P_0 \\ \tau^+ = 0, \end{cases} \tag{67a}$$

where $\overline{J}_o(\tau) = J_o(\tau) + J_{ono}T^\dagger B_{ono}$, with T^+ seen as a uniformly bounded input, and with the flow and jump sets

$$\mathbb{R}^{n_o} \times \mathbb{R}^{n_\eta} \times \mathbb{R}^{n_o \times n_o} \times [0, \tau_M], \qquad \mathbb{R}^{n_o} \times \mathbb{R}^{n_\eta} \times \mathbb{R}^{n_o \times n_o} \times \mathcal{I}. \tag{67b}$$

Since A is Schur, let $Q_\eta \in \mathcal{S}_{>0}^{n_\eta}$ be a solution to the inequality $A^\top Q_\eta A - Q_\eta < 0$. Using Corollary 1, we prove that there exist $\lambda^\star > 0$ and $0 < \gamma^\star \leq 1$ such that we have the arbitrarily fast GES of the error $(\tilde{z}_o, \tilde{\eta})$ with respect to the value $(\tilde{z}_o, \tilde{\eta})(t_{j_m+j^\star}, j_m + j^\star)$ when $\lambda > \lambda^\star$ and $0 < \gamma < \gamma^\star$. Then the GES in the z-coordinates with respect to $(\tilde{z}_o, \tilde{z}_{no})(t_{j_m+j^\star}, j_m + j^\star)$ is obtained thanks to the uniform left-invertibility of T. Last, the arbitrarily fast GES is recovered in the x-coordinates after hybrid time $(t_{\bar{j}}, \bar{j})$ where $\bar{j} = j_m + j^\star$ by seeing that $\tilde{z} = \mathcal{V}e^{-F\tau}\tilde{x}$ with $\tau \in [0, \tau_M]$. □

Remark 9 Note that it is the *rate* of convergence that can be arbitrarily fast and not the convergence time since we must anyway wait for j_m jumps for the flow lengths to be in $\mathcal{I}$ and, more importantly, for the $\max_i m_i$ jumps giving us uniform backward distinguishability (see Definition 5). Furthermore, speeding up the rate may make T poorer conditioned, thus increasing the bound ρ_1, which is the well-known *peaking* phenomenon. This type of result is typical in high-gain KKL designs (see [27]). Note though that this arbitrarily fast convergence rate is an advantage compared to the LMI-based design in Section 3 where the rate is fixed once the LMI is solved: Corollary 1 does not apply in that case because the parameters a and Q_η in (84) are not independent, i.e., Q_η is not such that (84) holds for any $a > 0$.

4.3 KKL-based observer design in the (z_o, z_{no})-coordinates

The rectangular shape of T in observer (60) makes the dimension of η larger than that of z_{no}, preventing us from easily writing the observer in the z-coordinates, unlike in Section 3. Following the spirit of [4], consider a map $\Gamma : \mathbb{R}^{n_\eta \times n_{no}} \to \mathbb{R}^{n_\eta \times (n_\eta - n_{no})}$ such that for any $T \in \mathbb{R}^{n_\eta \times n_{no}}$, $\Gamma(T)$ is a full-rank matrix such that its columns are orthogonal to those of T and $\Gamma^\top(T)\Gamma(T) \geq \mathrm{Id}$. Such a map always exists (see Remark 10 for an explicit construction method). Then, define $T_e : \mathbb{R}^{n_\eta \times n_{no}} \to \mathbb{R}^{n_\eta \times n_\eta}$ such that

$$T_e(T) = \begin{pmatrix} T & \Gamma(T) \end{pmatrix}, \tag{68}$$

which is a square matrix extension of T, that is invertible whenever T is full-rank.

Remark 10 One possible way to construct $\Gamma(T)$ is to exploit a singular value decomposition of T. Indeed, given $T \in \mathbb{R}^{n_\eta \times n_{no}}$ with $n_\eta \geq n_{no}$, consider orthonormal matrices $U(T) \in \mathbb{R}^{n_\eta \times n_\eta}$ and $V(T) \in \mathbb{R}^{n_{no} \times n_{no}}$, as well as a matrix $\Sigma(T) = \begin{pmatrix} \Sigma'(T) \\ 0 \end{pmatrix} \in \mathbb{R}^{n_\eta \times n_{no}}$, with $\Sigma'(T)$ diagonal, such that $T = U(T)\Sigma(T)V^\top(T)$. Let us split $U(T)$ as $U(T) = \begin{pmatrix} U_1(T) & U_2(T) \end{pmatrix}$ where $U_1(T) \in \mathbb{R}^{n_\eta \times n_{no}}$ and $U_2(T) \in \mathbb{R}^{n_\eta \times (n_\eta - n_{no})}$. By taking $\Gamma(T) = U_2(T)$, we see that $\Gamma(T)$ is orthogonal to T and verifies $\Gamma^\top(T)\Gamma(T) = \mathrm{Id}$.

An alternative KKL-based observer to observer (60) can then be implemented in the (z_o, z_{no})-coordinates, as

$$\begin{cases} \dot{\hat{z}}_o = G_o(\tau)u_c + P^{-1}H_{c,o}^\top(\tau)R^{-1}(\tau)(y_c - H_{c,o}(\tau)\hat{z}_o) \\ (\dot{\hat{z}}_{no}, \dot{\hat{\omega}}) = (G_{no}(\tau)u_c, 0) \\ \qquad + \mathrm{sat}_{s_{T_e}}((T_e(T))^\dagger)B_{ono}P^{-1}H_{c,o}^\top(\tau)R^{-1}(\tau)(y_c - H_{c,o}(\tau)\hat{z}_o) \\ \dot{P} = -\lambda P + H_{c,o}^\top(\tau)R^{-1}(\tau)H_{c,o}(\tau) \\ \dot{T} = 0 \\ \dot{\tau} = 1 \\ \\ \hat{z}_o^+ = J_o(\tau)\hat{z}_o + J_{ono}(\tau)\hat{z}_{no} + \mathcal{V}_o u_d \\ (\hat{z}_{no}^+, \hat{\omega}^+) = (J_{noo}(\tau)\hat{z}_o + J_{no}(\tau)\hat{z}_{no} + \mathcal{V}_{no}u_d, 0) + \mathrm{sat}_{s_{T_e}}((T_e(T^+))^\dagger) \\ \qquad \times (\gamma A\Gamma(T)\hat{\omega} + B_{d,no}(y_d - H_{d,o}(\tau)\hat{z}_o - H_{d,no}(\tau)\hat{z}_{no})) \\ P^+ = P_0 \\ T^+ = (\gamma AT + B_{d,no}H_{d,no}(\tau) + B_{ono}J_{ono}(\tau))\,\mathrm{sat}_{s_J}(J_{no}^\dagger(\tau)) \\ \tau^+ = 0, \end{cases} \tag{69}$$

with jumps still triggered at the same time as $\mathcal{H}$ and initialized as in Theorem 6 with any $\hat{\omega}(0,0) \in \mathbb{R}^{n_\eta - n_{no}}$. Still in the spirit of [4], $\hat{\omega} \in \mathbb{R}^{n_\eta - n_{no}}$ is an estimate of some fictitious extra state $\omega \in \mathbb{R}^{n_\eta - n_{no}}$ defined on $\mathrm{dom}\, x$ such that $\omega(t,j) = 0$ for all $(t,j) \in \mathrm{dom}\, x$, serving to equalize the dimension in the z_{no} and η coordinates. Note that along the solutions to observer (69), since T is constant during flows, $T_e(T)$ is also constant during flows and needs to be re-computed only at jumps.

Theorem 7 *Suppose Assumptions 1, 2, 4, and 5 hold. Define n_η and $(A, B_{d,no}, B_{ono})$ as in Theorem 6. Given any $\lambda_1 > 0$, any $P_0 \in \mathcal{S}_{>0}^{n_o}$, and any $T_0 \in \mathbb{R}^{n_\eta \times n_{no}}$, there exists $0 < \gamma^\star \leq 1$ such that there exists $\lambda^\star > 0$ such that for any $0 < \gamma < \gamma^\star$ and any $\lambda > \lambda^\star$, there exist $\bar{j} \in \mathbb{N}_{>0}$, saturation levels $s_{T_e} > 0$, $s_J > 0$, and scalar $\rho_1 > 0$ such that for any solution $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0, \mathcal{U})$ and any solution $(\hat{z}, \hat{\omega}, P, T, \tau)$ to observer (69) with $\hat{\omega}(0,0) \in \mathbb{R}^{n_\eta - n_{no}}$, $P(0,0) = P_0$, $T(0,0) = T_0$, $\tau(0,0) = 0$, the chosen $(A, B_{d,no}, B_{ono})$, $\mathrm{sat}_{s_{T_e}}$ at level s_{T_e}, sat_{s_J} at level s_J, and jumps triggered at the same time as in x, $(\hat{z}, \hat{\omega}, P, T, \tau)$ is complete and we have*

$$|x(t,j) - \hat{x}(t,j)| \leq \rho_1 |x(t_{\bar{j}}, \bar{j}) - \hat{x}(t_{\bar{j}}, \bar{j})| e^{-\lambda_1(t+j)}, \qquad \forall (t,j) \in \mathrm{dom}\, x, j \geq \bar{j}, \tag{70}$$

with $\hat{x}$ obtained by $\hat{x} = e^{F\tau}\mathcal{D}\hat{z}$ with $\mathcal{D}$ defined in (25)*.*

Proof First, because T in observer (69) verifies the same dynamics as in observer (60), for the same constant $\overline{c}_{T,m}$ as in the proof of Theorem 6, for every solution $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0, \mathcal{U})$ and any solution $(\hat{z}, \hat{\omega}, P, T, \tau)$ to observer (69) with $P(0,0) = P_0$, $T(0,0) = T_0$, $\tau(0,0) = 0$, and any $\gamma \in (0,1]$, with jumps triggered at the same time as in x, we have $\|T(t,j)\| \leq \overline{c}_{T,m}$ for all $(t,j) \in \operatorname{dom} x$ such that $j \leq j_m$. Moreover, considering the same $\gamma^\star, j^\star, \underline{c}_T$ as in the proof of Theorem 6, T becomes uniformly left-invertible after hybrid time $(t_{j_m+j^\star}, j_m + j^\star)$, i.e.,

$$T^\top(t,j)T(t,j) \geq \underline{c}_T \operatorname{Id}, \qquad \forall (t,j) \in \operatorname{dom} x, j \geq j_m + j^\star.$$

It follows that $T_e(T)$ also becomes uniformly invertible, since

$$T_e^\top(T)T_e(T) = \begin{pmatrix} T^\top T & 0 \\ 0 & \Gamma^\top(T)\Gamma(T) \end{pmatrix} \geq \min\{\underline{c}_T, 1\} \operatorname{Id}, \quad \forall (t,j) \in \operatorname{dom} x, j \geq j_m + j^\star.$$

After hybrid time $(t_{j_m+j^\star}, j_m + j^\star)$, we thus have

$$((T_e(T(t,j)))^\dagger)^\top (T_e(T(t,j)))^\dagger \leq \max\left\{\frac{1}{\underline{c}_T}, 1\right\} \operatorname{Id}, \tag{71}$$

so that there exists a saturation level $s_{T_e} > 0$ such that $\operatorname{sat}_{s_{T_e}}((T_e(T(t,j)))^\dagger) = (T_e(T(t,j)))^\dagger = (T_e(T(t,j)))^{-1}$ for all $(t,j) \in \operatorname{dom} x$ such that $j \geq j_m + j^\star$. Pick $0 < \gamma < \gamma^\star$ and consider a solution $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0, \mathcal{U})$ and a solution $(\hat{z}, \hat{\omega}, P, T, \tau)$ to observer (69) with $P(0,0) = P_0$, $T(0,0) = T_0$, $\tau(0,0) = 0$, the chosen $(A, B_{d,no}, B_{ono})$, the saturation level s_{T_e}, and jumps triggered at the same time as in x. Define for all $(t,j) \in \operatorname{dom} x$ the change of variables

$$\eta(t,j) = T_e(T(t,j)) \begin{pmatrix} z_{no}(t,j) \\ 0 \end{pmatrix} - B_{ono} z_o(t,j). \tag{72}$$

We see that for all $(t,j) \in \operatorname{dom} x$,

$$\eta(t,j) = T(t,j) z_{no}(t,j) - B_{ono} z_o(t,j), \tag{73}$$

and hence it verifies the dynamics of η in the proof of Theorem 6. Let us study the dynamics of the image of $(\hat{z}_o, \hat{z}_{no}, \hat{\omega})$ in observer (69) defined as

$$\hat{\eta}(t,j) = T_e(T(t,j)) \begin{pmatrix} \hat{z}_{no}(t,j) \\ \hat{\omega}(t,j) \end{pmatrix} - B_{ono} \hat{z}_o(t,j). \tag{74}$$

We see that after hybrid time $(t_{j_m+j^\star}, j_m + j^\star)$, $\dot{\hat{\eta}} = (TG_{no}(\tau) - B_{ono}G_o(\tau))u_c$ during flows and at jumps,

$$
\begin{aligned}
\hat{\eta}^+ &= T_e(T^+)\begin{pmatrix}\hat{z}_{no}^+\\ \hat{\omega}^+\end{pmatrix} - B_{ono}\hat{z}_o^+\\
&= \left(T^+\ \Gamma^+\right)\left(J_{noo}(\tau)\hat{z}_o + J_{no}(\tau)\hat{z}_{no}, 0\right) + \mathrm{sat}_{s_{Te}}\left((T_e(T^+))^\dagger\right)(\gamma A\Gamma(T)\hat{\omega}\\
&\quad + B_{d,no}(y_d - H_{d,o}(\tau)\hat{z}_o - H_{d,no}(\tau)\hat{z}_{no})) - B_{ono}(J_o(\tau)\hat{z}_o + J_{ono}(\tau)\hat{z}_{no})\\
&= T^+J_{noo}(\tau)\hat{z}_o + (\gamma AT + B_{d,no}H_{d,no}(\tau) + B_{ono}J_{ono}(\tau))\hat{z}_{no} + \gamma A\Gamma(T)\hat{\omega}\\
&\quad + B_{d,no}(y_d - H_{d,o}(\tau)\hat{z}_o - H_{d,no}(\tau)\hat{z}_{no}) - B_{ono}(J_o(\tau)\hat{z}_o + J_{ono}(\tau)\hat{z}_{no})\\
&= (T^+J_{noo}(\tau) - B_{ono}J_o(\tau))\hat{z}_o + \gamma A(T\hat{z}_{no} + \Gamma(T)\hat{\omega}) + B_{d,no}(y_d - H_{d,o}(\tau)\hat{z}_o)\\
&= (T^+J_{noo}(\tau) + \gamma AB_{ono} - B_{ono}J_o(\tau))\hat{z}_o + \gamma A\hat{\eta} + B_{d,no}(y_d - H_{d,o}(\tau)\hat{z}_o).
\end{aligned}
\tag{75}
$$

Therefore, $\hat{\eta}$ has the same dynamics as in the proof of Theorem 6 and so by proceeding similarly, we get the results in the (z_o, z_{no}, ω)-coordinates thanks to the uniform invertibility of the transformation. □

4.4 KKL-based observer design in the x-coordinates

In a similar manner as in Section 3.2, the KKL-based observer can equivalently be implemented directly in the original x-coordinates. Based on the proof of Theorem 4 by noting that $\dot{T} = 0$ during flows, we can derive the dynamics of the observer in the x-coordinates as

$$
\begin{cases}
\dot{\hat{\zeta}} = F'\hat{\zeta} + (u_c, 0) + \mathcal{P}H_c'^\top R^{-1}(\tau)(y_c - H_c'\hat{\zeta})\\
\dot{\mathcal{P}} = \lambda\mathcal{P} + F'\mathcal{P} + \mathcal{P}F'^\top - \mathcal{P}H_c'^\top R^{-1}(\tau)H_c'\mathcal{P}\\
\dot{T} = 0\\
\dot{\tau} = 1\\
\\
\hat{\zeta}^+ = J'\hat{\zeta} + (u_d, 0) + \mathcal{D}_{no}'\,\mathrm{sat}_{s_{Te}}\left((T_e(T^+))^\dagger\right)(B_{d,no}(y_d - H_d\hat{x}) + \gamma A\Gamma(T)\hat{\omega})\\
\mathcal{P}^+ = \star^\top P_0^{-1}(\mathcal{D}_o' + \mathcal{D}_{no}'\,\mathrm{sat}_{s_{Te}}\left((T_e(T))^\dagger\right)B_{ono})^\top\\
T^+ = (\gamma AT + B_{d,no}H_d e^{F\tau}\mathcal{D}_{no} + B_{ono}\mathcal{V}_o J e^{F\tau}\mathcal{D}_{no})\,\mathrm{sat}_{s_J}\left((\mathcal{V}_{no}Je^{F\tau}\mathcal{D}_{no})^\dagger\right)\\
\tau^+ = 0,
\end{cases}
\tag{76}
$$

where $\hat{\zeta} = (\hat{x}, \hat{\omega})$, with jumps still triggered at the same time as $\mathcal{H}$, where

$$
F' = \begin{pmatrix}F & 0\\ 0 & 0\end{pmatrix},\quad H_c' = \begin{pmatrix}H_c & 0\end{pmatrix},\quad J' = \begin{pmatrix}J & 0\\ 0 & 0\end{pmatrix},\quad \mathcal{D}_o' = \begin{pmatrix}\mathcal{D}_o\\ 0\end{pmatrix},\quad \mathcal{D}_{no}' = \begin{pmatrix}\mathcal{D}_{no} & 0\\ 0 & \mathrm{Id}\end{pmatrix}.
$$

The fact that observability is pumped from discrete time to continuous time via the interaction of $\mathcal{P}$ and T at the jump is recovered, while here it is interesting to see that instead of being reset to a constant as in Section 3.2, here $\mathcal{P}^+$ depends on T, which adapts to the successive flow lengths.

Theorem 8 *Suppose Assumptions 1, 2, 4, and 5 hold. Define n_η and $(A, B_{d,no}, B_{ono})$ as in Theorem 6. Given any $\lambda_1 > 0$, any $P_0 \in \mathcal{S}_{>0}^{n_o}$, and any $T_0 \in \mathbb{R}^{n_\eta \times n_{no}}$, there*

exists $0 < \gamma^\star \leq 1$ *such that there exists* $\lambda^\star > 0$ *such that for any* $0 < \gamma < \gamma^\star$ *and any* $\lambda > \lambda^\star$*, there exist* $\bar{j} \in \mathbb{N}_{>0}$*, saturation levels* $s_{T_e} > 0$*,* $s_J > 0$*, and scalar* $\rho_1 > 0$ *such that for any solution* $x \in \mathcal{S}_\mathcal{H}(\mathcal{X}_0, \mathcal{U})$ *and any solution* $(\hat{\xi}, \mathcal{P}, T, \tau)$ *to observer* (76) *with* $\mathcal{P}(0,0) = \star^\top P_0^{-1}(\mathcal{D}_o' + \mathcal{D}_{no}' \operatorname{sat}_{s_{T_e}}((T_e(T_0)^\dagger)B_{ono})^\top$*,* $T(0,0) = T_0$*,* $\tau(0,0) = 0$*, the chosen* $(A, B_{d,no}, B_{ono})$*,* $\operatorname{sat}_{s_{T_e}}$ *at level* s_{T_e}*,* sat_{s_J} *at level* s_J*, and jumps triggered at the same time as in* x*,* $(\hat{\xi}, \mathcal{P}, T, \tau)$ *is complete and we have*

$$|x(t,j) - \hat{x}(t,j)| \leq \rho_1 |x(t_{\bar{j}}, \bar{j}) - \hat{x}(t_{\bar{j}}, \bar{j})| e^{-\lambda_1(t+j)}, \qquad \forall (t,j) \in \operatorname{dom} x, j \geq \bar{j}. \tag{77}$$

Proof Define $\mathcal{D}' = \begin{pmatrix} \mathcal{D}_o' & \mathcal{D}_{no}' \end{pmatrix}$ and its inverse $\mathcal{V}'$. Consider $\lambda^\star$, $\gamma^\star$, $j^\star$, $\underline{c}_T$, and s_{T_e} given by Theorem 7. Pick a solution $x \in \mathcal{S}_\mathcal{H}(\mathcal{X}_0, \mathcal{U})$ and a solution $(\hat{\xi}, \mathcal{P}, T, \tau)$ to observer (76) with $\mathcal{P}(0,0) = \star^\top P_0^{-1}(\mathcal{D}_o' + \mathcal{D}_{no}' \operatorname{sat}_{s_{T_e}}((T_e(T_0)^\dagger)B_{ono})^\top$, $T(0,0) = T_0$, $\tau(0,0) = 0$, $0 < \gamma < \gamma^\star$, and $\lambda > \lambda^\star$, with jumps triggered at the same time as in x. Consider $(\hat{z}, \underline{\hat{\omega}}, P, \underline{T}, \underline{\tau})$ solution to observer (69), with $(\hat{z}(0,0), \underline{\hat{\omega}}(0,0)) = \mathcal{V}'\hat{\xi}(0,0)$, $P(0,0) = P_0$, $\underline{T}(0,0) = T(0,0)$, $\underline{\tau}(0,0) = 0$, and the same parameters, with jumps triggered at the same time as in x. First, notice from their dynamics that $\underline{T} = T$ and $\underline{\tau} = \tau$. Then, applying Theorem 7, we get

$$|x(t,j) - \underline{\hat{x}}(t,j)| \leq \rho_1 |x(t_{\bar{j}}, \bar{j}) - \underline{\hat{x}}(t_{\bar{j}}, \bar{j})| e^{-\lambda_1(t+j)}, \qquad \forall (t,j) \in \operatorname{dom} x, j \geq \bar{j},$$

where $\underline{\hat{x}} = e^{F\tau}\mathcal{D}\hat{z}$. The proof consists in showing that $\underline{\hat{x}} = \hat{x}$ where $\hat{\xi} = (\hat{x}, \hat{\omega})$, thus obtaining (77). Denote $\underline{\hat{\xi}} = (\underline{\hat{x}}, \underline{\hat{\omega}})$. We start by observing that $\underline{\hat{\xi}} = e^{F'\tau}\mathcal{D}'(\hat{z}, \underline{\hat{\omega}})$, so that

$$\dot{\underline{\hat{\xi}}} = F'\underline{\hat{\xi}} + (u_c, 0) + L_c'(P, T, \tau)(y_c - H_c'\underline{\hat{\xi}}), \tag{78a}$$

during flows and

$$\underline{\hat{\xi}}^+ = J'\underline{\hat{\xi}} + (u_d, 0) + L_{d,x}'(T)(y_d - H_d\underline{\hat{x}}) + L_{d,\omega}'(T)\underline{\hat{\omega}}, \tag{78b}$$

at jumps where

$$\begin{aligned} L_c'(P,T,\tau) &= e^{F'\tau}\mathcal{D}' \begin{pmatrix} P^{-1}H_{c,o}^\top(\tau)R^{-1}(\tau) \\ \operatorname{sat}_{s_{T_e}}((T_e(T))^\dagger)B_{ono}P^{-1}H_{c,o}^\top(\tau)R^{-1}(\tau) \end{pmatrix} \\ &= e^{F'\tau}(\mathcal{D}_o' + \mathcal{D}_{no}' \operatorname{sat}_{s_{T_e}}((T_e(T))^\dagger)B_{ono})P^{-1}H_{c,o}^\top(\tau)R^{-1}(\tau), \end{aligned} \tag{79a}$$

and

$$L_{d,x}'(T) = \mathcal{D}' \begin{pmatrix} 0 \\ \operatorname{sat}_{s_{T_e}}((T_e(T^+))^\dagger)B_{d,no} \end{pmatrix} = \mathcal{D}_{no}' \operatorname{sat}_{s_{T_e}}((T_e(T^+))^\dagger)B_{d,no}, \tag{79b}$$

$$L_{d,\omega}'(T) = \mathcal{D}' \begin{pmatrix} 0 \\ \gamma \operatorname{sat}_{s_{T_e}}((T_e(T^+))^\dagger)A\Gamma(T) \end{pmatrix} = \gamma\mathcal{D}_{no}' \operatorname{sat}_{s_{T_e}}((T_e(T^+))^\dagger)A\Gamma(T). \tag{79c}$$

From (23), we actually have $H_c'e^{F'\tau}\mathcal{D}_{no}' = 0$, so $H_c'e^{F'\tau}\mathcal{D}_{no}' \operatorname{sat}_{s_{T_e}}((T_e(T))^\dagger)B_{ono} = 0$ and we have $L_c'(P,T,\tau) = \underline{\mathcal{P}}H_c'^\top R^{-1}(\tau)$ where

$$\underline{\mathcal{P}} = \star^\top P^{-1}(\mathcal{D}'_o + \mathcal{D}'_{no}\,\mathrm{sat}_{s_{Te}}((T_e(T))^\dagger)B_{ono})^\top e^{F'^\top \tau}. \tag{80}$$

Calculating $\dot{\underline{\mathcal{P}}}$, we obtain the same flow/jump dynamics as $\mathcal{P}$ in observer (76). Besides, $\underline{\mathcal{P}}(0,0) = \mathcal{P}(0,0)$, so $\underline{\mathcal{P}} = \mathcal{P}$ thanks to the uniqueness of solutions. We deduce that $\underline{\hat{\zeta}}$ follows the same dynamics as $\hat{\zeta}$, and since

$$\underline{\hat{\zeta}}(0,0) = \mathcal{D}'(\hat{z}, \underline{\hat{\omega}})(0,0) = \hat{\zeta}(0,0),$$

we have $\underline{\hat{\zeta}} = \hat{\zeta}$, which implies $\underline{\hat{x}} = \hat{x}$ and concludes the proof. □

Example 5 Consider the same system as in Example 4. First, with $J_{no}(\tau) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for all $\tau \in \mathcal{I}$, Assumption 4 is satisfied. Since $H_{d,no}(\tau) = 0$ for all $\tau \in \mathcal{I}$, we discard the jump output and only consider the fictitious one described by the matrix $J_{ono}(\tau)$. We then see that with $m_1 = m_2 = 1$, for all sequences of flow lengths $(\tau_j)_{j\in\mathbb{N}}$,

$$\mathbb{O}_j^{bw} = J_{ono}(\tau_j)J_{no}^{-1}(\tau_j) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

satisfies $\mathbb{O}_j^{bw\top}\mathbb{O}_j^{bw} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} > 0$ for all $j \geq \max\{m_1, m_2\} = 1$. Therefore, Assumption 5 is satisfied and we can thus design a KKL-based observer where T is of dimension 2×2. Let us take $A = \mathrm{diag}(0.1, 0.2)$, an empty $B_{d,no}$, and $B_{ono} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then, a KKL-based observer as in observer (60), (69), or (76), with the mentioned $(A, B_{d,no}, B_{ono})$, a weighting matrix taken as $R = \mathrm{Id}$, a large enough λ, and a small enough γ, can be designed for this system.

5 Conclusion

This chapter presents and discusses new results on observer design for general hybrid systems with linear maps and known jump times. After defining and discussing the hybrid (pre-)asymptotic detectability and uniform complete observability conditions, we briefly presented again the Kalman-like observer in [28]. We then propose a decomposition of the state of a hybrid system with linear maps and known jump times into a part that is instantaneously observable during flows and a part that is not. A thorough analysis of the asymptotic detectability of the second part is performed, where we show that this part can actually be detectable from an extended output made of the jump output and a fictitious one thanks to the flow-jump combination. A high-gain Kalman-like observer with resets at jumps is proposed to estimate the first part, while two different jump-based algorithms are proposed for the second one. Several examples are provided to illustrate the methods.

A comparison among the mentioned designs, namely the Kalman-like observer [28] and the two designs depending on observability decomposition, is presented

in Table 1 at the end. While the KKL-based design requires stronger conditions and has a larger dimension than the LMI-based one, it can provide an arbitrarily fast convergence rate of the estimate (achieved after a certain time). Compared to the Kalman-like design in [28], the LMI-based observer has a smaller dimension, whereas the KKL-based one can be bigger or smaller depending on n_o versus n_{no}. Therefore, for a particular application, if n_o is large compared to n_{no}, going through a decomposition is advantageous dimension-wise. Note also that while the Kalman-like observer can easily deal with time-varying matrices in the system dynamics and output, the decomposition method must additionally assume the (uniform) existence of the transformation into system (26), for example, to solve the LMI (33) for the gains $L_{d,no}(\cdot)$ and K_{no}. Furthermore, even though the invertibility of $J_{no}(\tau)$ for all $\tau \in \mathcal{I}$ assumed for the KKL-based method may seem lighter than the invertibility of J at the jump times assumed in the Kalman-like approach [28, Assumption 2], it may turn out stronger since $J_{no}^{\dagger}(\tau)$ is used in the KKL implementation (see observer (60)), while J^{-1} is used only for analysis in the Kalman-like design, not in the implementation (see observer (16)).

Future work is to study hybrid systems with *nonlinear* maps (and known jump times) by finding transformations into coordinates of possibly higher dimensions and Lyapunov-based sufficient conditions to couple different observers [30], as well as those with *unknown* jump times.

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6 Appendix: Technical lemmas

6.1 Exponential stability of the error dynamics

Consider a hybrid system of the form

$$\begin{cases} \dot{\tilde{z}}_o = -P^{-1}H_{c,o}^{\top}(\tau)R^{-1}(\tau)H_{c,o}(\tau)\tilde{z}_o \\ \dot{\tilde{\eta}} = 0 \\ \dot{P} = -\lambda P + H_{c,o}^{\top}(\tau)R^{-1}(\tau)H_{c,o}(\tau) \\ \dot{\tau} = 1 \\ \\ \tilde{z}_o^{+} = M_o(u,\tau)\tilde{z}_o + M_{o\eta}(u,\tau)\tilde{\eta} \\ \tilde{\eta}^{+} = M_{\eta o}(u,\tau)\tilde{z}_o + M_{\eta}(\tau)\tilde{\eta} \\ P^{+} = P_0 \\ \tau^{+} = 0, \end{cases} \tag{81a}$$

where $u \in \mathcal{U}$ is the input, with the flow and jump sets

$$\mathbb{R}^{n_o} \times \mathbb{R}^{n_\eta} \times \mathbb{R}^{n_o \times n_o} \times [0, \tau_M], \qquad \mathbb{R}^{n_o} \times \mathbb{R}^{n_\eta} \times \mathbb{R}^{n_o \times n_o} \times \mathcal{I}, \tag{81b}$$

where $\mathcal{I}$ is a compact subset of $[\tau_m, \tau_M]$ for some positive τ_m and τ_M.

Lemma 2 *Assume that:*

1. *$H_{c,o}$ is continuous on $[0, \tau_M]$ and such that the pair $(0, H_{c,o}(\tau))$ satisfies (27);*
2. *R is continuous on $[0, \tau_M]$ and $R(\tau) > 0$ for all $\tau \in [0, \tau_M]$;*
3. *M_o, $M_{o\eta}$, and $M_{\eta o}$ are bounded on $\mathcal{U} \times \mathcal{I}$;*
4. *M_η is continuous on $\mathcal{I}$ and there exists $Q_\eta \in \mathcal{S}^{n_\eta}_{>0}$ such that*

$$M_\eta^\top(\tau) Q_\eta M_\eta(\tau) - Q_\eta < 0, \qquad \forall \tau \in \mathcal{I}. \tag{82}$$

Then for any $P_0 \in \mathcal{S}^{n_o}_{>0}$, there exists $\lambda^\star > 0$ such that for all $\lambda > \lambda^\star$, there exist $\rho_1 > 0$ and $\lambda_1 > 0$ such that any solution $(\tilde{z}_o, \tilde{\eta}, P, \tau)$ to system (81) with $P(0,0) = P_0$, $\tau(0,0) = 0$, and $u \in \mathcal{U}$, is complete and verifies

$$|(\tilde{z}_o, \tilde{\eta})(t,j)| \leq \rho_1 e^{-\lambda_1 (t+j)} |(\tilde{z}_o, \tilde{\eta})(0,0)|, \qquad \forall (t,j) \in \operatorname{dom}(\tilde{z}_o, \tilde{\eta}, P, \tau). \tag{83}$$

Proof First, due to the compactness of $\mathcal{I}$ and Item 4 in Lemma 2, there exists $a > 0$ such that for all $\tau \in \mathcal{I}$, (82) is strengthened into

$$M_\eta^\top(\tau) Q_\eta M_\eta(\tau) - Q_\eta \leq -a Q_\eta, \qquad \forall \tau \in \mathcal{I}. \tag{84}$$

Since $P(0,0) = P_0$, $P^+ = P_0$, and $\tau(0,0) = 0$, the component $(t,j) \mapsto P(t,j)$ of the solution to system (81) can actually be written as a closed form of the component $(t,j) \mapsto \tau(t,j)$ by defining

$$\mathbb{P}(\tau) = e^{-\lambda \tau} P_0 + \int_0^\tau e^{-\lambda(\tau - s)} H_{c,o}^\top(s) R^{-1}(s) H_{c,o}(s) ds, \tag{85}$$

namely $P(t,j) = \mathbb{P}(\tau(t,j))$ for all $(t,j) \in \operatorname{dom} x$. Note that since $P_0 > 0$, $\mathbb{P}(\tau)$ is invertible for all $\tau \in [0, \tau_M]$. It follows that the solution is complete and $(\tilde{z}_o, \tilde{\eta}, \tau)$ is solution to

$$\begin{cases} \dot{\tilde{z}}_o = -\mathbb{P}^{-1}(\tau) H_{c,o}^\top(\tau) R^{-1}(\tau) H_{c,o}(\tau) \tilde{z}_o \\ \dot{\tilde{\eta}} = 0 \\ \dot{\tau} = 1 \\ \\ \tilde{z}_o^+ = M_o(u,\tau)\tilde{z}_o + M_{o\eta}(u,\tau)\tilde{\eta} \\ \tilde{\eta}^+ = M_{\eta o}(u,\tau)\tilde{z}_o + M_\eta(\tau)\tilde{\eta} \\ \tau^+ = 0, \end{cases} \tag{86}$$

with the flow set $\mathbb{R}^{n_o} \times \mathbb{R}^{n_\eta} \times [0, \tau_M]$ and the jump set $\mathbb{R}^{n_o} \times \mathbb{R}^{n_\eta} \times \mathcal{I}$. Consider the Lyapunov function

$$V(\tilde{z}_o, \tilde{\eta}, \tau) = e^{\frac{\lambda}{2}\tau} \tilde{z}_o^\top \mathbb{P}(\tau) \tilde{z}_o + k e^{-\epsilon \tau} \tilde{\eta}^\top Q_\eta \tilde{\eta}, \tag{87}$$

where $k > 0$ and $\epsilon > 0$. We have for all $\tau \in [\tau_m, \tau_M] \supseteq \mathcal{I}$,

$$
\begin{aligned}
e^{\frac{\lambda}{2}\tau}\mathbb{P}(\tau) &\geq e^{\frac{\lambda}{2}\tau}\int_{\frac{3}{4}\tau}^{\tau} e^{-\lambda(\tau-s)}H_{c,o}^{\top}(s)R^{-1}(s)H_{c,o}(s)ds \\
&\geq r_m e^{\frac{\lambda}{4}\tau}\int_{\frac{3}{4}\tau}^{\tau}\mathcal{D}_o^{\top}e^{F^{\top}s}H_c^{\top}H_c e^{Fs}\mathcal{D}_o ds \\
&\geq r_m e^{\frac{\lambda}{4}\tau_m}\int_{\frac{3}{4}\tau}^{\frac{3}{4}\tau+\frac{1}{4}\tau_m}\mathcal{D}_o^{\top}e^{F^{\top}s}H_c^{\top}H_c e^{Fs}\mathcal{D}_o ds \\
&\geq e^{\frac{\lambda}{4}\tau_m}r_m\alpha\,\mathrm{Id} \\
&:= e^{\frac{\lambda}{4}\tau_m}\lambda_m\,\mathrm{Id},
\end{aligned}
\tag{88}
$$

where $r_m > 0$ is a lower bound of the continuous map R on the compact set $[0, \tau_M]$ (thanks to Item 2 in Lemma 2), $\alpha > 0$ (independent of λ) is obtained by applying (27) with $\delta = \frac{\tau_m}{4}$, and $\lambda_m := r_m\alpha$. On the other hand, from (85), for all $\tau \in [0, \tau_m]$, $\mathbb{P}(\tau) \geq e^{-\lambda\tau_m}P_0$, so $e^{\frac{\lambda}{2}\tau}\mathbb{P}(\tau) \geq e^{-\lambda\tau_m}P_0$. Besides, as $\mathbb{P}$ is continuous on the compact set $[0, \tau_M]$, there exists $p_M > 0$ such that $\mathbb{P}(\tau) \leq p_M\,\mathrm{Id}$ for all $\tau \in [0, \tau_M]$. It then follows that there exist $\underline{\rho} > 0$ and $\overline{\rho} > 0$ defined as

$$
\underline{\rho} = \min\left\{\mathrm{eig}\left(e^{-\lambda\tau_m}P_0\right), e^{\frac{\lambda}{4}\tau_m}\lambda_m, \mathrm{eig}\left(ke^{-\epsilon\tau_M}Q_\eta\right)\right\}, \tag{89a}
$$

$$
\overline{\rho} = \max\left\{e^{\frac{\lambda}{2}\tau_M}p_M, \mathrm{eig}\left(ke^{-\epsilon\tau_m}Q_\eta\right)\right\}, \tag{89b}
$$

such that

$$
\underline{\rho}|(\tilde{z}_o, \tilde{\eta})|^2 \leq V(\tilde{z}_o, \tilde{\eta}, \tau) \leq \overline{\rho}|(\tilde{z}_o, \tilde{\eta})|^2, \qquad \forall(\tilde{z}_o, \tilde{\eta}) \in \mathbb{R}^n, \forall\tau \in [0, \tau_M]. \tag{90}
$$

During flows, for all $(\tilde{z}_o, \tilde{\eta}) \in \mathbb{R}^n$ and $\tau \in [0, \tau_M]$,

$$
\begin{aligned}
\dot{V} &= e^{\frac{\lambda}{2}\tau}\tilde{z}_o^{\top}\left(\frac{\lambda}{2}\mathbb{P}(\tau) - 2H_{c,o}^{\top}(\tau)R^{-1}(\tau)H_{c,o}(\tau) + \dot{\mathbb{P}}(\tau)\right)\tilde{z}_o - \epsilon ke^{-\epsilon\tau}\tilde{\eta}^{\top}Q_\eta\tilde{\eta} \\
&= e^{\frac{\lambda}{2}\tau}\tilde{z}_o^{\top}\left(-\frac{\lambda}{2}\mathbb{P}(\tau) - H_{c,o}^{\top}(\tau)R^{-1}(\tau)H_{c,o}(\tau)\right)\tilde{z}_o - \epsilon ke^{-\epsilon\tau}\tilde{\eta}^{\top}Q_\eta\tilde{\eta} \\
&\leq -\frac{\lambda}{2}e^{\frac{\lambda}{2}\tau}\tilde{z}_o^{\top}\mathbb{P}(\tau)\tilde{z}_o - \epsilon ke^{-\epsilon\tau}\tilde{\eta}^{\top}Q_\eta\tilde{\eta} \\
&\leq -\min\left\{\frac{\lambda}{2}, \epsilon\right\}V.
\end{aligned}
\tag{91}
$$

At jumps, for all $(\tilde{z}_o, \tilde{\eta}) \in \mathbb{R}^n$, $u \in \mathcal{U}$, and $\tau \in \mathcal{I}$,

$$
\begin{aligned}
V^+ - V = \star^{\top}P_0(M_o(u,\tau)\tilde{z}_o + M_{o\eta}(u,\tau)\tilde{\eta}) - e^{\frac{\lambda}{2}\tau}\tilde{z}_o^{\top}\mathbb{P}(\tau)\tilde{z}_o \\
+ k\star^{\top}Q_\eta(M_{\eta o}(u,\tau)\tilde{z}_o + M_\eta(\tau)\tilde{\eta}) - ke^{-\epsilon\tau}\tilde{\eta}^{\top}Q_\eta\tilde{\eta}.
\end{aligned}
\tag{92}
$$

From Young's inequality, (88), (84), and Items 3 and 4 in Lemma 2, there exist non-negative constants $c_i, i = 1, 2, \ldots, 5$ independent of (λ, k, ϵ) such that for any

$\kappa > 0$, for all $(\tilde{z}_o, \tilde{\eta}) \in \mathbb{R}^n$, $u \in \mathcal{U}$, and $\tau \in \mathcal{I}$,

$$V^+ - V \leq \left(c_1 + kc_2 + \kappa c_3 - e^{\frac{\lambda}{4}\tau_m}\lambda_m\right)\tilde{z}_o^\top \tilde{z}_o - \left(k\left(a - (1 - e^{-\epsilon\tau_M})\right) - c_4 - \frac{k^2 c_5}{\kappa}\right)\tilde{\eta}^\top Q_\eta \tilde{\eta}. \quad (93)$$

We now show that this quantity can be made negative definite by successively picking the degrees of freedom. For the $\tilde{\eta}$ part, $\exists \epsilon^\star > 0$ such that $0 < \epsilon < \epsilon^\star \implies a - (1 - e^{-\epsilon\tau_M}) > 0$, $\exists k^\star > 0$ such that $k > k^\star \implies k\left(a - (1 - e^{-\epsilon\tau_M})\right) - c_4 > 0$, and $\exists \kappa^\star$ such that $\kappa > \kappa^\star \implies k\left(a - (1 - e^{-\epsilon\tau_M})\right) - c_4 - \frac{k^2 c_5}{\kappa} > 0$. Then, for the $\tilde{z}_o$ part, $\exists \lambda^\star > 0$ such that $\lambda > \lambda^\star \implies c_1 + kc_2 + \kappa c_3 - e^{\frac{\lambda}{4}\tau_m}\lambda_m < 0$. We deduce that for any $\lambda > \lambda^\star$, there exist $a_c > 0$ and $a_d > 0$ such that for all $(\tilde{z}_o, \tilde{\eta}) \in \mathbb{R}^n$,

$$\dot{V} \leq -a_c V, \qquad \forall \tau \in [0, \tau_M], \quad (94a)$$

$$V^+ - V \leq -a_d V, \qquad \forall u \in \mathcal{U}, \forall \tau \in \mathcal{I}. \quad (94b)$$

From (90) and (94), we conclude according to [13, Definition 7.29 and Theorem 7.30] that the set $\mathcal{A} = \{(\tilde{z}_o, \tilde{\eta}, \tau) \in \mathbb{R}^{n_o} \times \mathbb{R}^{n_{no}} \times [0, \tau_M] : \tilde{z}_o = 0, \tilde{\eta} = 0\}$ is GES for system (86). □

Corollary 1 *Let us now consider system* (81) *with* $M_\eta(\tau)$ *replaced by* $\gamma M_\eta(\tau)$ *for* $\gamma \in (0, \gamma_0^\star]$. *Under the same assumptions as in Lemma 2, for any* $\lambda_c > 0$ *and any* $P_0 \in \mathcal{S}^{n_o}_{>0}$, *there exists* $\gamma^\star > 0$ *such that there exists* $\lambda^\star > 0$ *such that for any* $0 < \gamma < \gamma^\star$ *and any* $\lambda > \lambda^\star$, *there exists* $\rho_c > 0$ *such that any solution* $(\tilde{z}_o, \tilde{\eta}, P, \tau)$ *to the new system* (81), *with* $P(0,0) = P_0$, $\tau(0,0) = 0$, *and* $u \in \mathcal{U}$, *is complete and verifies*

$$|(\tilde{z}_o, \tilde{\eta})(t, j)| \leq \rho_c e^{-\lambda_c(t+j)} |(\tilde{z}_o, \tilde{\eta})(0,0)|, \qquad \forall (t, j) \in \mathrm{dom}(\tilde{z}_o, \tilde{\eta}, P, \tau). \quad (95)$$

Proof This is a modification of the proof of Lemma 2. Consider the Lyapunov function in (87). First, let us show with an appropriate choice of ϵ that for λ sufficiently large and γ sufficiently small, we have for some $a_d > 0$,

$$\dot{V} \leq -2\lambda_c \left(\frac{1}{\tau_m} + 1\right) V, \qquad V^+ < V. \quad (96)$$

Following the same analysis as in the proof of Lemma 2, we obtain that during flows, $\dot{V} \leq -\min\left\{\frac{\lambda}{2}, \epsilon\right\} V$ for all $\tau \in [0, \tau_M]$, and at jumps thanks to (82), for all $u \in \mathcal{U}$ and $\tau \in \mathcal{I}$,

$$V^+ - V \leq \left(c_1 + kc_2 + \kappa c_3 - e^{\frac{\lambda}{4}\tau_m}\lambda_m\right)\tilde{z}_o^\top \tilde{z}_o - \left(k\left(e^{-\epsilon\tau_M} - \gamma^2\right) - c_4 - \frac{k^2 c_5}{\kappa}\right)\tilde{\eta}^\top Q_\eta \tilde{\eta}. \quad (97)$$

Let us pick $\epsilon = 2\lambda_c\left(\frac{1}{\tau_m}+1\right)$ and define $\lambda_0^\star := 4\lambda_c\left(\frac{1}{\tau_m}+1\right)$. Then, the first item in (96) holds as soon as $\lambda \geq \lambda_0^\star$. Now define $\gamma^\star := \min\left\{\gamma_0^\star, \sqrt{e^{-\epsilon\tau_M}}\right\}$. For any $0 < \gamma < \gamma^\star$, we have $e^{-\epsilon\tau_M} - \gamma^2 > e^{-\epsilon\tau_M} - (\gamma^\star)^2 > 0$, then k, κ, and λ are successively picked (based on $\gamma^\star$) as in the proof of Lemma 2. The final $\lambda^\star$ is the larger one between this and $\lambda_0^\star$. For any $\lambda > \lambda^\star$ and $0 < \gamma < \gamma^\star$, we obtain (96). Second, we deduce (95) from (96) and the dwell time condition. From (96), we get $V(t,j) \leq e^{-2\lambda_c\left(\frac{1}{\tau_m}+1\right)t}V(0,0)$ for all $(t,j) \in \mathrm{dom}(\tilde{z}_o, \tilde{\eta}, P, \tau)$. Since the flow lengths of system (81) are at least $\tau_m > 0$ for all $j \geq 1$, we have $j \leq \frac{t}{\tau_m}+1$ so that $t \geq \frac{t+j-1}{\frac{1}{\tau_m}+1}$, for all $(t,j) \in \mathrm{dom}(\tilde{z}_o, \tilde{\eta}, P, \tau)$. Therefore, $V(t,j) \leq e^{2\lambda_c}e^{-2\lambda_c(t+j)}V(0,0)$, for all $(t,j) \in \mathrm{dom}(\tilde{z}_o, \tilde{\eta}, P, \tau)$, implying (95). □

6.2 Boundedness in finite time

Lemma 3 *Consider a hybrid system with state $\eta \in \mathbb{R}^{n_\eta}$ and input $u \in \mathcal{U} \subset \mathbb{R}^{n_u}$:*

$$\begin{cases} \dot{\eta} = M_c(u)\eta & (\eta, u) \in C \\ \eta^+ = M_d(u)\eta & (\eta, u) \in D \end{cases} \tag{98}$$

with $M_c, M_d : \mathbb{R}^{n_u} \to \mathbb{R}^{n_\eta \times n_\eta}$. Consider positive scalars m_c, ρ_d, and τ_M, as well as $j_m \in \mathbb{N}$. Then, there exists $\rho > 0$ such that for any solution (η, u) to system (98) *with flow lengths in $[0, \tau_M]$ and such that $\|M_c(u(t,j))\| \leq m_c$ and $\|M_d(u(t,j))\| \leq \rho_d$ for all $u \in \mathcal{U}$ and $(t,j) \in \mathrm{dom}\, x$, we have $j_m \in \mathrm{dom}_j\, \eta$ and*

$$|\eta(t_{j_m}, j_m)| \leq \rho|\eta(0,0)|. \tag{99}$$

Proof First, the flow lengths of system (98) are in $[0, \tau_m]$ and solutions are both t- and j-complete. During flows, the evolution of η is characterized by the transition matrix $\Psi_{M_c(u)}$ as

$$\eta(t,j) = \Psi_{M_c(u), u\in\mathcal{U}}(t, t_{j-1})\eta(t_{j-1}, j-1). \tag{100}$$

If M_c is uniformly bounded for $u \in \mathcal{U}$, there exists $\rho_c > 0$ such that for all $(t,j) \in \mathrm{dom}\,\eta$ and all $u \in \mathcal{U}$, $|\eta(t,j)| \leq \rho_c|\eta(t_{j-1}, j-1)|$. Next, we have for all $j \in \mathrm{dom}_j\,\eta$, $|\eta(t_j, j)| \leq \rho_d|\eta(t_j, j-1)|$. Therefore, for any $j_m \in \mathbb{N}_{>0}$, we have $|\eta(t_{j_m}, j_m)| \leq \rho_c^{j_m}\rho_d^{j_m-1}|\eta(0,0)|$, which is (99) by seeing that $\rho = \rho_c^{j_m}\rho_d^{j_m-1}$. □

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	Kalman-like observer (16) (Section 2.1)	LMI-based observer (34) (Section 3)	KKL-based observer (60) (Section 4)
Time domain	No assumption	Persistence of both flows and jumps (Assumption 2)	
Hybrid model	Invertibility of J at the jump times ([28, Assumption 2])	The pair (F, H_c) constant	The pair (F, H_c) constant, invertibility of $J_{no}(\tau)$ for all $\tau \in \mathcal{I}$ (Assumption 4)
Observability	Uniform complete observability of system (1) (Definition 4)	Instantaneous observability of z_o and a form of quadratic detectability of system (32) for $\tau \in \mathcal{I}$ (Assumption 3)	Instantaneous observability of z_o and uniform backward distinguishability of system (32) for the $(\tau_j)_{j\in\mathbb{N}}$ generated by the time domain of x (Assumption 5)
Dimension	$n_x + \frac{n_x(n_x+1)}{2}$	$n_x + \frac{n_o(n_o+1)}{2} + 1$	$n_o + \left(\sum_{i=1}^{n_{d,\text{ext}}} m_i\right) + \frac{n_o(n_o+1)}{2} + \left(\sum_{i=1}^{n_{d,\text{ext}}} m_i\right) n_{no} + 1$
Convergence rate	Arbitrarily fast, achieved after a certain time ([28, Theorem 2])	Determined by the solution to a matrix inequality (Assumption 3)	Arbitrarily fast, achieved after a certain time (Theorem 6)
ISS Lyapunov function	Yes ([28, Theorem 3])	Yes	Yes

Table 1 Comparison of observer designs for hybrid systems with linear maps and known jump times. All the conditions here are *sufficient* conditions.