Observer Design based on Observability Decomposition for Hybrid Systems with Linear Maps and Known Jump Times

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***Abstract*— We propose an observer design method for hybrid systems with linear maps and known jump times based on decomposing the state into parts exhibiting different kinds of observability properties. Using a series of transformations depending on the time elapsed since the previous jump, the state may be decomposed into up to three parts, where the first one is instantaneously observable during flows from the flow output, the second one detectable at jumps from the jump output thanks to the combination of flows and jumps, and the remaining part naturally detectable at jumps still thanks to this combination of flows and jumps but implicitly from the flow output. An observer is then designed to estimate each part, relying on a flow-based Kalman-like observer exploiting the flow output for the first part, a jump-based observer exploiting the jump output for the second, and a jump-based observer exploiting a fictitious output for the third. Global exponential stability of the estimation error is proven using Lyapunov analysis.**

I. INTRODUCTION

Hybrid dynamical systems are widely studied and have a lot of applications, e.g., impulsive systems, walking robots, biology, and so on [1]. However, the observer design problem for this class of systems is still largely unsolved mainly because the time domain of each hybrid solution typically depends on its initial condition and is thus unknown to the observer. Hence, the time domain of the system and observer solutions typically differ, making both design and analysis of convergence challenging [2]. Even in the less complex case where the system jump times, namely the times at which discrete events appear, are known or detected [3], [4], e.g., impulsive systems [5], [6], [7], [8], [9] or continuous-time systems with sampled measurements [10], [11], [12], [13], existing results typically assume either: 1) Lyapunov/LMI-based sufficient conditions, see e.g., [3] or [4, Section 3], but without constructive observability-based criteria to check their solvability; or, 2) Observability of the full state during flows from the flow output only, exploiting continuous-time high-gain observers and *persistence of flows*, see e.g., [4, Section 4]; or, 3) Observability of the full state from the jump output only thanks to the combination of flows and jumps, exploiting discrete-time observers on an equivalent discrete-time system sampled at the jumps and *persistence of jumps*, see e.g., [4, Section 5] or [7], [12], [13]; or, 4) For switched systems, observability gained by accumulating information from individual non-observable subsystems under *persistent switching* [8], [9].

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However, in some hybrid systems, state components may exhibit different kinds of observability properties, associated with the flow and/or jump output or even hidden inside the flow-jump coupling. It has been suggested from [14] that for linear time-varying systems, components with different observability properties can be separated from each other using decomposition. In the context of output regulation, [15] extends these ideas to hybrid systems with linear maps and periodic jumps with output during flows only. Indeed, it is seen that a part of the dynamics is instantaneously observable during flows from the flow output, while a part of the non-observable dynamics becomes *visible* in the observable states at jumps. The rest of the dynamics, called the *invisible dynamics*, may be discarded for internal model design. This idea, however, is still limited in [15] to the case of periodic jumps and flow output only.

In this paper, we extend these ideas in the context of observer design for hybrid systems with linear maps and known jump times. Unlike in [15], jumps are typically not periodic and the full state needs to be reconstructed, and for that, one may rely on outputs available both during flows and at jumps, thus exploiting the full potential of hybrid systems.

Our main contribution is a series of linear transformations depending on the time elapsed since the previous jump that effectively decomposes the state into (up to) three parts with different observability properties: the first one is instantaneously observable during flows from the flow output, the second one is detectable at jumps from the jump output thanks to the combination of flows and jumps, and finally, the remaining part is implicitly detectable at jumps from the flow output. This decomposition allows us to explicitly construct an observer estimating each mentioned part of the state in the new coordinates depending on their observability properties, namely an arbitrarily fast flow-based Kalman-like observer exploiting the flow output for the first part, a jump-based observer exploiting the jump output for the second, and a jump-based observer exploiting a fictitious *hidden output* for the third. Using Lyapunov analysis, a high-gain-like result is obtained, where the global exponential stability (GES) of the estimation error is achieved if the flow-based Kalman-like observer is pushed sufficiently fast.

A. Notations

Let $\mathbb{R}$ (resp. $\mathbb{N}$) denote the set of real numbers (resp. natural numbers, i.e., $\{0,1,2,\ldots\}$). Let $\mathbb{R}_{\geq 0} = [0,+\infty)$ while $\mathbb{R}_{>0} = (0,+\infty)$ and $\mathbb{N}_{>0} = \mathbb{N}\backslash\{0\}$. Let $\mathbb{R}^{m\times n}$ be the set of real $(m \times n)$-dimensional matrices, while $\mathcal{S}_{>0}^{m\times n}$ denotes the set of (symmetric) positive definite real

$(m \times n)$-dimensional matrices. Let $\|z\|$ be the Euclidean norm of the vector z. For a solution $(t, j) \mapsto x(t, j)$ (see [1, Definition 2.6]) of a hybrid system, we denote $\mathrm{dom}\, x$ its domain, $\mathrm{dom}_t\, x$ (resp. $\mathrm{dom}_j\, x$) the domain's projection on the time (resp. jump) component, and for $j \in \mathbb{N}$, $t_j(x)$ the only time defined by $(t_j, j) \in \mathrm{dom}\, x$ and $(t_j, j-1) \in \mathrm{dom}\, x$. A solution x is *complete* if $\mathrm{dom}\, x$ is unbounded and *Zeno* if it is complete and $\sup \mathrm{dom}_t\, x < +\infty$. Let $\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ be the diagonal matrix operator and $\Re(\varphi) := \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}$ be the rotation matrix. Last, $\star$ in the matrix inequalities denotes the symmetric block.

II. PROBLEM FORMULATION

Consider a hybrid dynamical system with linear maps

$$\mathcal{H} \left\{ \begin{array}{lll} \dot{x} = Fx & x \in C & y_c = H_c x \\ x^+ = Jx & x \in D & y_d = H_d x \end{array} \right. \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $y_c \in \mathbb{R}^{p_c}$ and $y_d \in \mathbb{R}^{p_d}$ are the outputs known during the intervals of flow and at the jump times, and C and D are the flow and jump sets respectively.

Remark 1: The design of this paper still holds if the flow/jump dynamics contain extra known terms and if the flow/jump sets depend on a known exogenous input, which is neglected here for brevity.

The goal of this paper is to design an asymptotic observer for (1), assuming its jump times are known. Because in practice we may be interested in estimating only certain trajectories of "physical interest", we denote in the following $\mathcal{X}_0$ a set containing the initial conditions of the trajectories to be estimated. Following [4], our design only requires us to have an idea of the possible duration of flow intervals between successive jumps in the trajectories of interest as defined next.

Definition 1: Let $\mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$ denote the set of maximal solutions of $\mathcal{H}$ initialized in $\mathcal{X}_0$. For a closed subset $\mathcal{I}$ of $\mathbb{R}_{\geq 0}$, we say that solutions have flow lengths within $\mathcal{I}$, if, for any $x \in \mathcal{S}_{\mathcal{H}}(\mathcal{X}_0)$,

- $0 \leq t - t_j(x) \leq \sup(\mathcal{I})$, $\forall (t, j) \in \mathrm{dom}\, x$;
- $t_{j+1}(x) - t_j(x) \in \mathcal{I}$ holds $\forall j \in \mathbb{N}_{>0}$, if $\sup \mathrm{dom}_j\, x = +\infty$, and $\forall j \in \{1, 2, ..., \sup \mathrm{dom}_j\, x - 1\}$, otherwise.

In brief, $\mathcal{I}$ contains all the possible lengths of the flow intervals between successive jumps. The first item is to bound the length of the flow intervals not covered by the second item, namely possibly the first one, which is $[0, t_1]$, and the last one, which is $\mathrm{dom}_t\, x \cap [t_{J(x)}, +\infty)$ where $t_{J(x)}$ is the time when the last jump happens (when defined). If $\mathcal{I}$ is unbounded, the system may admit (eventually) continuous solutions, while $0 \in \mathcal{I}$ means the system can jump more than once at the same time instance or have flow lengths going to zero (including Zeno solutions).

In this paper, we tackle the general case where some state components may be observable during flows and others via the combination of flows and jumps. It follows that both the flow and jump outputs may need to be fully exploited to reconstruct the state so that neither (eventually) continuous nor discrete/Zeno trajectories are allowed: both flows and jumps need to be persistent as assumed next.

Assumption 1: Any maximal solution to $\mathcal{H}$ initialized in $\mathcal{X}_0$ is complete and has flow lengths within a compact set $\mathcal{I} \subseteq [\tau_m, \tau_M]$ where $\tau_m > 0$.

Our goal is thus to design an observer assuming we know: 1) the output(s) y_c during flows and/or y_d at jumps, 2) when the plant's jumps occur, and 3) some information about the possible flow lengths, as in Assumption 1. Because the jump times of $\mathcal{H}$ are known, we design an observer of the form

$$\hat{\mathcal{H}} \left\{ \begin{array}{l} \left. \begin{array}{l} \dot{\hat{x}} = F\hat{x} + L_c(\tau)(y_c - H_c\hat{x}) \\ \dot{\tau} = 1 \end{array} \right\} \text{when } \mathcal{H} \text{ flows} \\ \left. \begin{array}{l} \hat{x}^+ = J\hat{x} + L_d(\tau)(y_d - H_d\hat{x}) \\ \tau^+ = 0 \end{array} \right\} \text{when } \mathcal{H} \text{ jumps} \end{array} \right. \tag{2}$$

with jumps triggered at the same time as the plant's, a timer τ keeping track of the time elapsed since the previous jump, and the gains $L_c : [0, \tau_M] \to \mathbb{R}^{n \times p_c}$ and $L_d : [0, \tau_M] \to \mathbb{R}^{n \times p_d}$ function of this elapsed time τ such that the error $x - \hat{x}$ is GES. For clarification, the term "elapsed time" refers to the time that has passed since the last jump, while "flow length" means the duration of a full flow interval. These are equal at the end of each flow interval but not in between.

Exploiting the fact that the flow lengths of the trajectories to be estimated are known to be in $\mathcal{I}$, we solve this problem by designing the gains $L_c(\cdot)$ and $L_d(\cdot)$ such that every maximal solution to the interconnection

$$\left\{ \begin{array}{l} \left. \begin{array}{l} \dot{x} = Fx \\ \dot{\hat{x}} = F\hat{x} + L_c(\tau)H_c(x - \hat{x}) \\ \dot{\tau} = 1 \end{array} \right\} (x, \hat{x}, \tau) \in \hat{C}^\tau \\ \left. \begin{array}{l} x^+ = Jx \\ \hat{x}^+ = J\hat{x} + L_d(\tau)H_d(x - \hat{x}) \\ \tau^+ = 0 \end{array} \right\} (x, \hat{x}, \tau) \in \hat{D}^\tau \end{array} \right. \tag{3}$$

with $\hat{C}^\tau = \mathbb{R}^n \times \mathbb{R}^n \times [0, \tau_M]$ and $\hat{D}^\tau = \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{I}$, satisfies

$$|x(t, j) - \hat{x}(t, j)| \leq \rho_1 |x(0, 0) - \hat{x}(0, 0)| e^{-\lambda_1(t+j)}$$

on its domain for some positive scalars ρ_1 and λ_1. In that sense, our design depends only implicitly on the sets $\mathcal{X}_0$, C, and D through the choice of $\mathcal{I}$ satisfying Assumption 1, namely we choose to design the observer based on the information contained in $\mathcal{I}$ only. Finally, we omit the flow/jump sets when they are not explicitly needed.

III. OBSERVABILITY DECOMPOSITION

Let us start by noting that thanks to Assumption 1, all the solutions to $\mathcal{H}$ initialized in $\mathcal{X}_0$ are included in the set of solutions to

$$\mathcal{H}^\tau \left\{ \begin{array}{l} \left. \begin{array}{rcl} \dot{x} & = & Fx \\ \dot{\tau} & = & 1 \end{array} \right\} (x, \tau) \in C^\tau \qquad y_c = H_c x \\ \left. \begin{array}{rcl} x^+ & = & Jx \\ \tau^+ & = & 0 \end{array} \right\} (x, \tau) \in D^\tau \qquad y_d = H_d x \end{array} \right. \tag{4}$$

with $C^\tau := \mathbb{R}^n \times [0, \tau_M]$ and $D^\tau := \mathbb{R}^n \times \mathcal{I}$. Notice that $\mathcal{H}^\tau$ admits a larger set of solutions than $\mathcal{H}$ since the information of the flow and jump sets are replaced by the knowledge of flow lengths in $\mathcal{I}$ only. However, as can be seen in (3), we are actually designing an observer for $\mathcal{H}^\tau$ and it is thus relevant to consider $\mathcal{H}^\tau$ for observability analysis.

In view of observer design and motivated by [15], we start by proposing a change of variables decomposing the state x of $\mathcal{H}^\tau$ into components associated with different types of observability.

A. *Observability from y_c during Flows*

First, recall that if the whole state is observable during flows from the flow output, i.e., the pair (F, H_c) is observable (and still assuming $0 \notin \mathcal{I}$, i.e., a dwell-time condition), it is shown in [4] that a high-gain flow-based observer can be designed, namely using only y_c during flows with L_c constant and $L_d = 0$. In this paper, we consider the more general case where only part of the state is observable from y_c during flows. Let the (flow) observability matrix be

$$\mathcal{O} := \text{row}(H_c, H_cF, \dots, H_cF^{n-1}),$$

and assume it is of rank $d_o := \dim \text{Im}\, \mathcal{O} < n$. Consider a basis $(v_i)_{1\leq i\leq n}$ of $\mathbb{R}^n$ such that $(v_i)_{1\leq i\leq d_o}$ is a basis of the observable subspace and $(v_i)_{v_{d_o+1}\leq i\leq v_n}$ is a basis of the non-observable subspace $\ker \mathcal{O}$. Then, we define the matrix $\mathcal{D} := \begin{pmatrix}\mathcal{D}_o & \mathcal{D}_{no}\end{pmatrix}$ where

$$\begin{aligned}\mathcal{D}_o &:= \begin{pmatrix}v_1 & \dots & v_{d_o}\end{pmatrix} \in \mathbb{R}^{n\times d_o},\\ \mathcal{D}_{no} &:= \begin{pmatrix}v_{d_o+1} & \dots & v_n\end{pmatrix} \in \mathbb{R}^{n\times d_{no}},\end{aligned}$$

which satisfy in particular

$$\mathcal{O}\mathcal{D}_{no} = 0, \quad H_c e^{F\tau}\mathcal{D}_{no} = 0, \quad \forall \tau \geq 0. \tag{5}$$

We denote $\mathcal{V} := \mathcal{D}^{-1}$ which we decompose consistently into two parts $\mathcal{V} =: \begin{pmatrix}\mathcal{V}_o\\ \mathcal{V}_{no}\end{pmatrix}$, so that $\mathcal{V}_o x$ represents the part of the state that is instantaneously observable during flows (see [16, Theorem 6.O6]).

A first idea could be to design a sufficiently fast high-gain observer for $\mathcal{V}_o x$ and estimate the rest of the state $\mathcal{V}_{no}x$ either through y_d or detectability. However, as noticed in [15, Proposition 6], the fact that $\mathcal{V}_o x$ and $\mathcal{V}_{no}x$ possibly interact with each other during flows prevents us from achieving stability by pushing the high gain and no satisfactory result could be obtained via this road.

Actually, the estimation of any state that is not instantaneously observable during flows needs to take into account the combination of flows and jumps. That is why it is relevant to exhibit explicitly this combination via the change of coordinates

$$x \mapsto \begin{pmatrix}z_o\\ z_{no}\end{pmatrix} = \begin{pmatrix}\mathcal{V}_o e^{-F\tau}\\ \mathcal{V}_{no}e^{-F\tau}\end{pmatrix} x, \tag{6}$$

whose inverse is

$$x = e^{F\tau}\mathcal{D}\begin{pmatrix}z_o\\ z_{no}\end{pmatrix} = e^{F\tau}(\mathcal{D}_o z_o + \mathcal{D}_{no}z_{no}), \tag{7}$$

and which, according to (5), transforms the dynamics (4) into

$$\left\{\begin{array}{rcl}\dot{z}_o &=& 0\\ \dot{z}_{no} &=& 0\\ \dot{\tau} &=& 1\\ z_o^+ &=& J_o(\tau)z_o + J_{ono}(\tau)z_{no}\\ z_{no}^+ &=& J_{noo}(\tau)z_o + J_{no}(\tau)z_{no}\\ \tau^+ &=& 0,\end{array}\right. \tag{8a}$$

with the measurements

$$y_c = H_{c,o}(\tau)z_o, \quad y_d = H_{d,o}(\tau)z_o + H_{d,no}(\tau)z_{no}, \tag{8b}$$

where $J_o(\tau) = \mathcal{V}_o J e^{F\tau}\mathcal{D}_o$, $J_{ono}(\tau) = \mathcal{V}_o J e^{F\tau}\mathcal{D}_{no}$, $J_{noo}(\tau) = \mathcal{V}_{no}Je^{F\tau}\mathcal{D}_o$, $J_{no}(\tau) = \mathcal{V}_{no}Je^{F\tau}\mathcal{D}_{no}$, $H_{c,o}(\tau) = H_c e^{F\tau}\mathcal{D}_o$, $H_{d,o}(\tau) = H_d e^{F\tau}\mathcal{D}_o$, and $H_{d,no}(\tau) = H_d e^{F\tau}\mathcal{D}_{no}$. This idea of bringing at the jumps the whole combination of flows and jumps is similar to the so-called *equivalent discrete-time system* exhibited in [4] for jump-based observer design. Notice that the observability decomposition through $\mathcal{V}$ ensures that the flow dynamics of z_o and y_c are totally independent of z_{no}, which only impacts z_o at jumps. In other words, the whole dependence of the observable part on the non-observable part via flows and jumps has been gathered at the jumps. Besides, z_o is by definition observable from y_c. More precisely, for any $\delta > 0$, there exists $\alpha > 0$ such that the observability Gramian of the continuous pair $(0, H_{c,o}(\tau))$ satisfies

$$\begin{aligned}&\int_t^{t+\delta} H_{c,o}^\top(s)H_{c,o}(s)ds =\\ &\int_t^{t+\delta}\mathcal{D}_o^\top e^{F^\top s}H_c^\top H_c e^{Fs}\mathcal{D}_o ds \geq \alpha I, \qquad \forall t \geq 0.\end{aligned} \tag{9}$$

Indeed, this Gramian corresponds to the observability Gramian of the pair (F, H_c) projected onto the observable subspace. This condition is related to the *uniform complete observability* of the continuous pair $(0, H_{c,o}(\tau))$ in the Kalman literature [17] but here with an arbitrarily small window δ. Since z_o is observable via y_c, we propose to estimate z_o sufficiently fast during flows to compensate for the interaction with z_{no} at jumps.

B. *Detectability at Jumps from y_d and Implicit Output*

The next step is to notice that a part of z_{no}, more precisely $J_{ono}(\tau)z_{no}$, becomes *visible* in z_o at the jumps [15]. It naturally follows that this part of z_{no} could be indirectly estimated thanks to a sufficiently fast estimation of z_o and then used as an indirect measurement to estimate a larger part of z_{no} at jumps. This fact is also exploited in [15] in the context of regulation and is illustrated in an example in Section V-B, where some state components are not observable during flows from y_c and yet they are estimated thanks to these hidden dynamics at jumps, without relying on any other output y_d. Here, more generally, a part of z_{no} could also be estimated via the output y_d available at jumps. That is why we first decompose z_{no} into two parts, $z_{o'} \in \mathbb{R}^{d_{o'}}$ and $z_{no'} \in \mathbb{R}^{d_{no'}}$, where $z_{o'}$ is detectable via y_d while the rest $z_{no'}$ is required to be detectable from the fictitious output $J_{ono}(\tau)z_{no}$. Note that we could also have proceeded the other way around, namely first extract the detectable part from $J_{ono}(\tau)z_{no}$ and estimate the rest by y_d, but we choose here to prioritize the "physical" output y_d.

In the forthcoming analysis, we assume that this "splitting" can be done independently of the length of the flow intervals τ, as stated next.

Assumption 2: There exists a constant change of coordinates $\Upsilon := \begin{pmatrix}\Upsilon_{o'} \\ \Upsilon_{no'}\end{pmatrix}$ where $\Upsilon_{o'} \in \mathbb{R}^{d_{o'} \times d_{no}}$ and $\Upsilon_{no'} \in \mathbb{R}^{d_{no'} \times d_{no}}$ decomposing z_{no} into

$$z_{no} \mapsto \begin{pmatrix} z_{o'} \\ z_{no'} \end{pmatrix} = \begin{pmatrix} \Upsilon_{o'} \\ \Upsilon_{no'} \end{pmatrix} z_{no}, \tag{10}$$

with inverse $\Lambda := \Upsilon^{-1}$ decomposed consistently into $\Lambda =: \begin{pmatrix}\Lambda_{o'} & \Lambda_{no'}\end{pmatrix}$ such that

$$\Upsilon_{o'} J_{no}(\tau)\Lambda_{no'} = 0,\ H_{d,no}(\tau)\Lambda_{no'} = 0,\ \forall \tau \in \mathcal{I}, \tag{11}$$

and the (discrete) pair $(\Upsilon_{o'} J_{no}(\tau)\Lambda_{o'}, H_{d,o}(\tau)\Lambda_{o'})$ is detectable for all $\tau \in \mathcal{I}$.

Remark 2: For now, we require only the detectability of the pair $(\Upsilon_{o'} J_{no}(\tau)\Lambda_{o'}, H_{d,o}(\tau)\Lambda_{o'})$ for each individual $\tau \in \mathcal{I}$ in order to build the decomposition. However, τ, modeling here the length of flow in between jumps, may vary in $\mathcal{I}$ from one jump to the other throughout each solution. Therefore, unless jumps are periodic, stronger properties may be asked depending on the type of observer design. For the observer in this paper, *quadratic detectability* in (20a) is required.

As a result, (8) is transformed into

$$\left\{\begin{array}{rcl} \dot{z}_o &=& 0 \\ \dot{z}_{o'} &=& 0 \\ \dot{z}_{no'} &=& 0 \\ \dot{\tau} &=& 1 \\ z_o^+ &=& J_o(\tau)z_o + J_{oo'}(\tau)z_{o'} + J_{ono'}(\tau)z_{no'} \\ z_{o'}^+ &=& J_{o'o}(\tau)z_o + J_{o'}(\tau)z_{o'} + J_{o'no'}(\tau)z_{no'} \\ z_{no'}^+ &=& J_{no'o}(\tau)z_o + J_{no'o'}(\tau)z_{o'} + J_{no'}(\tau)z_{no'} \\ \tau^+ &=& 0, \end{array}\right. \tag{12a}$$

with the measurements

$$\begin{array}{l} y_c = H_{c,o}(\tau)z_o, \\ y_d = H_{d,o}(\tau)z_o + H_{d,o'}(\tau)z_{o'} + H_{d,no'}(\tau)z_{no'}, \end{array} \tag{12b}$$

where $J_{oo'}(\tau) = J_{ono}(\tau)\Lambda_{o'}$, $J_{ono'}(\tau) = J_{ono}(\tau)\Lambda_{no'}$, $J_{o'o}(\tau) = \Upsilon_{o'} J_{noo}(\tau)$, $J_{o'}(\tau) = \Upsilon_{o'} J_{no}(\tau)\Lambda_{o'}$, $J_{o'no'}(\tau) = \Upsilon_{o'} J_{no}(\tau)\Lambda_{no'}$, $J_{no'o}(\tau) = \Upsilon_{no'} J_{noo}(\tau)$, $J_{no'o'}(\tau) = \Upsilon_{no'} J_{no}(\tau)\Lambda_{o'}$, $J_{no'}(\tau) = \Upsilon_{no'} J_{no}(\tau)\Lambda_{no'}$, $H_{d,o'}(\tau) = H_{d,no}(\tau)\Lambda_{o'}$, and $H_{d,no'}(\tau) = H_{d,no}(\tau)\Lambda_{no'}$, such that for all $\tau \in \mathcal{I}$, the pair $(J_{o'}(\tau), H_{d,o'}(\tau))$ is detectable, and $J_{o'no'}(\tau) = 0$ and $H_{d,no'}(\tau) = 0$ according to (11).

This implies in particular that the jump dynamics of $z_{o'}$ and the jump output y_d become decoupled from $z_{no'}$ after the first jump. Since after the first jump onwards, no information on the $z_{no'}$ part is contained in y_d, the only way to access it is using the fictitious output $J_{ono'}(\tau)z_{no'}$ visible in z_o after the jump, and thus implicitly visible through y_c during flows. In other words, intuitively speaking, we see from this new form that if z_o is estimated sufficiently fast during flows through y_c, then the term $J_{ono'}(\tau)z_{no'}$ impacting its jump may be implicitly recovered and $z_{no'}$ may be estimated using this fictitious measurement. For that, we make the following Assumption 3.

Assumption 3: The discrete pair $(J_{no'}(\tau), J_{ono'}(\tau))$ is detectable for all $\tau \in \mathcal{I}$ and there exists $K_{no'} \in \mathbb{R}^{d_{no'} \times d_o}$ such that $J_{no'}(\tau) - K_{no'} J_{ono'}(\tau)$ is Schur for all $\tau \in \mathcal{I}$.

Remark 3: While the detectability of $(J_{no'}(\tau), J_{ono'}(\tau))$ serves as a necessary assumption to estimate $z_{no'}$, assuming $K_{no'}$ independent of τ is required here to perform the next change of variables. But depending on the type of observer, stronger or weaker properties may be asked. For the observer in this paper, the stronger property (20b) is required.

In order to further highlight this *hidden* detectability, we exploit the existence of $K_{no'}$ given by Assumption 3 and perform a third change of coordinates

$$\eta = z_{no'} - K_{no'} z_o \in \mathbb{R}^{d_{no'}}. \tag{13}$$

As a result, (12) is finally transformed into

$$\left\{\begin{array}{rcl} \dot{z}_o &=& 0 \\ \dot{z}_{o'} &=& 0 \\ \dot{\eta} &=& 0 \\ \dot{\tau} &=& 1 \\ z_o^+ &=& \overline{J}_o(\tau)z_o + J_{oo'}(\tau)z_{o'} + J_{ono'}(\tau)\eta \\ z_{o'}^+ &=& \overline{J}_{o'o}(\tau)z_o + J_{o'}(\tau)z_{o'} + J_{o'no'}(\tau)\eta \\ \eta^+ &=& J_{\eta o}(\tau)z_o + J_{\eta o'}(\tau)z_{o'} + J_\eta(\tau)\eta \\ \tau^+ &=& 0, \end{array}\right. \tag{14a}$$

with the measurements

$$\begin{array}{l} y_c = H_{c,o}(\tau)z_o, \\ y_d = \overline{H}_{d,o}(\tau)z_o + H_{d,o'}(\tau)z_{o'} + H_{d,no'}(\tau)\eta, \end{array} \tag{14b}$$

where the continuous pair $(0, H_{c,o}(\tau))$ is observable in the sense of (9) and for all $\tau \in \mathcal{I}$,

- $J_\eta(\tau)$ is Schur;
- The discrete pair $(J_{o'}(\tau), H_{d,o'}(\tau))$ is detectable;
- $J_{o'no'}(\tau) = 0$ and $H_{d,no'}(\tau) = 0$;

with the matrices defined by $\overline{J}_o(\tau) = J_o(\tau) + J_{ono'}(\tau)K_{no'}$, $\overline{J}_{o'o}(\tau) = J_{o'o}(\tau) + K_{no'} J_{o'no'}(\tau)$, $J_{\eta o}(\tau) = J_{no'o}(\tau) - K_{no'} J_o(\tau) + J_{no'}(\tau)K_{no'} - K_{no'} J_{ono'}(\tau)K_{no'}$, $J_{\eta o'}(\tau) = J_{no'o'}(\tau) - K_{no'} J_{oo'}(\tau)$, $J_\eta(\tau) = J_{no'}(\tau) - K_{no'} J_{ono'}(\tau)$, and $\overline{H}_{d,o}(\tau) = H_{d,o}(\tau) + K_{no'} H_{d,no'}$.

In those new coordinates, provided z_o and $z_{o'}$ are estimated sufficiently fast, η can naturally be estimated at jumps by a simple copy of its dynamics without any additional output injection thanks to $J_\eta(\tau)$ being Schur for all $\tau \in \mathcal{I}$.

We summarize the series of transformations and the observer above into Lemma 1.

Lemma 1: Under Assumptions 2 and 3, the change of coordinates

$$x \mapsto z := \begin{pmatrix} z_o \\ z_{o'} \\ \eta \end{pmatrix} = \underbrace{\begin{pmatrix} \mathcal{V}_o \\ \Upsilon_{o'}\mathcal{V}_{no} \\ \Upsilon_{no'}\mathcal{V}_{no} - K_{no'}\mathcal{V}_o \end{pmatrix} e^{-F\tau}}_{\mathcal{T}(\tau)} x \tag{15}$$

transforms the dynamics (4) into (14) with the same flow/jump maps. The inverse of this transformation is

$$x := \underbrace{e^{F\tau}\begin{pmatrix}\mathcal{D}_o + \mathcal{D}_{no}\Lambda_{no'}K_{no'} & \mathcal{D}_{no}\Lambda_{o'} & \mathcal{D}_{no}\Lambda_{no'}\end{pmatrix}}_{\mathcal{T}^{-1}(\tau)} z. \tag{16}$$

We want to insist that the detectability of the components estimated at jumps comes from the flow-jump combination and not due to jumps alone since the useful information contained in the flow dynamics and output is gathered at the jumps via the first transformation (6).

With the state in the z-coordinates now decomposed into components of different observability properties, we propose the following general observer structure in these coordinates

$$\left\{\begin{array}{rcl}\dot{\hat{z}}_o &=& L_{c,o}(\tau)(y_c - H_{c,o}(\tau)\hat{z}_o)\\ \dot{\hat{z}}_{o'} &=& 0\\ \dot{\hat{\eta}} &=& 0\\ \dot{\tau} &=& 1\\ \hat{z}_o^+ &=& \overline{J}_o(\tau)\hat{z}_o + J_{oo'}(\tau)\hat{z}_{o'} + J_{ono'}(\tau)\hat{\eta}\\ \hat{z}_{o'}^+ &=& \overline{J}_{o'o}(\tau)\hat{z}_o + J_{o'}(\tau)\hat{z}_{o'} + J_{o'no'}(\tau)\hat{\eta} + L_{d,o'}(\tau)\times\\ && (y_d - \overline{H}_{d,o}(\tau)\hat{z}_o - H_{d,o'}(\tau)\hat{z}_{o'} - H_{d,no'}(\tau)\hat{\eta})\\ \hat{\eta}^+ &=& J_{\eta o}(\tau)\hat{z}_o + J_{\eta o'}(\tau)\hat{z}_{o'} + J_\eta(\tau)\hat{\eta}\\ \tau^+ &=& 0.\end{array}\right. \tag{17}$$

IV. OBSERVER DESIGN

We now design the gains of the observer (17) written in the new coordinates in order to make the estimation error $z - \hat{z}$ GES. Exploiting the uniform invertibility of the transformation (15), GES is then recovered in the initial coordinates by taking

$$\hat{x} := \mathcal{T}^{-1}(\tau)\hat{z} \tag{18}$$

with $\hat{z} := (\hat{z}_o, \hat{z}_{o'}, \hat{\eta})$ and $\mathcal{T}^{-1}(\cdot)$ defined in (16).

Because the output matrix $H_{c,o}(\tau)$ varies but still satisfies the observability condition (9), we design the gain $L_{c,o}(\cdot)$ as the gain of a continuous-time Kalman-like observer. Its advantage over a Kalman observer is that it allows for direct Lyapunov analysis and a direct relationship between the Lyapunov matrix and the observability Gramian. More precisely, the dynamics of the observer are given by

$$\left\{\begin{array}{rcl}\dot{\hat{z}}_o &=& P^{-1}H_{c,o}^\top(\tau)R^{-1}(\tau)(y_c - H_{c,o}(\tau)\hat{z}_o)\\ \dot{\hat{z}}_{o'} &=& 0\\ \dot{\hat{\eta}} &=& 0\\ \dot{P} &=& -\lambda P + H_{c,o}^\top(\tau)R^{-1}(\tau)H_{c,o}(\tau)\\ \dot{\tau} &=& 1\\ \hat{z}_o^+ &=& \overline{J}_o(\tau)\hat{z}_o + J_{oo'}(\tau)\hat{z}_{o'} + J_{ono'}(\tau)\hat{\eta}\\ \hat{z}_{o'}^+ &=& \overline{J}_{o'o}(\tau)\hat{z}_o + J_{o'}(\tau)\hat{z}_{o'} + J_{o'no'}(\tau)\hat{\eta} + L_{d,o'}(\tau)\times\\ && (y_d - \overline{H}_{d,o}(\tau)\hat{z}_o - H_{d,o'}(\tau)\hat{z}_{o'} - H_{d,no'}(\tau)\hat{\eta})\\ \hat{\eta}^+ &=& J_{\eta o}(\tau)\hat{z}_o + J_{\eta o'}(\tau)\hat{z}_{o'} + J_\eta(\tau)\hat{\eta}\\ P^+ &=& P_0\\ \tau^+ &=& 0,\end{array}\right. \tag{19}$$

with jumps triggered at the same time as $\mathcal{H}$ in the same way as (2), $P_0 \in \mathcal{S}_{>0}^{d_o\times d_o}$ and where $\tau \mapsto R(\tau) \in \mathcal{S}_{>0}^{p_c\times p_c}$ is a weighting matrix that is defined and is continuous on $[0, \tau_M]$ to be chosen only for design purpose.

We first provide a sufficient condition on the jump map of observer (19) to guarantee the GES of the estimation error for a sufficiently large flow parameter λ.

Assumption 4: There exist $Q_{o'} \in \mathcal{S}_{>0}^{d_{o'}\times d_{o'}}$, $Q_\eta \in \mathcal{S}_{>0}^{d_{no'}\times d_{no'}}$, $L_{d,o'} : [0, \tau_M] \to \mathbb{R}^{d_{o'}\times p_d}$ bounded on $[0, \tau_M]$ and continuous on $\mathcal{I}$, and $K_{no'} \in \mathbb{R}^{d_{no'}\times d_o}$ such that for all $\tau \in \mathcal{I}$,

$$(J_{o'}(\tau) - L_{d,o'}(\tau)H_{d,o'}(\tau))^\top Q_{o'}(J_{o'}(\tau) - L_{d,o'}(\tau)H_{d,o'}(\tau)) - Q_{o'} < 0, \tag{20a}$$

$$(J_{no'}(\tau) - K_{no'}J_{ono'}(\tau))^\top Q_\eta(J_{no'}(\tau) - K_{no'}J_{ono'}(\tau)) - Q_\eta < 0. \tag{20b}$$

These conditions require in particular the detectability of the pairs $(J_{o'}(\tau), H_{d,o'}(\tau))$ and $(J_{no'}(\tau), J_{ono'}(\tau))$ for each frozen τ. But the fact that $Q_{o'}$, Q_η, and $K_{no'}$ are independent of τ makes this assumption stronger: this is related to the notion of *quadratic detectability* [18]. It allows us to build an observer for any sequence of flow lengths $(\tau_j)_{j\in\mathbb{N}} \in \mathcal{I}$ and thus requires in fact the detectability of the discrete pairs for any such sequences. A future question is to thoroughly investigate which part of Assumptions 1, 2, 3, and 4 are necessary observability/detectability conditions for observer design.

Theorem 1: Under Assumptions 1, 2, 3, and 4, given any matrix $P_0 \in \mathcal{S}_{>0}^{d_o\times d_o}$, there exists a scalar $\lambda^\star > 0$ such that for any $\lambda > \lambda^\star$, there exist scalars $\rho_1 > 0$ and $\lambda_1 > 0$ such that for any solution x of (1) initialized in $\mathcal{X}_0$ and any solution $\hat{z}$ of the observer (19) with $P(0,0) = P_0$ and $\tau(0,0) = 0$, and jumps triggered at the same time as in x, we have

$$|x(t,j) - \hat{x}(t,j)| \leq \rho_1|x(0,0) - \hat{x}(0,0)|e^{-\lambda_1(t+j)}, \tag{21}$$

for all $(t,j) \in \operatorname{dom} x$, with $\hat{x}$ obtained from $\hat{z}$ by (18).

Sketch of Proof: Define the error $\tilde{z} = z - \hat{z}$ and similarly for $\tilde{z}_o$, $\tilde{z}_{o'}$, and $\tilde{\eta}$. Let us consider the Lyapunov function

$$V(\tilde{z},\tau) = e^{\frac{\lambda}{2}\tau}\tilde{z}_o^\top\mathbb{P}(\tau)\tilde{z}_o + k_{o'}e^{-\epsilon_{o'}\tau}\tilde{z}_{o'}^\top Q_{o'}\tilde{z}_{o'} + k_\eta e^{-\epsilon_\eta\tau}\tilde{\eta}^\top Q_\eta\tilde{\eta}, \tag{22}$$

where $k_{o'}$, k_η, $\epsilon_{o'}$, and ϵ_η are positive scalars and $\mathbb{P}(\tau(t,j)) = P(t,j)$ for all $(t,j) \in \operatorname{dom} x$. The part $\tilde{z}_o$ contracts during flows thanks to the Kalman-like correction term, while $\tilde{z}_{o'}$ and $\tilde{\eta}$ contract at jumps thanks to the jump-based observers. The weights $k_{o'}$ and k_η are tuned to ensure negativity at jumps despite the interactions between those components. As for the exponential terms, the role of $e^{\frac{\lambda}{2}\tau}$ is to bring negativity from flows to jumps, while that of $e^{-\epsilon_{o'}\tau}$ and $e^{-\epsilon_\eta\tau}$ is instead to bring negativity from jumps to flows.

The proof consists in showing that:

- Using (9), there exist positive scalars $\underline{\rho}$ and $\overline{\rho}$ such that

$$\underline{\rho}\|\tilde{z}\|^2 \leq V(\tilde{z},\tau) \leq \overline{\rho}\|\tilde{z}\|^2, \ \forall\tilde{z} \in \mathbb{R}^n, \ \forall\tau \in [0,\tau_M];$$

- During flows, for all $\tilde{z} \in \mathbb{R}^n$ and for all $\tau \in [0, \tau_M] \supset [\tau_m, \tau_M] \supseteq \mathcal{I}$, we have $\dot{V} \leq -\min\{\frac{\lambda}{2}, \epsilon_{o'}, \epsilon_\eta\}V$;

- At jumps, there exist $c_i \geq 0, i = 1, 2, \ldots, 9$, $a_{o'} > 0$, $a_\eta > 0$, and $\lambda_m > 0$ all independent of λ such that for any positive scalars γ_1, γ_2, and γ_3, for all $\tilde{z} \in \mathbb{R}^n$ and for all $\tau \in \mathcal{I}$,

$$V^+ - V \leq (c_1 + \gamma_1 c_2 + \gamma_2 c_3 - e^{\frac{\lambda}{4}\tau_m}\lambda_m)\tilde{z}_o^\top \tilde{z}_o - \left(k_{o'}a_{o'} - k_{o'}(1 - e^{-\epsilon_{o'}\tau_M}) - c_4 - \frac{k_{o'}c_5}{\gamma_1} - k_\eta \gamma_3 c_6\right)\tilde{z}_{o'}^\top \tilde{z}_{o'} - \left(k_\eta a_\eta - k_\eta(1 - e^{-\epsilon_\eta \tau_M}) - c_7 - \frac{k_\eta c_8}{\gamma_2} - \frac{k_\eta c_9}{\gamma_3}\right)\tilde{\eta}^\top \tilde{\eta}.$$

We can then show that this quantity can be made negative definite by successively picking the degrees of freedom. That leads to λ having to be high enough.

V. EXAMPLES

We illustrate the proposed methods using academic examples. The LMIs are solved using the LMI Lab package in MATLAB. Consider the hybrid system of form (1) with state $x = (x_1, x_2, x_3, x_4)$ and

$$F = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$H_c = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix},$$

with random flow lengths varying in $\mathcal{I} = [\frac{\pi}{10}, \frac{4\pi}{10}]$. We obtain $e^{F\tau} = \text{diag}(\Re(\tau), \Re(2\tau))$. It can be seen that only x_1 and x_2 are observable during flows from y_c and thus we obtain $z_o = \begin{pmatrix} \Re(-\tau) & 0_{2\times 2} \end{pmatrix} x$ and $z_{no} = \begin{pmatrix} 0_{2\times 2} & \Re(-2\tau) \end{pmatrix} x$ with the matrices $J_o(\tau) = 0_{2\times 2}$, $J_{ono}(\tau) = \begin{pmatrix} \cos(2\tau) & -\sin(2\tau) \\ 0 & 0 \end{pmatrix}$, $J_{noo}(\tau) = 0_{2\times 2}$, $J_{no}(\tau) = \Re(2\tau)$, and $H_{c,o}(\tau) = \begin{pmatrix} \cos(\tau) & -\sin(\tau) \end{pmatrix}$.

A. z_{no} Estimated Using y_d Only

Consider $H_d = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$, which gives $H_{d,o}(\tau) = 0_{1\times 2}$ and $H_{d,no}(\tau) = \begin{pmatrix} \cos(2\tau) & -\sin(2\tau) \end{pmatrix}$. Because the pair $(J_{no}(\tau), H_{d,no}(\tau))$ is fully observable for all $\tau \in \mathcal{I}$, we attempt to estimate the full state z_{no} from y_d only using a jump-based observer. In other words, $z_{o'} = z_{no}$ and $z_{no'}$ is empty. While a varying gain is possible, here for simplicity, solving (20a) results in the constant gain $L_{d,no} = \begin{pmatrix} 1 & 0 \end{pmatrix}^\top$.

B. z_{no} Estimated Using the Fictitious Output Only

In fact, at each frozen τ, the pair $(J_{no}(\tau), J_{ono}(\tau))$ is also observable except when $\sin(2\tau) = 0 \iff \tau = k\frac{\pi}{2}$ with k integer. This means that z_{no} is actually observable through the fictitious measurement of z_o at jumps for all frozen $\tau \in \mathcal{I}$ and we can thus attempt to estimate it without using y_d. Solving (20b) has led to $K_{no} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. We see from this example that thanks to the flow-jump coupling, by using z_o as a fictitious measurement, we can estimate state components not observable during flows from the flow output even without any real measurements at jumps (hidden dynamics).

C. z_{no} Estimated Using Both y_d and the Fictitious Output

Consider now $H_d = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$ and $\frac{\pi}{2} \in \mathcal{I}$. Unfortunately, we are not able to decompose z_{no} with a constant Υ satisfying Assumption 3 when $\mathcal{I}$ is not reduced to $\frac{\pi}{2}$. But assume now that the jumps are $\frac{\pi}{2}$-periodic, so that $\mathcal{I} = \{\frac{\pi}{2}\}$, and decompose z_{no} into $z_{o'} = \begin{pmatrix} 0 & 1 \end{pmatrix} z_{no}$ and $z_{no'} = \begin{pmatrix} 1 & 0 \end{pmatrix} z_{no}$. The $z_{o'}$ part is actively corrected at jumps thanks to y_d with the constant gain $L_{d,o'} = 1$ while $\eta = z_{no'} - \begin{pmatrix} 1 & 0 \end{pmatrix} z_o$ contracts naturally and is estimated as a hidden dynamics with z_o as a fictitious measurement.

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