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Abstract

One considers the control problem of an ensemble of Bloch equations (non-interacting half-spins) in a static magnetic field B_0. The state $M(t,\cdot)$ belongs to the Sobolev space $H^1((\omega_*,\omega^*),S^2)$ where the parameter $\omega \in (\omega_*,\omega^*)$ is the Larmor frequency. Previous works have constructed a Lyapunov based stabilizing feedback in a convenient H^1-norm that assures local L^∞-convergence of the initial state $M_0(\omega)$ to the south pole, solving locally the approximate steering problem from M_0 towards the south pole. However, the corresponding control law contains a comb of periodic π-Rabi pulses (Dirac impulses), corresponding to strongly unbounded control. The present work propose smooth uniformly bounded time-varying controls for this local steering problem, where the Rabi pulses are replaced by adiabatic following smooth pulses. Furthermore, simulations show that this new strategy produces faster convergence, even for initial conditions "relatively far" from the south pole.

Key words: nonlinear systems; ensemble controllability; quantum systems; adiabatic control; Lyapunov feedback stabilization; Bloch equations; control of PDEs.

1 Introduction

The goal of ensemble controllability is to simultaneously steer a continuum of systems between two states of interest with the same control inputs. This concept has already been studied by two different ways. In [8] and [9] we have its characterization by the use of Lie algebra tools in the context of quantum systems that are described by Bloch equations depending continuously on a finite number of scalar parameters, and with a finite number of control inputs. In [2], these aspects are studied under a functional analysis setting, developed for infinite dimensional systems governed by partial differential equations. In particular, this last paper shows that a priori L^2-bounded controls are not sufficient to achieve the exact controllability, but unbounded controls (containing a sum of Dirac masses) are able to recover it. In [3] and [4] it is shown that the ensemble of Bloch equations is approximately controllable to the south pole of the Bloch sphere (in the Sobolev space H^1) in finite time with unbounded controls. In practice, it is impossible to reproduce exactly the unbounded controls. Therefore, we would like to investigate whether the same effect can be achieved by using bounded controls.

Ensemble controllability with bounded controls is considered in the literature under different approaches. In [5] we have a comprehensive study as well as the time-optimal solution for the transfer population problem for the Bloch equations without dispersion, using geometric methods. In the presence of dispersion, we have in [7] a solution for the asymptotic stabilization problem when ω is in a finite or at least countable set by using topological methods and in [1,11] results for ensemble controllability between eigenstates of generic Hamiltonians using adiabatic approximation techniques where the dispersion parameter lives in a continuum.

In this work we propose a solution with smooth and bounded control inputs of the local approximate steering problem for an ensemble of Bloch equations in the continuum case. This solution is proved to steer an ensemble of initial conditions close enough to the south pole to an arbitrary neighbourhood of the south pole (vector $-e_3$ here below). As far as we know, this is the

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first constructive and mathematical result solving locally motion planing towards the south pole with smooth and uniformly bounded control inputs for such ensemble of Bloch systems. This solution combines adiabatic techniques with Lyapunov stabilizing methods to construct open-loop bounded control inputs. Simulations reported here indicate that the domain of application of the proposed open-loop control algorithm includes a quite large set of initial-value profiles with a significant support in the north hemisphere of the Bloch sphere.
The paper is organized as follows. Section 2 presents the problem and the main results. Section 2 is not devoted to a very formal and complete presentation, but it is conceived only to introduce the main aspects and ideas of the proposed control law. Section 3 presents some numerical experiments of a studied example. Section 4 considers a mathematically oriented presentation of the main results. Section 5 presents some conclusions. Finally, all the proofs are deferred to the Appendices.

2 Statement of the problem and a summary of the main results

We consider the ensemble $M(t,\omega)$ of Bloch equations:

$$\dot{M}(t,\omega) = S(u(t)e_1 + v(t)e_2 + \omega e_3)M(t,\omega), \qquad (1)$$

where $-\infty < \omega_* < \omega^* < +\infty$, $\omega \in (\omega_*, \omega^*)$, $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^3$ and $S(\cdot)$ is the map that defines the wedge product (see Appendix A). For simplicity the partial derivative of M with respect to time is denoted by $\dot{M}$, and the partial derivative of M with respect to ω is denoted by M'. For any profile $\omega \in [\omega_*, \omega^*] \mapsto M(\omega) \in \mathbb{R}^3$, its H_1-norm reads

$$\|M\|_{H^1} = \sqrt{\int_{\omega_*}^{\omega^*} \left(\|M(\omega)\|^2 + \|M'(\omega)\|^2\right) d\omega}$$

We shall consider only the Larmor dispersion (represented by the parameter ω) and focus on the following controllability problem:

Local approximate steering with bounded control inputs. Show the existence of $\delta > 0$ and $g > 0$ with the following property: for every initial condition $M_0 \in H^1$ such that $\|M_0(\omega) + e_3\|_{H^1} < \delta$, and for every $\varepsilon > 0$, it is possible to choose $T_f > 0$ (depending on ε) and to construct bounded controls $u : [0, T_f] \to [-g, +g]$ and $v : [0, T_f] \to [-g, +g]$ in a way that $\|M(T_f, \cdot) + e_3\|_{L^\infty} \leq \varepsilon$.

First of all, the main ideas of the control strategy that are presented in this paper are roughly described. In [3,4], a control law that contains a T-periodic comb of Dirac pulses is considered. The purpose of these Dirac pulses is to assure the population inversion of $M(t,\omega)$ as commonly found in magnetic resonance techniques. Since an application of T-periodic adiabatic pulses $(\bar{u}(t), \bar{v}(t))$ can also perform an approximate population inversion, our control strategy relies on considering these adiabatic pulses as a reference control, and to consider an auxiliary transformed system that is obtained by writing (1) in the rotating frame of the corresponding adiabatic propagator $A(t,\omega)$, which is the solution of the differential equation

$$\dot{A}(t,\omega) = S(\bar{u}(t)e_1 + \bar{v}(t)e_2 + \omega e_3)A(t,\omega) \qquad (2)$$

where $A(t,\omega) \in SO(3)$, and the T-periodic adiabatic control $(\bar{u}(t), \bar{v}(t))$ is such that $A(0,\omega) = I$ and $A(kT,\omega) \approx I$, for $k = 1, 2, \ldots, \ell$, for some ℓ big enough. Define the auxiliary state $N(t,\omega)$ by the transformation

$$N(t,\omega) = A(t,\omega)^\top M(t,\omega). \qquad (3)$$

By time-differentiation the equation (3), it is easy to obtain the auxiliary system

$$\dot{N}(t,\omega) = S\left[A^\top(t,\omega)(\widehat{u}(t)e_1 + \widehat{v}(t)e_2)\right] N(t,\omega) \qquad (4)$$

and to show that an input $(\widehat{u}(t), \widehat{v}(t))$ applied to the auxiliary system (4) produces a solution $N(t,\omega)$ if and only if an input $(\bar{u}(t) + \widehat{u}(t), \bar{v}(t) + \widehat{v}(t))$ produces a solution $M(t,\omega) = A(t,\omega)N(t,\omega)$ of (1).

In this new frame, the drift term of the differential equation is eliminated , and then the idea is to apply the Lyapunov stabilizing techniques of [4] to the auxiliary system (4). This is not far from what is done in [4], and we shall return to this aspect later [1] .

A heuristic strategy to be applied would be:

- Compute a T-periodic adiabatic control $(\bar{u}(t), \bar{v}(t))$ and the associate adiabatic propagator $A(t,\omega)$ solution of (2) with $A(0,\omega) = I$. For k not too large, e.g. $k = 1, 2, \ldots, \ell$ for some $\ell \in \mathbb{N}$, $A(kT,\omega)$ will remain close enough to the identity matrix;
- Compute a stabilizing feedback control $(\widehat{u}(t), \widehat{v}(t))$ for the auxiliary driftless system based on (4) with control Lyapunov function $\|N(t,\omega) + e_3\|_{H^1}$ that assures that $N(\ell T,\omega)$ is close enough to $-e_3$;
- Apply the control law $(\bar{u}(t) + \widehat{u}(t), \bar{v}(t) + \widehat{v}(t))$ to system (1) in open loop.

The main issue in this heuristic strategy comes from the fact that the adiabatic propagator $A(t,\omega)$ is not exactly T-periodic: thus the transformed system (4) will not be periodic and the previous techniques of [4] cannot be applied to analyze the convergence of the stabilizing feedback.

[1] See equation (D.1) of Appendix D, that represents the auxiliar dynamics that is considered in [4] when $M_f = -e_3$.

Thus, we have to consider $\bar{A}(t,\omega)$, the solution of (2) on each interval $[kT,(k+1)T)$ for any integer k with $\bar{A}(kT,\omega)=I$. By construction, $\bar{A}(t,\omega)$ is T-periodic and admits a small discontinuity at each kT.

As a consequence, when A is replaced by $\bar{A}$ in the right side of (4), one gets a T-periodic and driftless system for which Lyapunov techniques can be applied directly to derive the stabilizing control inputs $\widehat{u}$ and $\widehat{v}$:

$$\dot{\bar{N}}(t,\omega)=S\left[\bar{A}^\top(t,\omega)(\widehat{u}(t)e_1+\widehat{v}(t)e_2)\right]\bar{N}(t,\omega) \quad (5)$$

with $\bar{N}(0,\omega)=M_0(\omega)$. Then, with the control inputs $u(t)=\bar{u}(t)+\widehat{u}(t)$ and $v(t)=\bar{v}(t)+\widehat{v}(t)$, the solution $M(t,\omega)$ of the original system (1), starting from $M_0(\omega)$, will not be given by $\bar{M}(t,\omega)=\bar{A}(t,\omega)\bar{N}(t,\omega)$ but will remain close to it. Thus an error analysis between $M(t,\omega)$ and $\bar{M}(t,\omega)$ will be needed (see Theorem 2).

2.1 *The adiabatic propagator*

Consider the adiabatic propagator equation (2), where:

- $\bar{A}(t,\omega)\in SO(3)$, and $\bar{A}(kT,\omega)=I$, $\forall\omega\in[\omega_*,\omega^*]$, $\forall k\in\mathbb{N}$;
- The pair $(\bar{u}(t),\bar{v}(t))$ is the adiabatic control (6) defined as follows:

$$\bar{u}(t)=B_1(t)\sin\phi(t) \quad (6a)$$
$$\bar{v}(t)=B_1(t)\cos\phi(t), \quad (6b)$$

  where $\phi(t)$ and $B_1(t)$ are defined by:

$$\dot{\phi}(t)=\bar{k}(t)\bar{a}(t),\quad \phi(0)=0 \quad (6c)$$

$$B_1(t)=\bar{k}(t)\bar{b}(t) \quad (6d)$$

  where $\bar{a}(\cdot)$, $\bar{b}(\cdot)$, and $\bar{k}(\cdot)$ are T-periodic functions defined by $\bar{a}(t)=a(t/T)$, $\bar{b}(t)=b(t/T)$, and $\bar{k}(t)=Kk(t/T)$, where $a(\cdot)$, $b(\cdot)$, and $k(\cdot)$ are 1-periodic normalized functions defined in the Appendix B, and $K>0$ is a chosen gain.

Figure 1 shows these functions, that are parameterized by s_0, which defines for instance the size of the interval $[0,s_0]$ on which $b(\cdot)=0$. By (6), it is clear that the T-periodic adiabatic control $(\bar{u}(t),\bar{v}(t))$ is null for $t\in[0,Ts_0]$ and bounded by K for any $t\in[0,T]$. This dead-band interval in the control input $(\bar{u},\bar{v})$ is crucial to prove of the stabilization result of the auxiliary system in Appendix D.

Definition 1 *Fix $T>0$. One let $\bar{A}:\mathbb{R}\times[\omega_*,\omega^*]\to SO(3)$ stands for the T-periodic map such that, in each interval $[t_{0_k},t_{0_{k+1}})=[kT,(k+1)T)$ then $\bar{A}(t,\omega)$ is the solution of system (2) with initial condition $\bar{A}(t_{0_k},\omega)=I$ for $k\in\mathbb{N}$ and with the T-periodic adiabatic input $\Omega(t)=(\bar{u}(t),\bar{v}(t))$ that is defined in (6) (see also Appendix B).*

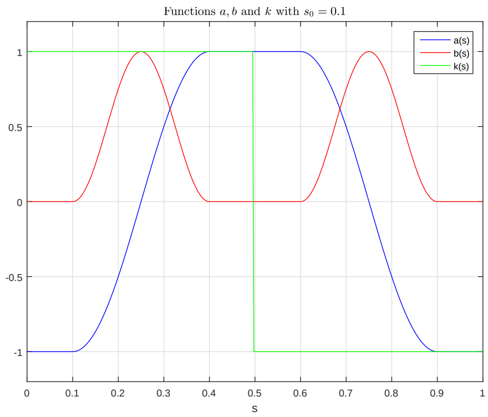

Fig. 1. Functions a, b, and k with $s_0=0.1$.

Since $\bar{A}(t,\omega)$ is not continuous at T, we denote $\lim_{t\to T^-}\bar{A}(t,\omega)$ by $\bar{A}(T^-,\omega)$.

Remark 1 *The maps $\bar{A}(\cdot,\omega)$, $\bar{a}(\cdot)$, $\bar{b}(\cdot)$, and $\bar{k}(\cdot)$ depend on the choice of T. This is not explicitly indicated for the sake of simplicity.*

The following adiabatic convergence result is proved [2] in [10]:

Theorem 1 *For $K>\max\{|\omega_*|,|\omega^*|\}$ the smooth and K-bounded control inputs $\bar{u}(t)$ and $\bar{v}(t)$ defined in (6)for $t\in[0,tT]$, one has $\lim_{T\to\infty}\max_{\omega\in[\omega_*,\omega_*]}\|\bar{A}(T^-,\omega)-I\|=0$.*

2.2 *The auxiliary system and an approximation result*

The auxiliary system is the T-periodic auxiliary system given by (5) with solution $\bar{N}(t,\omega)$. We shall study the relationship between the continuous solution $M(t,\omega)$ of the original system (1) and $\bar{N}(t,\omega)$.

Theorem 2 *Fix initial conditions $\bar{N}_0=M_0$ of systems (5) and (1), an integer k and a time $T>0$. Assume that $\bar{N}(t,\omega)$ is the (continuous) solution of (5) that is obtained by the application of some arbitrary control inputs $(\widehat{u}(t),\widehat{v}(t))$ on $[0,kT]$. Consider $\bar{A}(t,\omega)$ of definition 1 with arbitrary T-periodic inputs $\bar{u}(t)$ and $\bar{v}(t)$. Assume that $M(t,\omega)$ is the (continuous) solution of (1)) that is obtained by the application of the input $(u(t),v(t))$, where $u(t)=\widehat{u}(t)+\bar{u}(t)$, and $v(t)=\widehat{v}(t)+\bar{v}(t))$. Then we have*

$$\|M(kT,\cdot)+e_3\|_{L^\infty}\le k\|\bar{A}(T^-,\cdot)-I\|_{L^\infty}+\|\bar{N}(kT,\cdot)+e_3\|_{L^\infty}$$

Proof. See appendix C. □

This result clearly indicates that, if it is possible to stabilize locally the auxiliary system (5) uniformly with re-

[2] It may be also proved using the results of [12].

spect to the choice of T and k, then we have a constructive solution of the above approximate controllability problem with bounded control inputs. Such uniform stabilization with respect to T and k is due to the fact that the adiabatic control inputs $\bar{u}$ and $\bar{v}$ vanish on $[0, s_0 T]$ with $s_0 > 0$ and are uniformly bounded by K.

2.3 H^1 control law of auxiliary system (5)

Consider the Lyapunov functional

$$\mathcal{L} = \frac{1}{2}\|\bar{N} + e_3\|^2_{H^1} = \int_{\omega_*}^{\omega^*} \left[\frac{1}{2}\langle \bar{N}', \bar{N}'\rangle + 1 + \langle \bar{N}, e_3\rangle\right] d\omega \tag{7}$$

In order to compute $\dot{\mathcal{L}}$ note that $\xi = \widehat{u}(t)e_1 + \widehat{v}(t)e_2$ does not depend on ω. One has

$$\dot{\bar{N}}' = S(\bar{A}^\top \xi)\bar{N}' + S((\bar{A}')^\top \xi)\bar{N} \tag{8}$$

Hence

$$\begin{aligned}\dot{\mathcal{L}} &= \int_{\omega_*}^{\omega^*} \langle \bar{N}', [(\bar{A}^\top)'\xi \wedge \bar{N}]\rangle + \langle e_3, [(\bar{A}^\top)\xi \wedge \bar{N}]\rangle d\omega \\ &= H_1\widehat{u} + H_2\widehat{v}\end{aligned} \tag{9}$$

where

$$H_i(t) = \int_{\omega_*}^{\omega^*} \langle \bar{N}', [(\bar{A}^\top)' e_i \wedge \bar{N}]\rangle + \langle e_3, [(\bar{A}^\top)e_i \wedge \bar{N}]\rangle d\omega, \tag{10a}$$

for $i = 1, 2$.
One may construct the control law [3]

$$\begin{aligned}\widehat{u}(t) &= -H_1(\bar{A}(t,\cdot), \bar{N}(t,\cdot)), \\ \widehat{v}(t) &= -H_2(\bar{A}(t,\cdot), \bar{N}(t,\cdot)),\end{aligned} \tag{10b}$$

obtaining

$$\dot{\mathcal{L}} = -(H_1^2 + H_2^2) \leq 0 \tag{11}$$

2.4 The explicit steering control

The control strategy is summarized as follows:

- Fix $\ell > 0$. Choose $T > 0$ and $s_0 \in (0, 1/4)$ and construct the T-periodic adiabatic pulses $(\bar{u}(t), \bar{v}(t))$ of (6). Compute the adiabatic propagator $\bar{A}(t,\omega)$ of (2) in $[0, T)$ with initial condition $\bar{A}(0,\omega) = I$. Extend $A(t,\omega)$ to $[0, \ell T]$ in a way that $A(t,\omega)$ is T-periodic.
- Compute the (continuous) solution $\bar{N}(t,\omega)$ of the closed loop system (5)-(10a)-(10b), and save the corresponding control law $(\widehat{u}(t), \widehat{v}(t))$ given by (10b) in the interval $[0, \ell T]$.
- Apply the open loop control law $(u(t), v(t)) = (\bar{u}(t) + \widehat{u}(t), \bar{v}(t) + \widehat{v}(t))$ to system (1) in the interval $[0, \ell T]$.

The main result of the paper says that, for ℓ and T big enough, this control strategy provides a constructive answer for local approximate steering towards $-e_3$ with uniformly bounded and smooth control inputs. More precisely, the proof of Theorem 4 shows that, given $\varepsilon > 0$, if one chooses ε_1 and ε_2 such that $\varepsilon = \varepsilon_1 + \varepsilon_2$, then it is possible to fix some $T_0 > 0$ and to find ℓ big enough such that all the members of the family of closed loop auxiliary systems (5) are such that $\|\bar{N}(\ell T, \omega) + e_3\|_{L^\infty} \leq \varepsilon_1$ for all $T \geq T_0$. Furthermore, for T big enough, Theorem 1 implies that the adiabatic control (6) assures $\ell\|\bar{A}(T^-,\cdot) - I\|_{L^\infty} \leq \varepsilon_2$. Then Theorem 2 implies that this control strategy provides a solution of the proposed steering problem at $T_f = \ell T$, indeed.

3 Simulation Results and comparison with the old method

For simplicity we shall refer to the method of [4] as the "old method" and the method described in this paper will be referred as the "new method". One has chosen, $\epsilon = 0.2$, $T = 20$, $s_0 = 0.1$, $K = 10$, $\ell = 16$ and $T_f = \ell T$ for the new method. For convenience of the presentation of the legends of figures, the inputs $\widehat{u}(t)$ and $\widehat{v}(t)$ are denoted by $u_1(t)$ and $u_2(t)$. As defined in (10a)-(10b), we have chosen unitary gains of the feedback law (we mean, there is no gain multiplying H_i of (10b)). For the old method, we have chosen $T = 1$ and unitary gains as well. We have verified that greater values of T than 1 for the old method are worse, but smaller values of T will not improve the result. Figure 2 shows the simulation results in the Bloch sphere for these choices.

The obtained error of the adiabatic propagator (see Theorem 1) is $\|\bar{A}(T^-,\cdot) - I\|_{L^\infty} \leq 0.0009$ (see Figure B.1 in the appendix). So $\ell\|\bar{A}(T^-,\cdot) - I\|_{L^\infty} \leq 0.015$. It is very small in this case. In the simulations we have found that $\|\bar{N}(\ell T) + e_3\|_{L^\infty}$ is more than ten times greater than $\ell\|\bar{A}(T^-,\cdot) - I\|_{L^\infty}$. Hence one will show only the behaviour of the auxiliary state $\bar{N}(t,\omega)$.

In figure 2 one may see the initial condition $\bar{N}_0 = M_0$ and the final condition $\bar{N}(\ell T)$. One has obtained $\|\bar{N}(T_f) + e_3\|_{L^\infty} = 0.185$ with our new method, and $\|\bar{N}(T_f) + e_3\|_{L^\infty} = 0.58$ with our old method, using the same unitary gains multiplying H_i. The expressions of the feedback of the old method is analogous to (10a)-(10b), with the difference that $\bar{A}(t,\omega)$ is replaced by the matrix $\exp(\sigma(t)S(\omega e_3))$, where $\dot{\sigma}(t) = (-1)^{E(t/T)}$, $E(s)$ is the integer part of s, and $\sigma(0) = 0$ (see the proof of Lemma 1). Our new method have produced a result that is more

[3] Although $\bar{A}$ and $\bar{N}$ depend on ω, $\widehat{u}$ and $\widehat{v}$ do not.

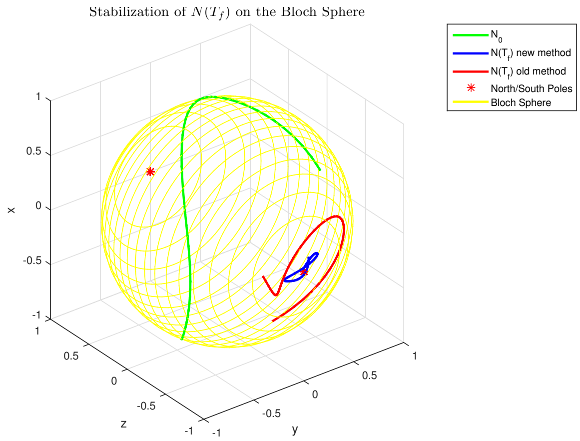

Fig. 2. Results on the Bloch sphere for both the new and the old method.

than 3 times better than the old method with respect to the final L^∞ norm. Note that, for the new method, $\ell\|\bar{A}(T^-,\cdot)-I\|_{L^\infty}+\|\bar{N}(T_f)+e_3\|_{L^\infty}\leq 0.015+0.185 = 0.2=\varepsilon$. So Theorem 3 assures that the problem is solved for the given ε.

Figure 3 regards only the new method. It shows the evolution of the Lyapunov functional $\mathcal{L}(t)=\dfrac{1}{2}\|\bar{N}(t)+e_3\|^2_{H^1}$. In that figure one shows also the evolution of $\|\bar{N}(t)+e_3\|_{L^\infty}$. The controls $u_1(t)$ and $u_2(t)$ are also depicted in that figure. The Figure 4 is a "zoom" of the last one. This allows to see the "microstructure" of the control of the new strategy.

The Figure 6 presents a comparison of the input norms of the old and the new method.

The Figure 5 shows the plot of $\log(\|\bar{N}(t)+e_3\|^2_{H^1})$ versus time. The slope of the curves of $\log(\|\bar{N}(t)+e_3\|^2_{H^1})$ would give a measure of the exponential rate of decaying of $\|\bar{N}(t)+e_3\|^2_{H^1}=2\mathcal{L}(t)$. The slope is much bigger for the first method in the beginning, and this inclination decreases faster for the old method with respect to the new one. This indicates that the new method seems to be more effective than the old one.

4 Main Results

The following result is the heart of the proof of our stabilization result (Corollary 1). It implies that, if the initial condition is at least ε far from $-e_3$ in the L^∞ norm, then the Lyapunov function $\mathcal{L}(t)$ of the auxiliary system will decrease at least of a quantity c for each period, at least while $\|\bar{N}+e_3\|_{L^\infty}$ is bigger than ε. The value of s_0 that appears in Theorem 3 is related to definition of adiabatic controls (6)-(B.1).

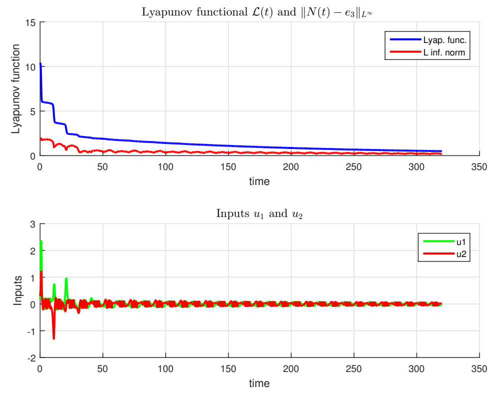

Fig. 3. Lyapunov functional and inputs for the new method.

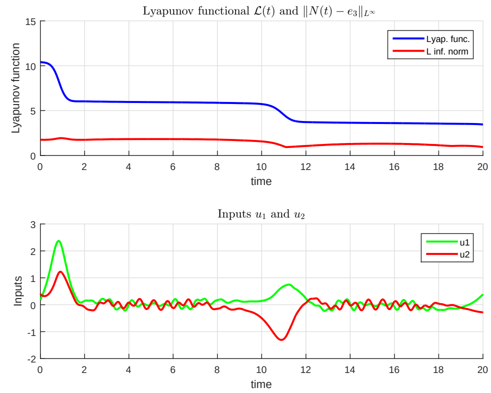

Fig. 4. Lyapunov functional and inputs for the new method (zoom).

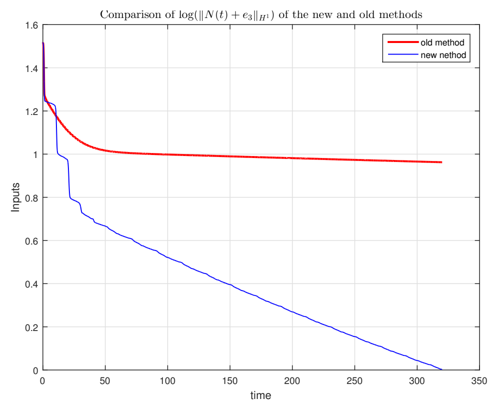

Fig. 5. Plot of the natural logarithm of the square of the H^1 norm of $(\bar{N}(t)+e_3)$.

Theorem 3 *Fix $t_0=kT$ for some $k\in\mathbb{N}$. Let*[4] *$T_0>0$ and $\tau_0=s_0T_0<T/4$ with $s_0\in(0,1/4)$. It is possible to construct $\delta>0$ with the following property: for all $\varepsilon>0$, there exists $c>0$ (depending on ε) such that, for every $T\geq T_0$, and for every initial condition $\bar{N}_0=\bar{N}(t_0)$ such that $\|\bar{N}_0+e_3\|_{H^1}\leq\delta$, and $\|\bar{N}_0+e_3\|_{L^\infty}\geq\varepsilon$, then, one*

[4] By construction, for $T\geq T_0$ the T-periodic adiabatic control $(\bar{u},\bar{v})$ is null for $t\in[0,\tau_0])$.

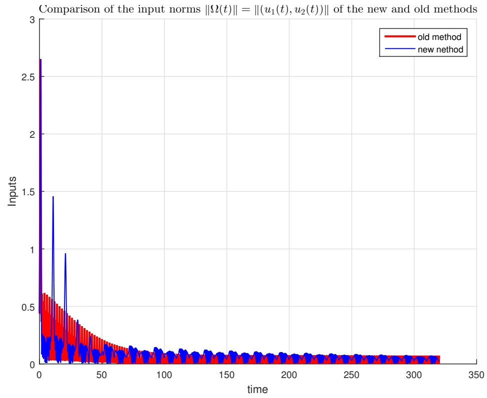

Fig. 6. Plot of input norms $\|u_1(t) + u_2(t)\|$.

will have $\mathcal{L}(\bar{N}(t_0 + \tau_0)) \leq \mathcal{L}(\bar{N}(t_0)) - c$ *for system* (5) *in closed loop with the smooth control law* (10a)-(10b) *that is uniformly bounded versus* T*.*

Proof. See Appendix D. □

The control law (10a)-(10b) stabilizes the auxiliary system uniformly with respect to the choice of T, as shown in the following result.

Corollary 1 *Consider the auxiliary system* (5) *in closed-loop with feedback control defined in* (10a)-(10b)*. Fix an initial condition* $\bar{N}_0$ *such that* $\|\bar{N}_0 + e_3\|_{H^1} < \delta$ *(where* δ *is defined in the statement of Theorem 3). Fix* $\varepsilon > 0$ *and* $T_0 > 0$*. There exists* $\ell > 0$ *, such that, for all* $T \geq T_0$*, the corresponding closed-loop system is such that* $\|\bar{N}(\ell T, \cdot) + e_3\|_{L^\infty} < \varepsilon$*.*

Proof. Let $c > 0$ (that depends on ε) be the constant defined by Theorem 3. Let $p \in \mathbb{N}$ such that $\mathcal{L}(\bar{N}_0) - pc < 0$. By contradiction, assume that $\|\bar{N}(\ell T, \cdot) + e_3\|_{L^\infty} \geq \varepsilon$ for all $\ell \in \{0, 1, \ldots, p\}$. Since the Lyapunov functional $\mathcal{L}(t)$ is nonincreasing, the repetitive application of Theorem 3 at the instants $t = kT$ for $k = 0, 1, \ldots, p$ would give $\mathcal{L}(\bar{N}(pT)) \leq \mathcal{L}(\bar{N}_0) - pc < 0$. This is not possible since the Lyapunov functional is always nonnegative. So there must exist some $\ell \in \{0, 1, \ldots, p\}$ with the claimed property. □

From Theorem 2, Corollary 1 and Theorem 1, one may establish the following strategy for solving our control problem:

(1) Fix $\varepsilon > 0$. Choose $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that [5] $\varepsilon = \varepsilon_1 + \varepsilon_2$.

(2) Find T^* big enough (depending on ℓ) such that $\ell \|\bar{A}(T^-, \cdot) - I\|_{L^\infty} \leq \varepsilon_2$ for all $T \geq T^*$ (application of Theorem 1).

(3) From Corollary 1 of section 2.3, it is possible to find $\ell \in \mathbb{N}$, $T_0 > T^*$ and a control law $\Omega_T : [0, T_f] \to \mathbb{R}^2$ (depending on T), with $T_f = \ell T$, in a way that the application of $(u_1(t), u_2(t)) = \Omega_T(t)$ to system (5) furnishes

$$\|\bar{N}(\ell T, \cdot) + e_3\|_{L^\infty} \leq \varepsilon_1$$

for all $T > T_0$.

(4) Apply the open loop control $(u(t), v(t)) = \Omega_T(t) + (\bar{u}(t), \bar{v}(t)))$ to system (1), obtaining (consequence of Theorem 2):

$$\|M(\ell T, \cdot) + e_3\|_{L^\infty} \leq \varepsilon_1 + \varepsilon_2 = \varepsilon.$$

Notice that by construction the resulting control inputs u and v are smooth time-functions and bounded by a constant g, independent of ϵ, time T and integer ℓ.

One may state the main result of this paper

Theorem 4 *The strategy of the previous steps (1), (2), (3) and (4) always works for solving the local approximate steering problem with smooth bounded control inputs for* ℓ *and* T *big enough. In particular, there exists* ℓ *big enough and* $T^*(\ell) > 0$ *such that the proposed control law furnishes a solution of this problem for all* $T \geq T^*(\ell)$ *at* $T_f = \ell T$*.*

Proof. Easy consequence of Corollary 1 of section 2.3, Theorem 2 and Theorem 1. □

5 Conclusions

The main result of this work indicates that the Rabi pulses that are commonly encountered in *Nuclear Magnetic Ressonance (NMR)* techniques (for instance spin-echo pulses) are not a mandatory ingredient for an efficient open loop control law. One might ask if this could imply that one may develop NMR methods with pulses with less intensity than the ones that are found in the present state of the art. This could be an interesting topic of future research, which may lead to produce less "agressive" NMR techniques for medical (and other possible) applications.

[5] The numerical experiments have shown that the convergence of $\|\bar{A}(T^-, \omega) - I\|$ to zero (when $T \to \infty$) is much faster than the convergence of $\|\bar{N}(\ell T, \cdot) + e_3\|_{L^\infty}$ to zero (when $\ell \to \infty$). Hence it is reasonable to choose ε_1 much larger than ε_2.

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A Vector (wedge) product and map S

Let $c = (c_1\ c_2\ c_3)^\top \in \mathbb{R}^3$ and define the map $S : \mathbb{R}^3 \to \mathfrak{so}(3)$ by

$$S(c) = \begin{bmatrix} 0 & -c_3 & c_2 \\ c_3 & 0 & -c_1 \\ -c_2 & c_1 & 0 \end{bmatrix} \tag{A.1}$$

Note that $S(c)$ is the 3×3 matrix such that $c \wedge v = S(c)v$ for all $c, v \in \mathbb{R}^3$. From the invariance of the dot and the vector products it follows that for all $c, v \in \mathbb{R}^3$ and $A \in SO(3)$ one has:

$$\langle c, v\rangle = \langle Ac, Av\rangle$$
$$A(c \wedge v) = (Ac) \wedge (Av)$$

In particular, $AS(c) = S(Ac)A$ for all $c \in \mathbb{R}^3$ and $A \in SO(3)$.

B Definition of normalized functions a, b, and k

In this appendix we define the functions $a(\cdot)$, $b(\cdot)$, and $k(\cdot)$ that are used in the adiabatic control. A computer simulation is presented in order to illustrate the convergence result (Theorem 1). For this, let $s_0 \in (0, 1/4)$. Define the function $a : [0, 1] \to \mathbb{R}$ by (see Figure 1):

$$a(s) = \begin{cases} -1, \text{if } s \in [0, s_0]; \\ -\cos\left[\dfrac{2\pi(s - s_0)}{1 - 4s_0}\right], \text{if } s \in \left(s_0, \dfrac{1}{2} - s_0\right]; \\ 1, \text{if } s \in \left(\dfrac{1}{2} - s_0, \dfrac{1}{2} + s_0\right]; \\ -\cos\left[\dfrac{2\pi(s - 3s_0)}{1 - 4s_0}\right], \text{if } s \in \left(\dfrac{1}{2} + s_0, 1 - s_0\right]; \\ -1, s \in (1 - s_0, 1]. \end{cases} \tag{B.1a}$$

Define the function $b(\cdot)$ by

$$b(s) = 1 - [a(s)]^2 \tag{B.1b}$$

and $k(\cdot)$ by

$$k(s) = \begin{cases} 1, \text{if } s \in [0, 0.5), \\ -1, \text{if } s \in [0.5, 1] \end{cases} \tag{B.1c}$$

One may extend these functions a, b, k to be 1-periodic functions in a natural way. A computer simulation of the adiabatic propagator $A(t, \omega)$ was done for $T = 10$, $T = 15$ and $T = 20$, with $s_0 = 0.1$ and $K = 10$. The values of $\|A(T^-, \omega) - I\|$ as a function of ω is given in Figure B.1. The fast convergence of the maximum value of this norm to zero when $T \to \infty$ is easily seen in that figure.

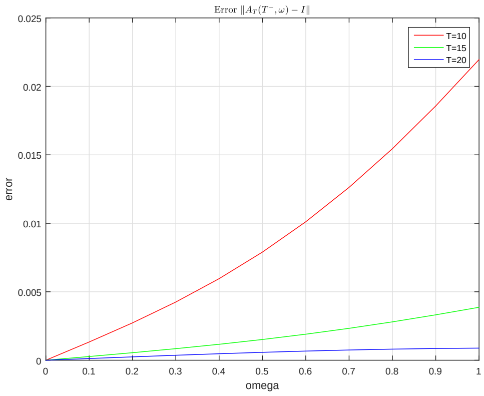

Fig. B.1. Plot of the frobenius norm $\|A(T^-,\omega)-I\|$ as a function of ω with $s_0=0.1$ and $K=10$, for $T=10$, $T=15$ and $T=20$.

C Proof of Theorem 2

The proof of Theorem 2 is a consequence of the following results:

Proposition 1 *Fix $\omega\in(\omega_*,\omega^*)$. Let $J=[\tau_0,\tau_1)\subset\mathbb{R}$ and assume that a continuous input $(u(t),v(t))$ defined in J is applied to system*

$$\dot M(t)=S(u(t)e_1+v(t)e_2+\omega e_3)M(t)$$

Let $M_a(t)$ (respectively $M_b(t)$) be the solution of this system defined on J with initial condition $M_a(\tau_0)$ (resp. $M_b(\tau_0)$). Then $\|M_a(t)-M_b(t)\|=\|M_a(\tau_0)-M_b(\tau_0)\|,\forall t\in J$.

Proof. By time-differentiation, since $S(\cdot)$ is anti-symmetric, it is easy to show that the scalar product $M_a(t)^\top M_b(t)$ is constant in J. Since $M_a(t)$ and $M_b(t)$ are unitary vectors for all $t\in J$, it follows that the angle between them is constant, then $\|M_a(t)-M_b(t)\|$ is constant. □

Proposition 2 *Assume that $\bar N(t,\omega)$ and $M(t,\omega)$ are defined as in the statement of Theorem 2. Let $M_1(t,\omega)=\bar A(t,\omega)\bar N(t,\omega)$. Since $\bar A(kT,\omega)=I$, note that, in each interval $J_k=[kT,(k+1)T)$, $M_1(t,\omega)$ is a solution of (1) with initial condition $\bar N(kT,\omega)$. Assume that $L_k=\lim_{t\to kT^-}\|M(t,\omega)-M_1(t,\omega)\|$. Then $L_k=\|M((k-1)T,\omega)-M_1((k-1)T,\omega)\|$ and $L_{k+1}\le L_k+\|\bar A(kT^-,\cdot)-I\|_{L^\infty}$.*

Proof. In the interval J_k, both curves $M(t,\omega)$ and $M_1(t,\omega)$ are solutions with the same applied input for $k\in\mathbb{N}$. By Proposition 1, the distance $\|M(t,\omega)-M_1(t,\omega)\|$ is constant on $J_k,k\in\mathbb{N}$. By Prop. 1 it follows that $L_k=\|M(kT^-,\omega)-M_1(kT^-,\omega)\|=\|M((k-1)T,\omega)-M_1((k-1)T,\omega)\|$.

As $M(t,\omega)$ and $\bar N(t,\omega)$ are continuous in time, then $M(kT^-,\omega)=M(kT,\omega)$ and $\bar N(kT^-,\omega)=\bar N(kT,\omega)$. Now note that $\|M_1(kT^-,\omega)-M_1(kT,\omega)\|=\|\bar A(kT^-,\omega)\bar N(kT,\omega)-\bar A(kT,\omega)\bar N(kT,\omega)\|=\|(\bar A(kT^-,\omega)-I)\bar N(kT,\omega)\|\le\|\bar A(kT^-,\omega)-I\|_{L^\infty}\|\bar N(kT,\omega)\|=\|\bar A(kT^-,\omega)-I\|_{L^\infty}$. In particular, $\|M(kT,\omega)-M_1(kT,\omega)\|=\|M(kT,\omega)-M_1(kT^-,\omega)+M_1(kT^-,\omega)-M_1(kT,\omega)\|\le\|M(kT,\omega)-M_1(kT^-,\omega)\|+\|M_1(kT^-,\omega)-M_1(kT,\omega)\|$. Now, from the continuity of $M(t,\omega)$ in t and by Prop. 1 applied in J_{k-1}, it follows that $\|M(kT,\omega)-M_1(kT^-,\omega)\|=\|M(kT^-,\omega)-M_1(kT^-,\omega)\|=\|M((k-1)T,\omega)-M_1((k-1)T,\omega)\|=L_{k-1}$. This concludes the proof. □

Now, to prove Theorem 2, note that $N(0,\omega)=M(0,\omega)=M_0$ and so Prop. 1 implies that $L_1=0$. Then, by induction, it follows from the last Proposition that $\|M(kT,\omega)-M_1(kT,\omega)\|=\|M(kT,\omega)-\bar N(kT,\omega)\|=L_k\le k\|\bar A(kT^-,\cdot)-I\|_{L^\infty}$. Hence, $\|M(k,T,\omega)+e_3\|\le\|M(k,T,\omega)-\bar N(kT,\omega)+\bar N(kT,\omega)+e_3\|\le\|M(k,T,\omega)-\bar N(kT,\omega)\|+\|\bar N(kT,\omega)+e_3\|\le k\|\bar A(kT^-,\cdot)-I\|_{L^\infty}+\|\bar N(kT,\omega)+e_3\|$, showing theorem 2.

D Proof of Theorem 3

The Proof of Theorem 3 relies on Lemmas 1 an 2 stated in the sequel.

Lemma 1 *Fix $t_0=kT$. It is possible to construct $\delta>0$ such that, if an initial condition $\bar N_0=\bar N(t_0,\cdot)$ of the auxiliary system (5) is such that $\|\bar N_0+e_3\|_{H^1}<\delta$ and the control law $(u_1(t),u_2(t))$ defined by (10a)-(10b) is null in $[t_0,t_0+\tau_0]$. Then $N_0=-e_3$.*

Proof. Since the auxiliary system (5) is T-periodic, it suffices to show the result for $t_0=0$; The idea is to show that, in the interval $[0,\tau_0]$, Lemma 1 is a particular case of [4, Prop. 3]. For this, note that the dynamics that is considered in that paper when $M_f=e_3$ (that implies that $R(\omega)=I$) is

$$\dot{\bar N}(t,\omega)=S[F(t,\omega)(u_1e_1+u_2e_2)]\bar N(t,\omega)\tag{D.1}$$

where $F(t,\omega)=\exp(\sigma(t)S(\omega e_3))$, and $\dot\sigma(t)=(-1)^{E(t/T)}$, $\sigma(0)=0$ and $E(x)$ denotes the integer part of x. In particular, for $t\in[0,\tau_0)$, one has $\sigma(t)=t$ for $\tau_0<T/2$. As the Proof of [4, Prop. 3] refers only to small neighborhood of t_0, it suffices to note that, for null control $\bar u(t)=\bar v(t)=0$, the solution of (2) is $A(t,\omega)=\exp(tS(\omega e_3))$, hence the dynamics of the auxiliary system (4) is analogous to the dynamics of (D.1), but with $F(t,\omega)=\exp(tS(\omega e_3))$ replaced by $F(-t,\omega)$. Hence similar arguments of the proof of [4, Prop. 3] may be applied to (4). □

Since the auxiliary system is T-periodic, we shall state the next result only for $t_0 = 0$.

Lemma 2 *Consider a sequence of initial conditions* $\bar{N}_0^n \in H^1((\omega_*, \omega_*), S^2)$ *such that* $\bar{N}_0^n \rightharpoonup \bar{N}_0^\infty$ *in* H^1 *weakly. Then the solution* $\bar{N}^n(t, \cdot) \rightharpoonup \bar{N}^\infty(t, \cdot)$ *weakly in* H^1 *and the control* $\Omega^n(t) = (\bar{u}^n(t), \bar{v}^n(t)) \to \Omega^\infty(t)$ *for* $t \in [0, \tau_0]$*, where* $\Omega^\infty(t)$ *is the control* (10a)-(10b) *that is obtained with the initial condition* $\bar{N}_0^\infty$*.*

Proof. Using the same arguments of the Proof of Lemma 1, it suffices to apply the same arguments of the proof of [4, Prop. 4] in the interval $[0, \tau_0]$ in the particular case where $\tau_n = 0, \forall n \in \mathbb{N}$. □

Proof. (of Theorem 3) Since the auxiliary system is T-periodic, there is no loss of generality in considering $t_0 = 0$. The proof of this theorem is based on Lemmas 1 and 2. By contradiction, if the result does not hold, one may construct a sequence $\bar{N}_0^n, n \in \mathbb{N}$ of initial conditions of the auxiliary system with the following properties [6] :
(i) $\|\bar{N}_0^n + e_3\|_{L^\infty} \geq \varepsilon, \forall n \in \mathbb{N}$;
(ii) $\|\bar{N}_0^n + e_3\|_{H^1} \leq \delta, \forall n \in \mathbb{N}$;
(iii) $\int_0^{\tau_0} [(\bar{u}^n)^2(t) + (\bar{v}^n)^2(t)] dt \leq 1/n, \forall n \in \mathbb{N}, n > 0$.
By (ii), passing to a convenient subsequence if necessary, one may assume $\bar{N}_0^n \rightharpoonup \bar{N}_0^\infty$ weakly in H^1. In particular, $\bar{N}_0^n \to \bar{N}_0^\infty$ strongly in the L^∞ norm. By (i), this strong convergence gives $\|\bar{N}_0^\infty + e_3\|_{L^\infty} \geq \varepsilon$. Hence, it is clear that $\|\bar{N}_0^\infty + e_3\|_{H^1} \geq \epsilon(\omega^* - \omega_*) > 0$.

Now, due to weak convergence, $\|\bar{N}_0^\infty + e_3\|_{H^1} \leq \lim_{n\to\infty} \|\bar{N}_0^n + e_3\|_{H^1} \leq \delta$ [6, Prop. 3.5]. Then, we shall show that the initial condition $\bar{N}_0^\infty$ produces null controls for $t \in [0, \tau_0]$. Then Lemma 1 will imply that $\|\bar{N}_0^\infty + e_3\|_{H^1} = 0$, which is a contradiction. By Lemma 2, one has that $\Omega_n(t) = (\bar{u}^n(t), \bar{v}^n(t)) \to \Omega_\infty(t)$ where $\Omega_\infty(t)$ is the control that is obtained with the initial condition $\bar{N}_0^\infty$. An extra work (that is analogous to the first step of the proof of [3, Theorem 1]) shows that the controls are of class C^1, and they are uniformly bounded, as well as their time-derivatives. In particular, the sequence of controls $\Omega_n(t)$ are uniformly bounded and equicontinuous, and so by Ascoli-Arzela theorem, passing to a subsequence if necessary, Ω_n converges to Ω_∞ in C^0 with the sup norm. Assuming that Ω_∞ is not identically null, this gives a contradiction with the fact that the L^2 norm of Ω_n tends to zero (assured by (iii)). □

[6] Note that (iii) follows from (10b) and (11).