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Analysis of the MUSIC estimator for the detection of frequencies in sinusoidal signals

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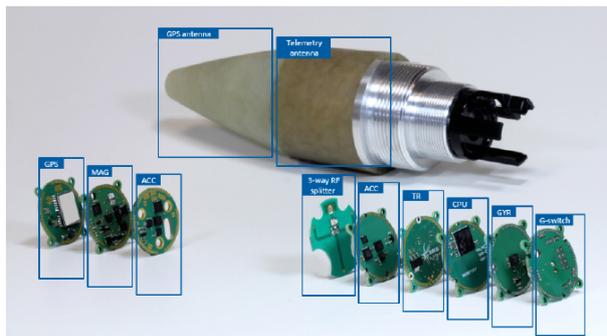
Key words : signal processing ; frequency estimation ; multisinusoids ; noise ; subspace methods ; MUSIC (Multiple Signal Classification) ; statistical analysis ; autocorrelation matrix; spectral decomposition

Abstract : *The paper considers the application of the subspace method MUSIC to the estimation of rolling, nutation and precession frequencies of a high velocity spinning artillery shell. The influence of the various parameters such as signal-to-noise ratio and slow drift of frequencies is studied. The formula giving the variance of the estimation is detailed at the light of the specificities of the application, and a novel result is established stating that when the sought-after frequency is varying in a time-affine manner over the considered data window, the MUSIC estimate recovers the average value of the frequency. A typical case-study serves as illustration, assessing that, at the expense of a increase of the computational load, MUSIC can recover the nutation frequency of the shell with a standard deviation below 1 rad/s.*

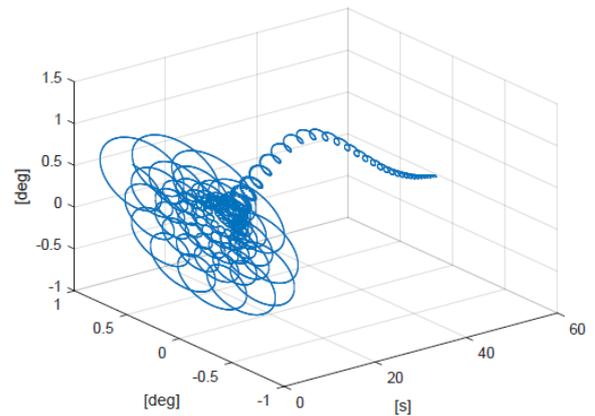
1 Introduction and context

The present research work focuses on one particular technique employed for determining the attitude of smart artillery shells from onboard sensors. In the thesis [3, 4, 5], a new method has been developed to estimate the attitude from strapdown magnetometers and accelerometers (see Figure 1a). Attitude determination from direction vectors is a central question for guidance and control applications of such projectiles. Mathematically it can be formulated as the Wahba's problem [14]. At any given instant, determining the attitude requires the measurement of two linearly independant directions. Interestingly, in free-flight (also termed exterior ballistic phase) the accelerometers can not be used to determine the direction of gravity as is commonly done on the ground, as they solely measure the aerodynamic forces which have no common factor with the gravity (see [3, section 2.5]). This causes an under-determination of the attitude, as one angle (corresponding to the rotation around the single measured direction) is unknown. The methodology proposed in [3] is to recover one angle from an unorthodox use of accelerometer measurements. In a nutshell, the method considers the direction given by the magnetometer and the pitch angle to determinate the attitude. The pitch angle can be obtained from a study of the dynamics of the velocity of the shell by identifying it with the slope angle. The velocity itself can be estimated from a specific treatment of accelerometers measurements. Indeed, a detailed aerodynamic model of the free flight rotational dynamics of the symmetric projectiles reveals that it is subjected to an epicyclic motion (see Figure 1b) that is measured by the accelerometers. The frequencies of the epicyclic motion are directly (and analytically) related

to the translational velocity of the shell [9]. For this reason, the estimation of these frequencies is a solution to estimate the desired angle enabling the full attitude determination. Estimating these frequencies can be achieved using various techniques, among which are windowed Fast-Fourier-Transform (FFT) [2], or, advantageously, subspaces methods such as MUSIC [11]. This last class of methods is the topic under consideration in this paper.



(a) Artillery shell and captors



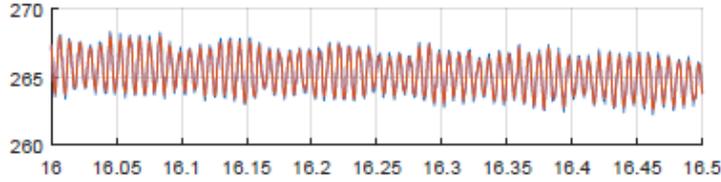
(b) Epicyclic motion

Figure 1

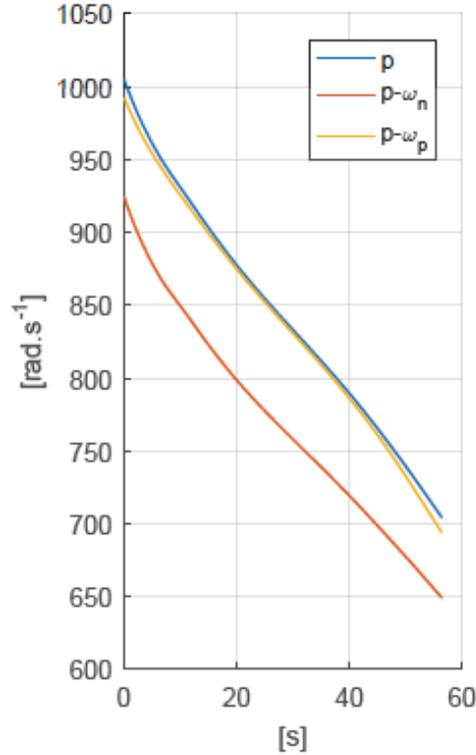
There are 3 frequencies in the epicyclic motion (precession, nutation and spin). Those frequencies are visible in the accelerometer measurements (Figure 2a). The frequencies are relatively close and vary (drift) over time (Figure 2b). Their magnitudes significantly decay during the flight and so does the signal. As a consequence, it is difficult to use traditional FFT method. Instead, we use a subspace method MUSIC (Multiple Signal Classification) [11] that is reported to work well with small windows (over which the drift of frequency is hopefully small), close frequencies and high level of noise, or equivalently low signal-to-noise ratios. Subspace methods rely on the spectral decomposition of the autocorrelation matrix. The general question under study in this article is the estimation of the level of accuracy one can expect in this particular application, considering the noise, the finite size of windows of measurements and the drifting frequencies over the window.

The contributions of this article are two-fold. On the one hand, a numerical estimation of the theoretical variances of the estimation of the frequencies in the studied case is proposed. On the other hand, a theoretical result on the impact of frequency drift is established.

The article is organized as follows. In Section 2, notations are introduced and the problem studied is stated (section 2.1). In Section 2.2, the MUSIC algorithm is presented. In Section 3, the bias, the variance and the dependence on the variability of the frequencies are studied. For the bias and the variance, the proofs follow the classic presentation of subspace methods and stress the role of the variation on the first order of the eigenstructure of the autocorrelation matrix estimation. The main results are Theorem 1 and Theorem 2. Concerning the frequency drift, we study the convergence of the autocorrelation matrix and use trigonometric developments to establish the novel Proposition 12. In Section 4, simulation results are presented, showing that the relation between the error on the frequency estimation and the size of the autocorrelation matrix (m) is not simple and has to be considered to choose the best m possible. In general, the



(a) Example of accelerometer measurement from [3]



(b) Evolution of the frequencies from [3]

Figure 2

error on the frequency estimation decreases then increases when the size of the autocorrelation matrix (m) increases, which increases the computational burden. In practice, an upper bound for m has to be considered (m has to remain smaller than the number of observations $N = 4032$ in the considered window) and for $m = 2000$ and for the parameters corresponding to our study case, the standard deviation is approximatively $0.0004 \text{ rad.s}^{-1}$.

2 Subspace methods

Roughly speaking, to estimate frequencies of a stationary random process, one needs to estimate the power spectral density of the process. There exist two main types of methods for this : non-parametric or classical methods and parametric methods based on a model for the signal [6]. For a sum of sinusoids (usually referred to as multisinusoids) covered in noise, the *subspace* methods are appealing. They rely on the spectral decomposition of the autocorrelation matrix.

Historically, the first subspace method was introduced by Pisarenko [10] and is now seen as a particular case of the MUSIC method introduced by Schmidt [11]. There exist other subspace methods such as minimum norm algorithm or eigenvector method [6]. Due to its popularity and reported solid performances, the algorithm chosen in [3] was the MUSIC algorithm and from now on we will focus on this algorithm.

2.1 Notations and autocorrelation matrix

The problem we study is the following :

Problem 1. Consider a random process s with N samples : $s[0], s[1], \dots, s[N-1]$. At time t , the signal is

$$s(t) = \sum_{k=1}^p x_k(t) + e(t) \text{ with } x_k(t) = \alpha_k e^{i(\omega_k t + \varphi_k)} \quad (1)$$

where $e(t)$ is an additive noise, $(\omega_k)_{k=1, \dots, p}$ are p unknown frequencies that we want to determine, the $(\alpha_k)_{k=1, \dots, p}$ are unknown amplitudes and $(\varphi_k)_{k=1, \dots, p}$ are unknown phases. The following assumptions are made:

- p is known
- ω_k are distinct constants
- α_k are constants
- φ_k are independent random variables uniformly distributed over $[0, 2\pi]$
- $e(t)$ is a real white noise with zero mean and variance σ^2 , its covariance is $E[e(t)e(v)] = \delta_t^v \sigma^2$, where δ is the Kronecker symbol.
- $x_k(t)$ and $e(t)$ are independant

The aim is to **determine the frequencies** $(\omega_k)_{k=1, \dots, p}$.

The MUSIC algorithm rely on the spectral decomposition of the autocorrelation matrix of the signal and we need a few notations before introducing the definition of the autocorrelation matrix.

Notations For a complex valued vector $X \in \mathbb{C}^m$, we denote X^+ its complex conjugate, X^* its conjugate transpose, X^T its transpose. A matrix $M \in \mathbb{C}^{m \times m}$ is hermitian if $M^* = M$. The considered norm is $\|X\| = \sqrt{X^* X}$.

Definition 1. Let $m > p$ be an integer. The autocorrelation matrix of size m of the signal is defined as

$$R = E[y(t)y^*(t)] \in \mathbb{C}^{m \times m} \text{ with } y(t) = \begin{pmatrix} s(t) \\ \vdots \\ s(t+m-1) \end{pmatrix}$$

From the (1), an expression of the autocorrelation matrix depending on the parameters of the signal can be found. Most of the following properties are recalled from [12].

Proposition 1. Note $M = N - m + 1$, and consider $x(t) = \left(\alpha_1 e^{i(\omega_1 t + \varphi_1)} \dots \alpha_p e^{i(\omega_p t + \varphi_p)} \right)^T$, $\varepsilon(t) = \left(e(t) \dots e(t + m - 1) \right)^T$, $a(\omega) = \left(1 e^{i\omega} \dots e^{i\omega(m-1)} \right)^T$, $A = \left(a(\omega_1) \dots a(\omega_p) \right) \in \mathbb{C}^{m \times p}$ and $P = \text{diag}(\alpha_1^2, \dots, \alpha_p^2)$. One has

$$R = APA^* + \sigma^2 I_m \quad (2)$$

The term APA^* is associated with the signal part of the process and the term $\sigma^2 I_m$ is associated with the noise part of the signal.

Proof. From $y(t) = Ax(t) + \varepsilon(t)$, one deduces

$$\begin{aligned} R &= E[y(t)y^*(t)] \\ &= E[(Ax(t) + \varepsilon(t))(Ax(t) + \varepsilon(t))^*] \\ &= E[Ax(t)x^*(t)A^* + Ax(t)\varepsilon^*(t) + \varepsilon(t)x^*(t)A^* + \varepsilon(t)\varepsilon(t)^*] \\ &= AE[x(t)x^*(t)]A^* + 0 + 0 + E[\varepsilon(t)\varepsilon(t)^*] \end{aligned}$$

Moreover, $E[x(t)x^*(t)]_{j,k} = E[\alpha_j e^{i(\omega_j t + \varphi_j)} \alpha_k e^{-i(\omega_k t + \varphi_k)}] = \alpha_j \alpha_k \delta_j^k = P_{j,k}$ so we have $R = APA^* + \sigma^2 I_m$. \square

Proposition 2. Note $\lambda_1, \dots, \lambda_m$ the eigenvalues of R in decreasing order. They are all reals and we have

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p > \lambda_{p+1} = \lambda_{p+2} = \dots = \lambda_m = \sigma^2$$

Proof. APA^* is hermitian positive-definite so the eigenvalues of R are real and superior to σ^2 . Moreover, the rank of APA^* is p so $m - p$ eigenvalues of APA^* are null and $m - p$ eigenvalues of R are equal to σ^2 . \square

Definition 2 (Eigenstructure of the autocorrelation matrix). Note $S = \begin{pmatrix} s_1 & \dots & s_p \end{pmatrix}$ the orthonormal eigenvectors associated with $\{\lambda_1, \dots, \lambda_p\}$ and $G = \begin{pmatrix} g_1 & \dots & g_{m-p} \end{pmatrix}$ the orthonormal eigenvectors associated with $\{\lambda_{p+1}, \dots, \lambda_m\}$. We denote $U = \begin{pmatrix} S & G \end{pmatrix}$ the entire set of eigenvectors of R .

Proposition 3. The vectors of A are orthogonal to the vectors of G : $G^* a(\omega_i) = 0$ for $i \in \{1, \dots, p\}$. And :

$$a^*(\omega_i) G G^* a(\omega_i) = 0 \text{ for } i \in \{1, \dots, p\} \quad (3)$$

To prove this property, we need the following lemma on hermitian matrices.

Lemma 1. Consider $A \in M_{m,n}(\mathbb{C})$ and note W its column space. If $A^* A$ is invertible, then $A(A^* A)^{-1} A^*$ is the orthogonal projection on W .

Proof of lemma 1. Consider $x \in \mathbb{C}^m$. x can be written $x = x_W + x_{W^\perp}$ with $x_W \in W$ and $x_{W^\perp} \in W^\perp$ where $W^\perp = \text{Ker}(A^*)$ is the orthogonal space of W . There exist $c \in \mathbb{C}^n$ such that $x_W = Wc$. $x_{W^\perp} \in W^\perp = \text{Ker}(A^*)$ so

$$A^* x_{W^\perp} = 0 \iff A^*(x - x_W) = 0 \iff A^* x = A^* x_W \iff A^* x = A^* A c \iff (A^* A)^{-1} A^* x = c$$

Hence : $x_W = Ac = A(A^* A)^{-1} A^* x$ and $A(A^* A)^{-1} A^*$ is the orthogonal projection on W . \square

Proof of proposition 3. $RS = S\Lambda$ with $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_p)$ (because S are eigenvectors of R) so from (2) $APA^*S = S\tilde{\Lambda}$ with $\tilde{\Lambda} = \Lambda - \sigma^2 I$ so

$$S = AC \text{ with } C = PA^*S\tilde{\Lambda}^{-1} \text{ and } C^{-1} = S^*A \quad (4)$$

because $S^*S = I$ so $I = S^*AC$ by multiplying the first term of (4) by S^* . We have $GG^* + SS^* = I$ and $SS^* = S(S^*S)^{-1}S^* = A(A^*A)^{-1}A^*$ because the column space of A and S coincide (by the previous lemma). Hence, $GG^* = I - A(A^*A)^{-1}A^*$ and $a^*(\omega_i)GG^*a(\omega_i) = a^*(\omega_i)a(\omega_i) - \underbrace{a^*(\omega_i)A(A^*A)^{-1}A^*a(\omega_i)}_{a^*(\omega_i)} = 0$ and $\|G^*a(\omega_i)\|^2 = 0$ so $G^*a(\omega_i) = 0$. \square

Proposition 4. $G^*a(\omega) \neq 0$ for $\omega \notin (\omega_k)_{k \in \{1, \dots, p\}}$

Proof. If there exists $\omega_{p+1} \notin (\omega_k)_{k=1, \dots, p}$ such that $G^*a(\omega_{p+1}) = 0$ then $SS^*a(\omega_{p+1}) = a(\omega_{p+1}) - GG^*a(\omega_{p+1}) = a(\omega_{p+1})$ ($SS^* = I - GG^*$) and $\begin{pmatrix} a(\omega_1) & \dots & a(\omega_p) & a(\omega_{p+1}) \end{pmatrix}$ is a basis of the column space of S which is of rank p so it is impossible. \square

The idea behind subspace methods is to use Propositions 3 and 4 to build a (frequency-discriminating) function :

$$h(\omega) = \frac{1}{\sum_{i=1}^{m-p} c_i \|g_i^*a(\omega)\|^2}$$

(with $(c_i)_{i=1, \dots, m-p}$ coefficients to choose) that will have local maximas at the frequencies $(\omega_k)_{k=1, \dots, p}$. A particular choice will be made next.

2.2 MUSIC Algorithm

Definition 3. *The (frequency-discriminating) function used in the MUSIC algorithm is the function where the c_i are equal to 1 :*

$$h_{MU}(\omega) = \frac{1}{\sum_{i=1}^{m-p} \|g_i^*a(\omega)\|^2} = \frac{1}{\|G^*a(\omega)\|^2} = \frac{1}{a^*(\omega)GG^*a(\omega)} \quad (5)$$

To build this function, we need an estimation of G from the samples of the signal so we need an estimation of the autocorrelation matrix:

Proposition 5. $\hat{R} = \frac{1}{M} \sum_{n=0}^{M-1} y[n]y^*[n]$ is an unbiased estimator of R and

$$\forall i, j \in \{1, \dots, m\}, \lim_{M \rightarrow \infty} E(|\hat{R}_{i,j} - R_{i,j}|^2) = 0$$

Proof. This is exactly the definition of a process being autocorrelation ergodic and $s(t)$ is autocorrelation ergodic. More details about ergodicity can be found in [6, chapter 3]. \square

Lemma 2. *As $\hat{R} \rightarrow R$ in mean-square sense when $M \rightarrow \infty$ ($\forall i, j \in \{1, \dots, m\}, \lim_{M \rightarrow \infty} E(|\hat{R}_{i,j} - R_{i,j}|^2) = 0$), $\hat{S} \rightarrow S$, $\hat{\Lambda} \rightarrow \Lambda$, $\hat{\Sigma} = \text{diag}(\hat{\lambda}_{p+1}, \dots, \hat{\lambda}_m) \rightarrow \sigma^2 I$, $\hat{S}^*G = (\hat{S} - S)^*G \rightarrow 0$, $S^*\hat{G} = (S - \hat{S})^*\hat{G} \rightarrow 0$, $\hat{G}\hat{G}^* \rightarrow GG^*$, $A^*\hat{G} \rightarrow 0$ and $\hat{\omega}_k \rightarrow \omega_k$.*

Proof. This is due to the continuity of the eigenvalues of R and the continuity of the eigenvectors S (because the multiplicity of their associated eigenvalues is 1). More details can be found in [8, page 130]. \square

Gathering all the ingredients above, one can formulate a streamlined version of MUSIC algorithm.

MUSIC ALGORITHM

- Compute $\hat{R} = \frac{1}{M} \sum_{n=0}^{M-1} y[n]y^*[n]$
- Compute the eigendecomposition¹ of \hat{R}
- Compute \hat{G} with the eigenvectors of \hat{R} associated with the $m - p$ smallest eigenvalues.
- Compute $\hat{h}_{MU}(\omega) = \frac{1}{a^*(\omega)\hat{G}\hat{G}^*a(\omega)}$ ²
- Compute the p (dominant) peaks of \hat{h}_{MU} . These are the frequencies estimates $(\hat{\omega}_k)_{k=1,\dots,p}$

3 Statistical analysis of MUSIC

We are interested in the behaviour of the estimator MUSIC when the parameters m , N are varied and when the frequencies $(\omega_k)_{k=1,\dots,p}$ and the noise change. To address these questions, we study the bias and the variance of the frequencies given by MUSIC.

3.1 Bias

3.1.1 Perturbation matrix

We need to study the evolution of the eigenstructure of a matrix when it is perturbed by a ε -small random matrix and we have the following proposition (inspired by [7]):

Proposition 6. *Consider a hermitian matrix A and a hermitian matrix with zero-mean random entries B . Note ε a small parameter and consider the perturbed matrix is $\hat{A} = A + \varepsilon B$. Note $\lambda_1, \dots, \lambda_m$ the eigenvalues of A in decreasing order, $\hat{\lambda}_1, \dots, \hat{\lambda}_m$ the eigenvalues of \hat{A} , U the entire set of orthonormal eigenvectors of A and \hat{U} the orthonormal eigenvectors of \hat{A} . We suppose, for some p , $\lambda_1 > \lambda_2 > \dots > \lambda_p > \lambda_{p+1} = \lambda_{p+2} = \dots = \lambda_m$. Then, one has :*

$$\forall i \in \{1, \dots, p\}, \hat{u}_i = u_i + \sum_{\substack{j=1 \\ j \neq i}}^m \varepsilon t_{1j}^i u_j + O(\varepsilon^2) \quad (6)$$

$$\text{with } t_{1j}^i = \frac{u_j^* B u_i}{\lambda_i - \lambda_j}$$

Proof. As the p largest eigenvalues are distinct, the eigenvectors $\hat{u}_1, \dots, \hat{u}_p$ tend to u_1, \dots, u_p when ε tends to zero (from [8]). For all $i \in \{1, \dots, p\}$, we can write $\hat{u}_i = u_i + \sum_{\substack{j=1 \\ j \neq i}}^m \varepsilon t_{1j}^i u_j + O(\varepsilon^2)$ an eigenvector of \hat{A} for the eigenvalue $\hat{\lambda}_i$. We have to verify that : $\|\hat{u}_i\|^2 = 1 + O(\varepsilon^2)$:

$$\|\tilde{u}_i\|^2 = (u_i^* + \sum_{\substack{j=1 \\ j \neq i}}^m \varepsilon (t_{1j}^i)^+ u_j^* + O(\varepsilon^2))(u_i + \sum_{\substack{j=1 \\ j \neq i}}^m \varepsilon t_{1j}^i u_j + O(\varepsilon^2))$$

$$\|\tilde{u}_i\|^2 = u_i^* u_i + \sum_{\substack{j=1 \\ j \neq i}}^m \varepsilon^2 (t_{1j}^i)^+ t_{1j}^i u_j^* u_j + O(\varepsilon^2)$$

¹ \hat{R} being hermitian, its eigenvalues are real, further they are positive.

² or alternatively $\hat{h}_{MU}(\omega) = \frac{1}{a^*(\omega)(I - \hat{S}\hat{S}^*)a(\omega)}$ because $\hat{G}\hat{G}^* = I - \hat{S}\hat{S}^*$

$$\|\tilde{u}_i\|^2 = 1 + O(\varepsilon^2)$$

Now, we need to find the coefficients t_{1j}^i . For $i \in \{1, \dots, p\}$ and $j \neq i$ in $\{1, \dots, m\}$, we have (up to ε^2 terms)

$$u_j^* \hat{A} \hat{u}_i = u_j^* \hat{\lambda}_i \hat{u}_i = u_j^* \hat{\lambda}_i (u_i + \sum_{\substack{k=1 \\ k \neq i}}^m \varepsilon t_{1k}^i u_k) = \hat{\lambda}_i \varepsilon t_{1j}^i = \varepsilon \lambda_i t_{1j}^i$$

We also have :

$$u_j^* \hat{A} \hat{u}_i = u_j^* (A + \varepsilon B) (u_i + \sum_{\substack{k=1 \\ k \neq i}}^m \varepsilon t_{1k}^i u_k) = u_j^* A u_i + u_j^* A \sum_{\substack{k=1 \\ k \neq i}}^m \varepsilon t_{1k}^i u_k + \varepsilon u_j^* B u_i = \varepsilon \lambda_j t_{1j}^i + \varepsilon u_j^* B u_i$$

This yields $t_{1j}^i = \frac{u_j^* B u_i}{\lambda_i - \lambda_j}$. □

We can apply this property to R and \hat{R} with $B = \sqrt{M}(\hat{R} - R)$ and $\varepsilon = \frac{1}{\sqrt{M}}$. Thanks to the central limit theorem, $\sqrt{M}(\hat{R} - R)$ is zero-mean with elements asymptotically jointly Gaussian.

Before deriving the expression of the bias, we need the expression of the statistical expectation of the coefficients t_{1j}^i and of \hat{S} and a development of $\hat{G}\hat{G}^*$:

Proposition 7.

$$E[t_{1j}^i] = 0 \text{ and } E[\hat{S} - S] = O\left(\frac{1}{M}\right) \quad (7)$$

Proof. $E[t_{1j}^i] = E\left[\frac{u_j^* B u_i}{\lambda_i - \lambda_j}\right] = 0$ because $E[B] = 0$. It follows that : $E[\hat{S} - S] = O\left(\frac{1}{M}\right)$. □

Proposition 8.

$$\hat{G}\hat{G}^* = GG^* + \frac{1}{\sqrt{M}}Z + O\left(\frac{1}{M}\right) \quad (8)$$

where Z is a zero-mean random matrix.

Proof. We have $\hat{G}\hat{G}^* = I_m - \hat{S}\hat{S}^*$. Thanks to (6), we can develop the expression of $\hat{S}\hat{S}^*$:

$$\begin{aligned} (\hat{S}\hat{S}^*)_{i,j} &= \sum_{k=1}^p \hat{s}_{i,k} \hat{s}_{j,k}^+ = \sum_{k=1}^p [u_{i,k} + \sum_{\substack{q=1 \\ q \neq k}}^m \varepsilon t_{1q}^k u_{i,q} + O(\varepsilon)] [u_{j,k}^+ + \sum_{\substack{r=1 \\ r \neq k}}^m \varepsilon (t_{1r}^k)^+ u_{j,r}^+ + O(\varepsilon)] \\ \hat{S}\hat{S}_{i,j}^* &= \sum_{k=1}^p [u_{i,k} u_{j,k}^+ + \sum_{\substack{r=1 \\ r \neq k}}^m \varepsilon (t_{1r}^k)^+ u_{i,k} u_{j,r}^+ + \sum_{\substack{q=1 \\ q \neq k}}^m \varepsilon t_{1q}^k u_{i,q} u_{j,k}^+ + O(\varepsilon^2)] = (SS^*)_{i,j} - \varepsilon Z_{i,j} + O(\varepsilon^2) \end{aligned}$$

Where $Z_{i,j} = -\sum_{k=1}^p [\sum_{\substack{r=1 \\ r \neq k}}^m (t_{1r}^k)^+ u_{i,k} u_{j,r}^+ + \sum_{\substack{q=1 \\ q \neq k}}^m \varepsilon t_{1q}^k u_{i,q} u_{j,k}^+]$. From (7) : $E[Z] = 0$.

So we have, $\hat{S}\hat{S}^* = SS^* - \frac{1}{\sqrt{M}}Z + O\left(\frac{1}{M}\right)$ and $\hat{G}\hat{G}^* = GG^* + \frac{1}{\sqrt{M}}Z + O\left(\frac{1}{M}\right)$. □

3.1.2 Bias of the estimator

We need one more result before proving that it is an asymptotically unbiased estimator.

Lemma 3. $A^* \hat{G} \hat{G}^* = -A^* S (\hat{S} - S)^* G G^* + O(\frac{1}{M})$

Proof. $A^* \hat{G} \hat{G}^* = A^* \hat{G} \hat{G}^* \hat{G} \hat{G}^* = A^* \hat{G} \hat{G}^* (G G^* + \frac{1}{\sqrt{M}} Z + O(1/M))$ from (8) and because $\hat{G}^* \hat{G} = I$. Then, $A^* \hat{G} \hat{G}^* = A^* (G G^* + \frac{1}{\sqrt{M}} Z + O(1/M)) = \frac{1}{\sqrt{M}} A^* Z + O(1/M)$ so $A^* \hat{G} \hat{G}^* = A^* \hat{G} \hat{G}^* G G^* + O(\frac{1}{M})$. It follows that $A^* \hat{G} \hat{G}^* G G^* = A^* (I - \hat{S} \hat{S}^*) G G^* = -A^* \hat{S} \hat{S}^* G G^*$. Because $S^* G = 0$, one has $A^* \hat{G} \hat{G}^* G G^* = -A^* \hat{S} (\hat{S} - S)^* G G^*$. And from $\hat{S} = S + (\hat{S} - S)$ one gets $A^* \hat{G} \hat{G}^* G G^* = -A^* S (\hat{S} - S)^* G G^* - A^* (\hat{S} - S) (\hat{S} - S)^* G G^*$. Moreover, $(\hat{S} - S) (\hat{S} - S)^* = O(\frac{1}{M})$ which yields the conclusion. \square

With this lemma, we can find a simple expression of $\hat{\omega}_k - \omega_k$ and deduce the bias with this new notations :

- $\beta_k = S^* a(\omega_k)$
- $v_k = \begin{pmatrix} \mu_k^*(\hat{s}_1 - s_1) \\ \vdots \\ \mu_k^*(\hat{s}_p - s_p) \end{pmatrix}$ with $\mu_k = G G^* d(\omega_k)$ with $d(\omega_k) = \frac{da}{d\omega}(\omega)$
- $h_k = d^*(\omega_k) G G^* d(\omega_k)$
- $f(\omega) = a^*(\omega) G G^* a(\omega)$

Proposition 9 (adapted from [12]). *The frequencies estimates are such that, for all $k = 1, \dots, p$,*

$$\hat{\omega}_k - \omega_k = \frac{\text{Re}(\beta_k^T v_k)}{h_k} + O(\frac{1}{M}) \quad (9)$$

Proof. The proof derives from a perturbation analysis of the maximums. By definition, we have $f'(\hat{\omega}_k) = 0$. Thus, an expansion gives $f'(\hat{\omega}_k) \simeq f'(\omega_k) + f''(\omega_k)(\hat{\omega}_k - \omega_k)$ with $f'(\omega) = 2 \text{Re}(a^*(\omega) \hat{G} \hat{G}^* d(\omega))$ and $f''(\omega) = 2d^*(\omega) \hat{G} \hat{G}^* d(\omega) + 2a^*(\omega) \hat{G} \hat{G}^* d'(\omega)$ so $f''(\omega_k) = 2h_k + O(\frac{1}{\sqrt{M}})$. With lemma 3, this gives

$$\begin{aligned} f'(\omega_k) &= -2 \text{Re}(a^*(\omega_k) S (\hat{S} - S)^* G G^* d(\omega_k)) + O(1/M) \\ &= -2 \text{Re}(\beta_k^* (\hat{S} - S)^* \mu_k) + O(1/M) \\ &= -2 \text{Re}(\beta_k^* v_k^+) + O(1/M) \\ &= -2 \text{Re}(\beta_k^T v_k) + O(1/M) \end{aligned}$$

Hence

$$\begin{aligned} f'(\hat{\omega}_k) = f'(\omega_k) + f''(\omega_k)(\hat{\omega}_k - \omega_k) &\iff 0 = -2 \text{Re}(\beta_k^T v_k) + O(1/M) + (2h_k + O(\frac{1}{\sqrt{M}}))(\hat{\omega}_k - \omega_k) \\ &\iff \hat{\omega}_k - \omega_k = \frac{\text{Re}(\beta_k^T v_k)}{h_k} + O(1/M) \end{aligned}$$

\square

It is now easy to prove that the bias of the estimator approaches 0 when M grows.

Theorem 1. $E(\hat{\omega}_k - \omega_k) = O(\frac{1}{M})$ so the estimation of frequencies by MUSIC is asymptotically unbiased.

Proof. We have : $\hat{\omega}_k - \omega_k = \frac{\text{Re}(\beta_k^*(\hat{S}-S)^*\mu_k)}{h_k} + O(1/M)$ so : $h_k \times E(\hat{\omega}_k - \omega_k) = \text{Re}(\beta_k^* E[(\hat{S} - S)^*] \mu_k) + O(1/M)$. Moreover, $E[(\hat{S} - S)^*] = O(\frac{1}{M})$ (7) so $E(\hat{\omega}_k - \omega_k) = O(\frac{1}{M})$. \square

3.2 Variance and covariance

3.2.1 Preliminary results

To derive the expression of the covariance matrix, we need the statistical expectation of $t_{1k}^i t_{1l}^j$ and $t_{1k}^i (t_{1l}^j)^+$ and before deriving their expressions we need the expectation of $Bu_i u_j^* B$ with $B = \sqrt{M}(\hat{R} - R)$ that can be found in Proposition 10 (from [13]). We denote :

- $Q_k = E[\varepsilon(t)\varepsilon^*(t-k)] = E[\varepsilon(t)\varepsilon^T(t-k)] = \sigma^2 J_k$ where $(J_k)_{i,j} = \delta_j^{k+i}$ (and $Q_k = E[y(t)y^T(t-k)]$)
- $R_k = E[y(t)y^*(t-k)] = AP \text{diag}(e^{i\omega_1}, \dots, e^{i\omega_p})^k A^* + Q_k = APD^k A^* + Q_k$

Proposition 10. With $B = \sqrt{M}(\hat{R} - R)$, then for all $i, j \in \{1, \dots, m\}$,

$$E(Bu_i u_j^* B) = \sum_{l=-(m-1)}^{m-1} (u_j^* R_{-l} u_i) R_l + Q_l u_j^+ u_i^T Q_l \quad (10)$$

Proof. We have

$$\begin{aligned} E[\hat{R}u_i u_j^* \hat{R}]_{k,l} &= E[(\hat{R}_k)^* u_i u_j^* \hat{R}_l] \\ &= \frac{1}{M^2} \sum_{q=0}^{M-1} \sum_{r=0}^{M-1} E[y^*(q)s(q+k-1)u_i u_j^* y(r)s^+(r+l-1)] \\ &= \frac{1}{M^2} \sum_{q=0}^{M-1} \sum_{r=0}^{M-1} E[y^*(q)u_i s(q+k-1)u_j^* y(r)s^+(r+l-1)] \end{aligned}$$

We use the following lemma (that we can apply thanks to the central limit theorem that states that when M tends to the infinity, $y^*(q)u_i$, $s(q+k-1)$, $u_j^* y(r)$ and $s^+(r+l-1)$ are Gaussian random variables) :

Lemma 4. (from [1]) If x_1, x_2, x_3 and x_4 are four Gaussian random variables with a least one with zero-mean, we have :

$$E[x_1 x_2 x_3 x_4] = E[x_1 x_2]E[x_3 x_4] + E[x_1 x_3]E[x_2 x_4] + E[x_1 x_4]E[x_2 x_3]$$

$$\begin{aligned}
 E[\hat{R}u_i u_j^* \hat{R}]_{k,l} &= \frac{1}{M^2} \sum_{q=0}^{M-1} \sum_{r=0}^{M-1} E[y^*(q)u_i s(q+k-1)] E[u_j^* y(r) s^+(r+l-1)] \\
 &\quad + E[y^*(q)u_i u_j^* y(r)] E[s(q+k-1) s^+(r+l-1)] \\
 &\quad + E[y^*(q)u_i s^+(r+l-1)] E[s(q+k-1) u_j^* y(r)] \\
 &= \frac{1}{M^2} \sum_{q=0}^{M-1} \sum_{r=0}^{M-1} (R_0)_k^* u_i u_j^* (R_0)_l + u_j^* R_{r-q} u_i (R_{q-r})_{k,l} + (Q_{r-q})_l u_i u_j^* (Q_{r-q})_k \\
 &= (Ru_i u_j^* R)_{k,l} + \frac{1}{M^2} \sum_{|t|<M} (M-|t|) [u_j^* R_{-t} u_i (R_t)_{k,l} + (Q_{-t} u_i u_j^* Q_{-t})_{l,k}] \\
 &= (Ru_i u_j^* R)_{k,l} + \frac{1}{M} \sum_{|t|<M} [u_j^* R_{-t} u_i (R_t)_{k,l} + (Q_t u_j^+ u_i^T Q_t)_{k,l}]
 \end{aligned}$$

So we have : $E[\hat{R}u_i u_j^* \hat{R}]_{k,l} = Ru_i u_j^* R + \frac{1}{M} \sum_{|t|<M} u_j^* R_{-t} u_i R_t + Q_t u_j^+ u_i^T Q_t$. Moreover

$$\begin{aligned}
 E(Bu_i u_j^* B) &= ME((\hat{R} - R)u_i u_j^* (\hat{R} - R)) \\
 &= ME(\hat{R}u_i u_j^* \hat{R}) - ME(\hat{R}u_i u_j^* R) - ME(Ru_i u_j^* \hat{R} - R) \\
 &= MRu_i u_j^* R + \sum_{|t|<M} u_j^* R_{-t} u_i R_t + Q_t u_j^+ u_i^T Q_t - MRu_i u_j^* R - 0 \\
 &= \sum_{l=-(m-1)}^{m-1} (u_j^* R_{-l} u_i) R_l + Q_l u_j^+ u_i^T Q_l
 \end{aligned}$$

□

The statistical expectations of $t_{1k}^i t_{1l}^j$ and $t_{1k}^i (t_{1l}^j)^+$ are now straightforward to establish.

Proposition 11.

$$E[t_{1k}^i (t_{1l}^j)^+] = \frac{1}{(\lambda_i - \lambda_k)(\lambda_j - \lambda_l)} \sum_{|t|<m} (u_k^* R_t u_l u_j^* R_{-t} u_i + u_k^* Q_t u_j^+ u_i^T Q_t u_l) \quad (11)$$

$$E[t_{1k}^i t_{1l}^j] = \frac{1}{(\lambda_i - \lambda_k)(\lambda_j - \lambda_l)} \sum_{|t|<m} (u_l^* R_{-t} u_i u_k^* R_t u_j + u_k^* Q_t u_l^+ u_i^T Q_t u_j) \quad (12)$$

Proof. First, $E[t_{1k}^i (t_{1l}^j)^+] = E[\frac{u_k^* B u_i}{\lambda_i - \lambda_k} \frac{u_j^* B u_l}{\lambda_j - \lambda_l}] = \frac{1}{(\lambda_i - \lambda_k)(\lambda_j - \lambda_l)} \sum_{|t|<m} \underbrace{u_k^* (u_j^* R_{-t} u_i R_t + Q_t u_j^+ u_i^T Q_t) u_l}_{(u_j^* R_{-t} u_i)(u_k^* R_t u_l) + u_k^* Q_t u_j^+ u_i^T Q_t u_l}$

which yields $E[t_{1k}^i t_{1l}^j] = E[\frac{u_k^* B u_i}{\lambda_i - \lambda_k} \frac{u_l^* B u_j}{\lambda_j - \lambda_l}] = \frac{1}{(\lambda_i - \lambda_k)(\lambda_j - \lambda_l)} \sum_{|t|<m} \underbrace{u_k^* (u_l^* R_{-t} u_i R_t + Q_t u_l^+ u_i^T Q_t) u_j}_{(u_l^* R_{-t} u_i)(u_k^* R_t u_j) + u_k^* Q_t u_l^+ u_i^T Q_t u_j}$ □

We need two last lemmas (from [12]) before deriving the expression of the variance :

Lemma 5. For all z_1, z_2 in the column space of G , we denote $v_k = \begin{pmatrix} z_k^*(\hat{s}_1 - s_1) \\ \vdots \\ z_k^*(\hat{s}_p - s_p) \end{pmatrix}$.

We have :

$$E[v_1 v_2^*] = \frac{1}{M} \sum_{k=-(m-1)}^{m-1} \tilde{\Lambda}^{-1} (z_1^* Q_k z_2 S^T R_k^+ S^+ + S^T Q_k z_2 z_1^* Q_k S^+) \tilde{\Lambda}^{-1} \quad (13)$$

$$E[v_1 v_2^T] = \frac{1}{M} \sum_{k=-(m-1)}^{m-1} \tilde{\Lambda}^{-1} (z_1^* Q_k z_2^+ S^T Q_k S + S^T Q_k z_2^+ z_1^* Q_k S) \tilde{\Lambda}^{-1} \quad (14)$$

Proof. First,

$$\begin{aligned} (\lambda_i - \sigma^2)(\lambda_j - \sigma^2) E[v_1 v_2^*]_{i,j} &= (\lambda_i - \sigma^2)(\lambda_j - \sigma^2) E[z_1^*(\hat{s}_i - s_i)(\hat{s}_j - s_j)^* z_2] \\ &= (\lambda_i - \sigma^2)(\lambda_j - \sigma^2) z_1^* E\left[\sum_{\substack{k=1 \\ k \neq i}}^m \varepsilon t_{1k}^i u_k \sum_{\substack{l=1 \\ l \neq j}}^m \varepsilon (t_{1l}^j)^+ u_l^*\right] z_2 \\ &= (\lambda_i - \sigma^2)(\lambda_j - \sigma^2) \frac{1}{M} \sum_{\substack{k=1 \\ k \neq i}}^m \sum_{\substack{l=1 \\ l \neq j}}^m z_1^* u_k \times \\ &\quad \left[\frac{1}{(\lambda_i - \lambda_k)(\lambda_j - \lambda_l)} \sum_{|t| < m} (u_k^* R_t u_l u_j^* R_{-t} u_i + u_k^* Q_t u_j^+ u_i^T Q_t u_l) \right] u_l^* z_2 \end{aligned}$$

Since z_1 and z_2 are in the column space of G , $z_1^* u_k = 0$ if $k \in \{1, \dots, p\}$ and $\sum_{k=p+1}^m \frac{z_1^* u_k u_k^*}{\lambda_i - \lambda_k} = \frac{z_1^* G G^*}{\lambda_i - \sigma^2} = \frac{z_1^*}{\lambda_i - \sigma^2}$ because $G G^*$ is the orthogonal projection on the column space of G . So, one has $E[v_1 v_2^*]_{i,j} = \frac{1}{M} \sum_{k=p+1}^m \sum_{l=p+1}^m z_1^* u_k \left[\frac{1}{(\lambda_i - \lambda_k)(\lambda_j - \lambda_l)} \sum_{|t| < m} (u_k^* R_t u_l u_j^* R_{-t} u_i + u_k^* Q_t u_j^+ u_i^T Q_t u_l) \right] u_l^* z_2$

$$\begin{aligned} (\lambda_i - \sigma^2)(\lambda_j - \sigma^2) E[v_1 v_2^*]_{i,j} &= \frac{1}{M} \sum_{|t| < m} z_1^* R_t z_2 u_j^* R_{-t} u_i + z_1^* Q_t u_j^+ u_i^T Q_t z_2 \\ &= \frac{1}{M} \sum_{|t| < m} z_1^* Q_t z_2 u_i^T R_t^+ u_j^+ + u_i^T Q_t z_2 z_1^* Q_t u_j^+ \end{aligned}$$

because $z_1^* A = 0$, we have

$$E[v_1 v_2^*] = \frac{1}{M} \sum_{k=-(m-1)}^{m-1} z_1^* Q_k z_2 \tilde{\Lambda}^{-1} S^T R_k^+ S^+ \tilde{\Lambda}^{-1} + \tilde{\Lambda}^{-1} S^T Q_k z_2 z_1^* Q_k S^+ \tilde{\Lambda}^{-1}$$

Then,

$$\begin{aligned}
 (\lambda_i - \sigma^2)(\lambda_j - \sigma^2)E[v_1 v_2^T]_{i,j} &= (\lambda_i - \sigma^2)(\lambda_j - \sigma^2)E[z_1^*(\hat{s}_i - s_i)(\hat{s}_j - s_j)^T z_2^+] \\
 &= (\lambda_i - \sigma^2)(\lambda_j - \sigma^2)z_1^* E\left[\sum_{\substack{k=1 \\ k \neq i}}^m \varepsilon t_{1k}^i u_k \sum_{\substack{l=1 \\ l \neq j}}^m \varepsilon t_{1l}^j u_l^T\right] z_2^+ \\
 &= (\lambda_i - \sigma^2)(\lambda_j - \sigma^2) \frac{1}{M} \sum_{\substack{k=1 \\ k \neq i}}^m \sum_{\substack{l=1 \\ l \neq j}}^m z_1^* u_k \times \\
 &\quad \left[\frac{1}{(\lambda_i - \lambda_k)(\lambda_j - \lambda_l)} \sum_{|t| < m} (u_l^* R_{-t} u_i u_k^* R_t u_j + u_k^* Q_t u_l^+ u_i^T Q_t u_j) \right] u_l^T z_2^+
 \end{aligned}$$

Since z_1 and z_2 are in the column space of G , $z_1^* u_k = 0$ if $k \in \{1, \dots, p\}$ and $\sum_{k=p+1}^m \frac{z_1^* u_k u_k^*}{\lambda_i - \lambda_k} = \frac{z_1^* G G^*}{\lambda_i - \sigma^2} = \frac{z_1^*}{\lambda_i - \sigma^2}$ is the orthogonal projection on the column space of G . So :

$$E[v_1 v_2^T]_{i,j} = \frac{1}{M} \sum_{k=p+1}^m \sum_{l=p+1}^m z_1^* u_k \left(\frac{1}{(\lambda_i - \lambda_k)(\lambda_j - \lambda_l)} \sum_{|t| < m} [u_k^* R_t u_j u_i^T R_t^+ u_l^+ + u_k^* Q_t u_l^+ u_i^T Q_t u_j] \right) u_l^T z_2^+$$

$$\begin{aligned}
 (\lambda_i - \sigma^2)(\lambda_j - \sigma^2)E[v_1 v_2^T]_{i,j} &= \frac{1}{M} \sum_{|t| < m} z_1^* R_t u_j u_i^T R_t^+ z_2^+ + z_1^* Q_t z_2^+ u_i^T Q_t u_j \\
 &= \frac{1}{M} \sum_{|t| < m} z_1^* Q_t u_j u_i^T Q_t z_2^+ + z_1^* Q_t z_2^+ u_i^T Q_t u_j
 \end{aligned}$$

because $z_1^* A = 0$, so we have

$$E[v_1 v_2^T] = \frac{1}{M} \sum_{k=-(m-1)}^{m-1} \tilde{\Lambda}^{-1} S^T Q_k z_2^+ z_1^* Q_k S \tilde{\Lambda}^{-1} + z_1^* Q_k z_2^+ \tilde{\Lambda}^{-1} S^T Q_k S \tilde{\Lambda}^{-1}$$

□

Lemma 6. We denote $X = S \tilde{\Lambda}^{-1} S^* A$. We have $S \tilde{\Lambda}^{-1} \beta_k = x_k$, $A^* X = P^{-1}$ and $X = A(A^* A)^{-1} P^{-1}$.

Proof. • $x_k = S \tilde{\Lambda}^{-1} S^* A_k = S \tilde{\Lambda}^{-1} \beta_k$ by definition of $\beta_k = S^* a(\omega_k)$

- $PA^* X = PA^* S \tilde{\Lambda}^{-1} S^* A = CS^* A = CC^{-1} = I$ (4) so $A^* X = P^{-1}$
- X is in the column space of S which coincides with the column space of A so there exists a matrix H such that $X = AH$. We have : $I = PA^* X = PA^* AH$ so $H = (A^* A)^{-1} P^{-1}$ and $X = A(A^* A)^{-1} P^{-1}$.

□

3.2.2 Expression of the covariance

We now have all the elements to establish the expression of the covariance matrix.

Theorem 2. $E[(\hat{\omega}_j - \omega_j)(\hat{\omega}_i - \omega_i)] = \frac{\sigma^4}{2Mh_i h_j \alpha_i^2 \alpha_j^2} \operatorname{Re} \sum_{k=-(m-1)}^{m-1} \mu_j^* J_k \mu_i^+ b_j^T J_k b_i + b_j^T J_k \mu_i^+ \mu_j^* J_k b_i + \mu_j^* J_k \mu_i b_j^T J_k b_i^+ + b_j^T J_k \mu_i \mu_j^* J_k b_i^+$ with b_j the j^{th} column of $B = A(A^* A)^{-1}$.

Proof. From

$$E[(\hat{\omega}_j - \omega_j)(\hat{\omega}_i - \omega_i)] = \frac{1}{h_i h_j} E[\operatorname{Re}(\beta_j^T v_j) \operatorname{Re}(\beta_i^T v_i)]$$

one gets

$$\operatorname{Re}(x) \operatorname{Re}(y) = \frac{1}{2} (\operatorname{Re}(xy) + \operatorname{Re}(xy^+))$$

and $E[(\hat{\omega}_j - \omega_j)(\hat{\omega}_i - \omega_i)] = \frac{1}{2h_i h_j} E[\operatorname{Re}(\beta_j^T v_j v_i^T \beta_i) + \operatorname{Re}(\beta_j^T v_j v_i^* \beta_i^+)] = \frac{1}{2h_i h_j} [\operatorname{Re}(\beta_j^T E[v_j v_i^T] \beta_i) + \operatorname{Re}(\beta_j^T E[v_j v_i^*] \beta_i^+)]$.

$$\begin{aligned} M \beta_j^T E[v_j v_i^T] \beta_i &= \sum_{k=-(m-1)}^{m-1} \beta_j^T \tilde{\Lambda}^{-1} (\mu_j^* Q_k \mu_i^+ S^T Q_k S + S^T Q_k \mu_i^+ \mu_j^* Q_k S) \tilde{\Lambda}^{-1} \beta_i \quad (13) \\ &= \sigma^4 \sum_{k=-(m-1)}^{m-1} \mu_j^* J_k \mu_i^+ x_j^T J_k x_i + x_j^T J_k \mu_i^+ \mu_j^* J_k x_i \quad (\text{lemma 6}) \\ &= \sigma^4 \sum_{k=-(m-1)}^{m-1} \mu_j^* J_k \mu_i^+ \frac{b_j^T}{\alpha_j} J_k \frac{b_i}{\alpha_i} + \frac{b_j^T}{\alpha_j} J_k \mu_i^+ \mu_j^* J_k \frac{b_i}{\alpha_i} \quad (\text{lemma 6}) \\ &= \frac{\sigma^4}{\alpha_i \alpha_j} \sum_{k=-(m-1)}^{m-1} \mu_j^* J_k \mu_i^+ b_j^T J_k b_i + b_j^T J_k \mu_i^+ \mu_j^* J_k b_i \end{aligned}$$

$$\begin{aligned} M \beta_j^T E[v_j v_i^*] \beta_i^+ &= \sum_{k=-(m-1)}^{m-1} \beta_j^T \tilde{\Lambda}^{-1} (\mu_j^* Q_k \mu_i S^T R_k^+ S^+ + S^T Q_k \mu_i \mu_j^* Q_k S^+) \tilde{\Lambda}^{-1} \beta_i^+ \quad (14) \\ &= \sum_{k=-(m-1)}^{m-1} \mu_j^* Q_k \mu_i \underbrace{x_j^T R_k^+ x_i^+}_{=x_i^* R_k^* x_j} + x_j^T Q_k \mu_i \mu_j^* Q_k x_i^+ \\ &= x_i^* (AD^{-k} P A^* + \sigma^2 J_{-k}) x_j \\ &= (P^{-1} D^{-k} P P^{-1})_{ij} + \sigma^2 x_i^* J_{-k} x_j \quad (\text{lemma 6}) \\ &= \frac{\delta_i^j e^{-ik\omega_j}}{\alpha_j^2} + \frac{\sigma^2}{\alpha_i^2 \alpha_j^2} b_i^* J_{-k} b_j \end{aligned}$$

We have: $\sum_{k=-(m-1)}^{m-1} J_k e^{-ik\omega_j} = a(\omega_j) a^*(\omega_j)$ so : $\mu_j^* \sum_{k=-(m-1)}^{m-1} J_k e^{-ik\omega_j} \mu_i = 0$ because $\mu_j^* A = 0$.

$$M \beta_j^T E[v_j v_i^*] \beta_i^+ = \frac{\sigma^4}{\alpha_i^2 \alpha_j^2} \sum_{k=-(m-1)}^{m-1} \mu_j^* J_k \mu_i b_j^T J_k b_i^+ + b_j^T J_k \mu_i \mu_j^* J_k b_i^+$$

Finally : $E[(\hat{\omega}_j - \omega_j)(\hat{\omega}_i - \omega_i)] = \frac{\sigma^4}{2Mh_i h_j \alpha_i^2 \alpha_j^2} \operatorname{Re}[\sum_{k=-(m-1)}^{m-1} \mu_j^* J_k \mu_i^+ b_j^T J_k b_i + b_j^T J_k \mu_i^+ \mu_j^* J_k b_i + \mu_j^* J_k \mu_i b_j^T J_k b_i^+ + b_j^T J_k \mu_i \mu_j^* J_k b_i^+]$. \square

Remark 1. The covariance can be written $E[(\hat{\omega}_j - \omega_j)(\hat{\omega}_i - \omega_i)] = \frac{d}{NSNR_j SNR_i}$ where d depends only on m and the frequencies and $SNR_k = \frac{\alpha_k}{\sigma^2}$ is the k^{th} signal to noise ratio.

Proof. Consider $d = \frac{N}{2Mh_i h_j} \text{Re} \sum_{k=-(m-1)}^{m-1} \mu_j^* J_k \mu_i^+ b_j^T J_k b_i + b_j^T J_k \mu_i^+ \mu_j^* J_k b_i + \mu_j^* J_k \mu_i b_j^T J_k b_i^+ + b_j^T J_k \mu_i \mu_j^* J_k b_i^+ \simeq \frac{1}{2h_i h_j} \text{Re} \sum_{l=-(m-1)}^{m-1} \mu_j^* J_k \mu_i^+ b_j^T J_k b_i + b_j^T J_k \mu_i^+ \mu_j^* J_k b_i + \mu_j^* J_k \mu_i b_j^T J_k b_i^+ + b_j^T J_k \mu_i \mu_j^* J_k b_i^+$. One has $E[(\hat{\omega}_j - \omega_j)(\hat{\omega}_i - \omega_i)] = \frac{d}{NSNR_j SNR_i}$. Interestingly, d depends only on m and on the frequencies because G and A depends only on m and on the frequencies so does B , μ_k and h_k . \square

3.3 Variation of frequency over a window

Below we study the impact of frequency drift. For simplicity, we consider that only one sinusoid is present in the signal. Let p be equal to 2 and we suppose $s(t) = \alpha_0 e^{i((\omega_0 + \varepsilon_0 t)t + \varphi_0)} + \alpha_0 e^{-i((\omega_0 + \varepsilon_0 t)t + \varphi_0)} + e(t)$ with ε_0 a small parameter. As formulated below, the frequency found by MUSIC will be the instantaneous frequency in the middle of the considered window (the instantaneous frequency at time t is $\frac{d((\omega_0 + \varepsilon_0 t)t)}{dt} = \omega_0 + 2\varepsilon_0 t$).

Proposition 12. The estimation of the autocorrelation matrix \hat{R} tends to the matrix \tilde{R}_M associated with the signal : $\tilde{s}(t) = \alpha_0 e^{i(\tilde{\omega}_0 t + \varphi_0)} + \alpha_0 e^{-i(\tilde{\omega}_0 t + \varphi_0)} + e(t)$ with $\tilde{\omega}_0 = \omega_0 + (N - 1)\varepsilon_0$.

Proof. Let \hat{R} be the estimation of the autocorrelation matrix of the signal s and $\hat{R}_{k,l}$ the coefficient of \hat{R} at the k^{th} line and l^{th} column.

$$\begin{aligned} \hat{R}_{k+1,l+1} &= \frac{1}{M} \sum_{n=0}^{M-1} s(n+k) s^+(n+l) \\ &= \frac{1}{M} \sum_{n=0}^{M-1} [2\alpha_0 \cos((\omega_0 + \varepsilon_0(n+k))(n+k) + \varphi_0) + e(n+k)] \times \\ &\quad [2\alpha_0 \cos((\omega_0 + \varepsilon_0(n+l))(n+l) + \varphi_0) + e(n+l)] \\ &= \frac{1}{M} \sum_{n=0}^{M-1} 4\alpha_0^2 \cos((\omega_0 + \varepsilon_0(n+k))(n+k) + \varphi_0) \cos((\omega_0 + \varepsilon_0(n+l))(n+l) + \varphi_0) \\ &\quad + 2\alpha_0 \cos((\omega_0 + \varepsilon_0(n+k))(n+k) + \varphi_0) e(n+l) \\ &\quad + 2\alpha_0 \cos((\omega_0 + \varepsilon_0(n+l))(n+l) + \varphi_0) e(n+k) + e(n+k)e(n+l) \end{aligned}$$

As $-1 \leq 2\alpha_0 \cos((\omega_0 + \varepsilon_0(n+k))(n+k) + \varphi_0) \leq 1$ and $\frac{1}{M} \sum_{n=0}^{M-1} e(n+l) \xrightarrow{M \rightarrow \infty} E[e(t)] = 0$ we have : $\frac{1}{M} \sum_{n=0}^{M-1} 2\alpha_0 \cos((\omega_0 + \varepsilon_0(n+k))(n+k) + \varphi_0) e(n+l) \xrightarrow{M \rightarrow \infty} 0$ and $\frac{1}{M} \sum_{n=0}^{M-1} 2\alpha_0 \cos((\omega_0 + \varepsilon_0(n+l))(n+l) + \varphi_0) e(n+k) \xrightarrow{M \rightarrow \infty} 0$.

Moreover $\frac{1}{M} \sum_{n=0}^{M-1} e(n+k)e(n+l) \xrightarrow{M \rightarrow \infty} E[e(t+k)e(t+l)] = \sigma^2 \delta_k^l$.

We denote $\Pi = E[4\alpha_0^2 \cos((\omega_0 + \varepsilon_0(n+k))(n+k) + \varphi_0) \cos((\omega_0 + \varepsilon_0(n+l))(n+l) + \varphi_0)]$.

We have :

$$\begin{aligned}
 \Pi &= E[2\alpha_0^2 \times \cos((\omega_0 + \varepsilon_0(n+k))(n+k) + \varphi_0 - (\omega_0 + \varepsilon_0(n+l))(n+l) - \varphi_0) \\
 &\quad + 2\alpha_0^2 \times \cos((\omega_0 + \varepsilon_0(n+k))(n+k) + \varphi_0 + (\omega_0 + \varepsilon_0(n+l))(n+l) + \varphi_0)] \\
 &= 2\alpha_0^2 \cos(\omega_0(k-l) + \varepsilon_0(k-l)(2n+k+l))
 \end{aligned}$$

If $k \neq l$:

$$\begin{aligned}
 \frac{1}{M} \sum_{n=0}^{M-1} \Pi &= \operatorname{Re} \left(\frac{1}{M} \sum_{n=0}^{M-1} 2\alpha_0^2 e^{i(\omega_0(k-l) + \varepsilon_0(k-l)(2n+k+l))} \right) \\
 &= \operatorname{Re} \left(\frac{1}{M} 2\alpha_0^2 e^{i(\omega_0(k-l) + \varepsilon_0(k-l)(k+l))} \underbrace{\sum_{n=0}^{M-1} e^{i\varepsilon_0(k-l)2n}}_{\substack{= \frac{1-e^{i2M\varepsilon_0(k-l)}}{1-e^{i2\varepsilon_0(k-l)}} \\ = \frac{e^{iM\varepsilon_0(k-l)} \sin(M\varepsilon_0(k-l))}{e^{i\varepsilon_0(k-l)} \sin(\varepsilon_0(k-l))}} \right) \\
 &= \frac{2\alpha_0^2 \sin(M\varepsilon_0(k-l))}{M \sin(\varepsilon_0(k-l))} \cos(\omega_0(k-l) + \varepsilon_0(k-l)(M+k+l-1))
 \end{aligned}$$

If $k = l$:

$$\frac{1}{M} \sum_{n=0}^{M-1} \Pi = 2\alpha_0^2 \cos(\omega_0(k-l) + \varepsilon_0(k-l)(N-1))$$

We supposed that : $M\varepsilon_0 \ll 1$ and $\varepsilon_0 \ll 1$ so : $\frac{\sin(M\varepsilon_0(k-l))}{M \sin(\varepsilon_0(k-l))} \simeq \frac{M\varepsilon_0(k-l)}{M(\varepsilon_0(k-l))} \simeq 1$.
 Moreover, as $1 \leq k, l \leq m$ for $m \ll M$, we have $M+k+l-1 \simeq M-1 \simeq M+m-1 = N-1$.
 So :

$$\hat{R}_{k+1, l+1} \simeq 2\alpha_0^2 \cos(\omega_0(k-l) + \varepsilon_0(k-l)(N-1)) + \sigma^2 \delta_k^l$$

When M tends to the infinity and if $M\varepsilon_0 \ll 1$, \hat{R} is almost the autocorrelation matrix associated with the signal $\tilde{s}(t) = 2\alpha_0 \cos(\tilde{\omega}_0 t + \varphi_0) + e(t)$ with $\tilde{\omega}_0 = \omega_0 + (N-1)\varepsilon_0$. \square

4 Simulation results

All the simulations³ were done on real-valued signals of the form

$$s(t) = \sum_{k=1}^{2\rho=p} \alpha_k e^{i(\omega_k t + \varphi_k)} + e(t) \text{ with } \forall k \in \{1, \dots, \rho\}, \alpha_{k+\rho} = \alpha_k, \omega_{k+\rho} = -\omega_k \text{ and } \varphi_{k+\rho} = -\varphi_k$$

The signal can be written: $s(t) = \sum_{k=1}^{\rho} 2\alpha_k \cos(\omega_k t + \varphi_k) + e(t)$.

The results are not modified even if the phases are not independant (see appendix for more details).

³<https://github.com/JulietteGerbaux/MUSIC.git>

4.1 Validation of theoretical results

Simulations were made to illustrate the 3 theoretical results presented in this article : bias (Theorem 1), variance (Theorem 2) and variation of frequency (Proposition 12).

To validate that the method was asymptotically unbiased when M tends to the infinity (or equivalently $N \rightarrow \infty$), we made a large number (100) of frequency estimations of the signal

$$s[n] = 2\alpha \cos(\omega n + \phi) + e(t), n = 0, \dots, N - 1$$

for every N between 50 and 2500 with $\alpha = \sigma = 0.5$ and $\omega = \frac{960}{f_s}$ rad ($f_s = 8000$ Hz is the sampling frequency). These values are chosen because they are close to the values of our case-study. With $m = 20$, we plot the mean frequency (multiplied by f_s) that was found by MUSIC for each N in Figure 3. One can see that the method is indeed asymptotically unbiased.

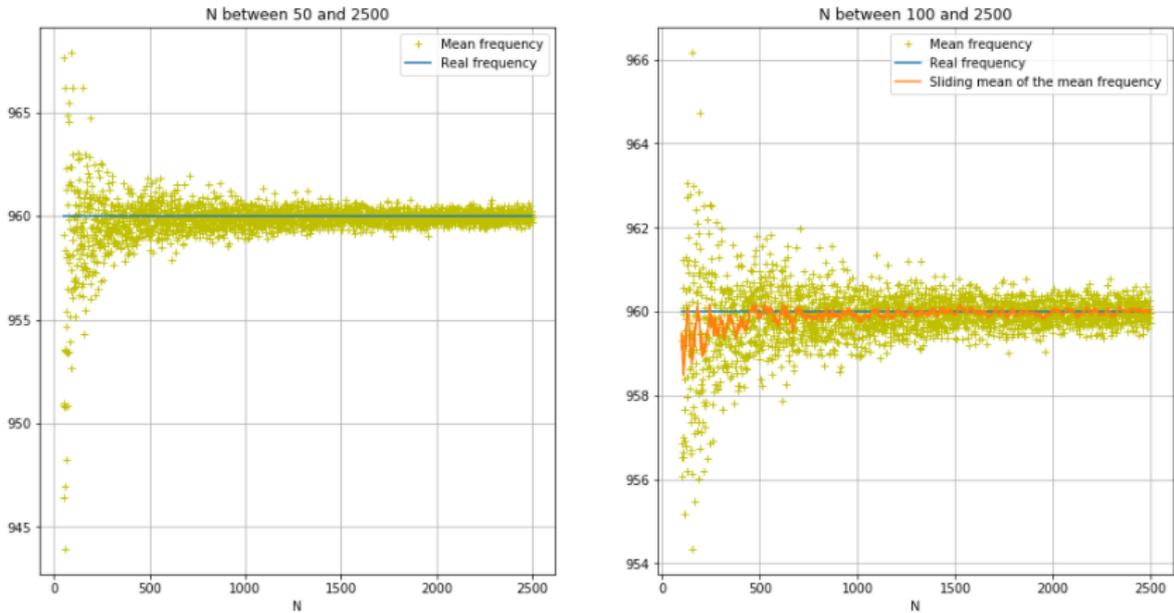


Figure 3 – Frequency estimation by MUSIC with 100 tests for every N with $s(t) = \cos(960t + \varphi) + e(t)$, $m = 20$, $\sigma = 0.5$, $\text{SNR}=1$ and $f_s = 8000$ Hz

Using the same data, we plot the variance of the estimation for every N in Figure 4. One can see that the experimental variance converges to the theoretical variance when N tends to infinity which is the expected result.

To have an idea of the influence of m on the variance, we plot for the same signal and $N = 1000$, the theoretical variance for different m in Figure 5. One can see that the relation between the variance and m is not simple and that numerical computation of the variance is useful to choose the appropriate m .

Finally, to verify the result on the variation of frequency, we applied the MUSIC algorithm to a signal s such that :

$$s(t) = \cos\left(\left(960 + 5\frac{f_s}{N}t\right)t + \varphi\right) + e(t)$$

The frequency found by MUSIC is $965 \text{ rad}\cdot\text{s}^{-1}$ approximately (Figure 6).

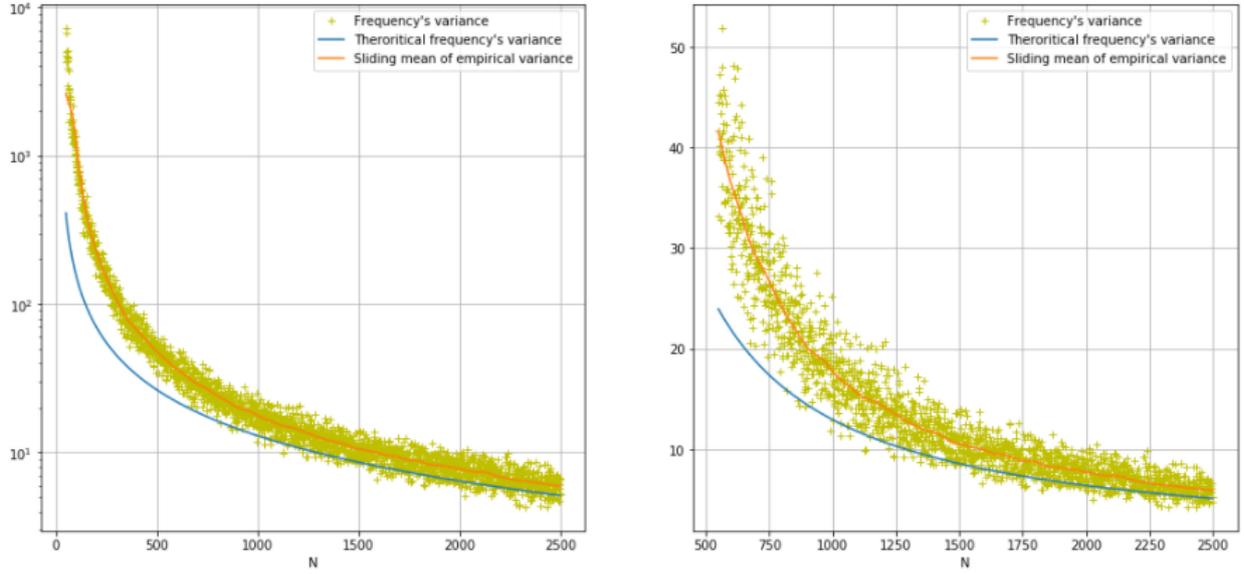


Figure 4 – Variance of the frequency estimation by MUSIC with 100 tests for every N with $s(t) = \cos(960t + \varphi) = e(t)$, $m = 20$, $\sigma = 0.5$, $\text{SNR}=1$ and $f_s = 8000$ Hz

4.2 Case-study

We now show that on a signal typical of our study case, the MUSIC algorithm performs well. A typical signal is [3]:

$$s(t) = \cos(940t + \varphi_1) + 0.91 \cos(958t + \varphi_2) + 0.78 \cos(880t + \varphi_3) + e(t) \quad (15)$$

with $s[n] = \cos(\frac{940}{f_s}n + \varphi_1) + 0.91 \cos(\frac{958}{f_s}n + \varphi_2) + 0.78 \cos(\frac{880}{f_s}n + \varphi_3) + e(t)$ for $n \in \{0, \dots, N-1\}$ with $N = 4032$ and $f_s = 8064$ Hz. The frequencies 940, 958 and 880 are in $\text{rad}\cdot\text{s}^{-1}$. The SNR is 25 so $\sigma^2 = \frac{1}{\sqrt{50}}$.

We tested the MUSIC algorithm on a typical signal (Figure 7) and we obtained the pseudo spectrum in Figure 8. The frequencies found are : 880.2, 940.9 and 958.4 $\text{rad}\cdot\text{s}^{-1}$.

The parameter m has to be chosen and we plot the pseudo-spectrum obtained by MUSIC for different m in Figure 9.

Thanks to the theoretical formula of the variance (Theorem 2), we can compute the expected standard deviation of each typical frequency for different m (Tab. 1) and it appears that $m = 2000$ seems to be a good choice. We could compute the variance for more m to choose the m corresponding to the minimal variance but a standard deviation of 0.0004 satisfies our exigences.

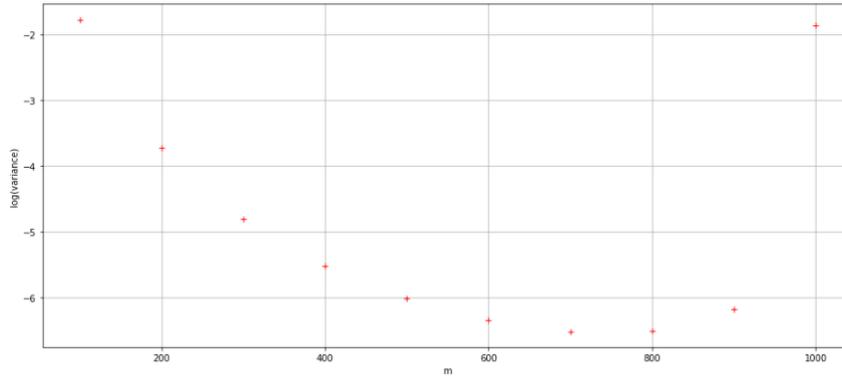


Figure 5 – Variance of the frequency estimation by MUSIC with $s(t) = \cos(960t + \varphi) + e(t)$, $N = 1000$, $\sigma = 0.5$, $\text{SNR}=1$ and $f_s = 8000$ Hz

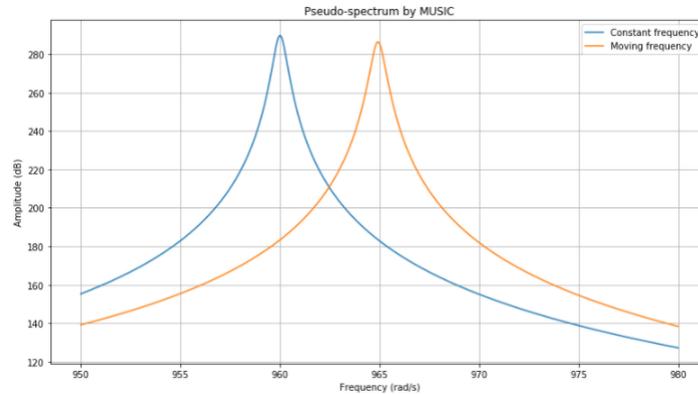


Figure 6 – Frequency detection with MUSIC for $s_1(t) = \cos(960t + \varphi_1)$, $s_2(t) = \cos((960 + 10\frac{f_s}{N}t)t + \varphi_2)$, $N = 20000$, $\sigma = 0.1$ and $f_s = 8000$ Hz

5 Conclusion

When employed on experimental signals, frequency detection methods are facing a list of strong challenges. Some are data acquisition issues such as missing data due to packet losses, irregular sampling rates. In our application, some further issues are caused by the physics of the systems under consideration: closeness of two of the three frequencies of the studied epicyclic motion, drift of frequencies and decay of the magnitudes, high level of noises. All these issues are plausible causes for the performance issues observed in [3] which led to discard the MUSIC implementation and, instead, favor ad-hoc techniques based on envelope filtering.

The present study sheds a new light on this conclusion. It appears that the decrease of the SNR (caused by the decay of the epicyclic motion) has a nonnegligible effect but that it can be circumvented by heavier computations (by increasing the parameter m in the algorithm). It also appears that the drift has, assuming that it remains small, an effect that can be predicted, and if necessary compensated for. However, the closeness of frequencies is indeed a problem, and it is likely that in practice the two close frequencies will be detected as one.

In this new analysis, the formula of the variance is instrumental as it quantifies the accuracy one can expect from MUSIC in function of the various parameters (see Tab.1 for parameters typical of our study case of attitude estimation). Moreover, the result on the variation of

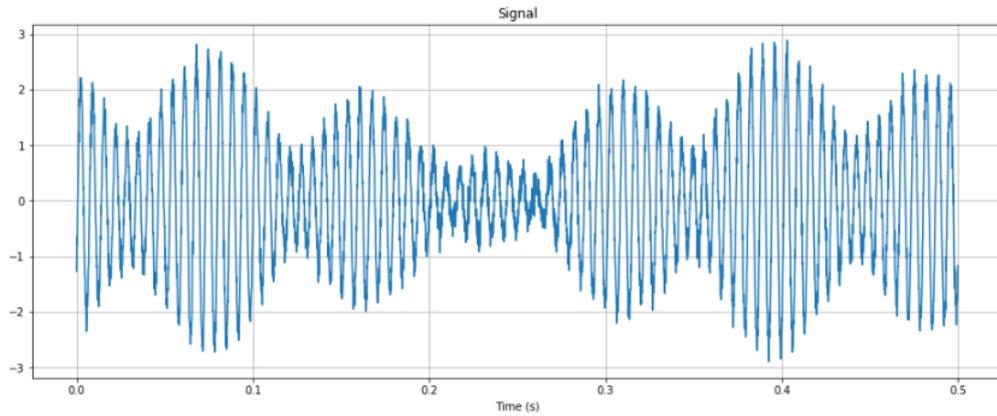


Figure 7 – Typical signal

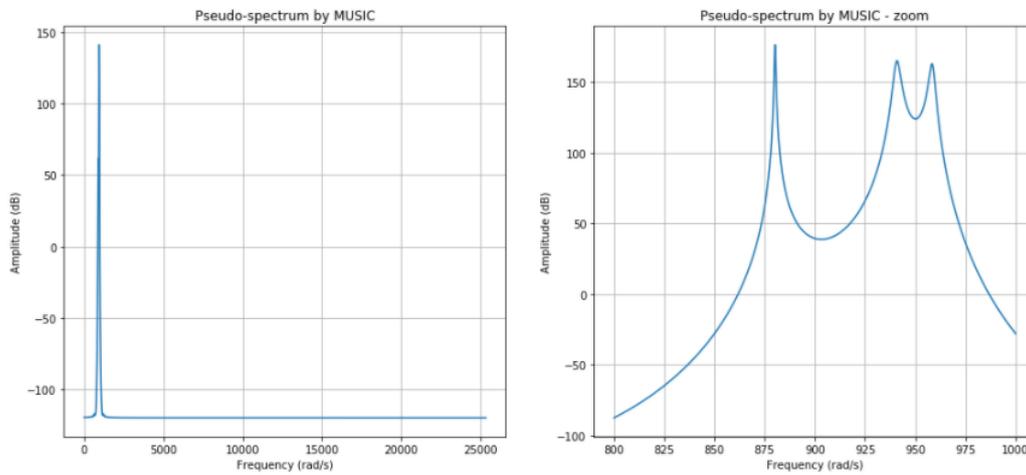


Figure 8 – MUSIC function associated with the typical signal with $m = 400$.

frequency help us to choose appropriate windows for the frequency estimation if we have an idea of how much the frequency vary.

Rather than resorting to black-box implementation of MUSIC, it is concluded that it is better to implement the streamlined algorithm described in §2.2 so that the m parameter can be directly tuned to the requirements of the specific case-study. Particular attention should be paid however in determining with a sufficient level of accuracy the eigenstructure of the empirical autocorrelation matrix.

The effects of other troubles listed above are still unsolved such as the influence of a bad sampling on the MUSIC algorithm.

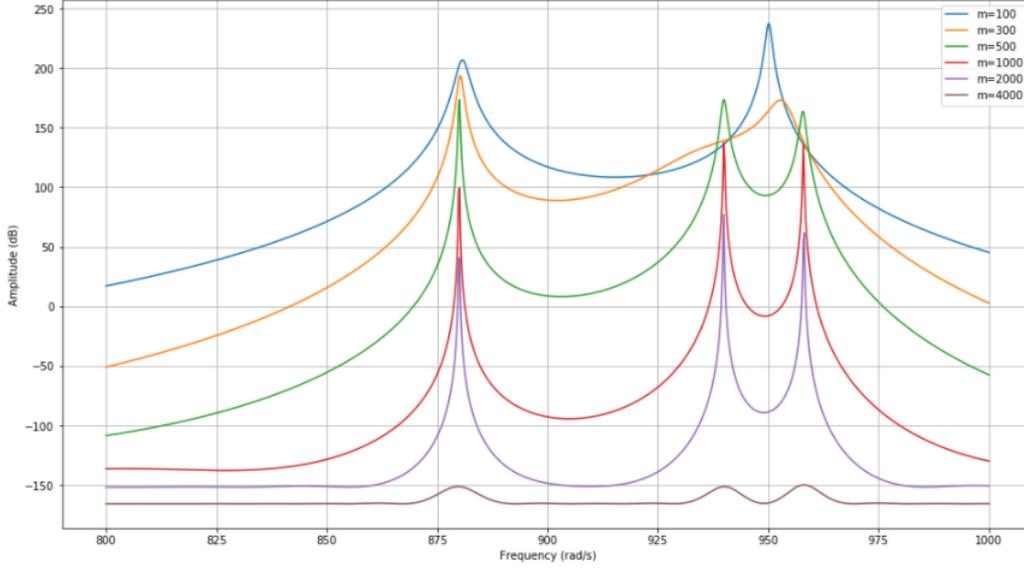


Figure 9 – Pseudo-spectrum of the typical signal obtained by MUSIC for different m

m	$\omega_1 = 940$	$\omega_2 = 958$	$\omega_3 = 880$
200	30.7	22.5	2.86
500	0.169	0.142	0.0215
1000	0.00320	0.00393	0.00107
2000	0.000375	0.000444	0.000419
4000	0.000738	0.000889	0.00114

Table 1 – Standard deviation for the typical signal

Appendix

In this appendix, we are going to show that the results does not change if we consider real signals.

$$s(t) = \sum_{k=1}^{2\rho=p} \alpha_k e^{i(\omega_k t + \varphi_k)} + e(t) \text{ with } \forall k \in \{1, \dots, \rho\}, \alpha_{k+\rho} = \alpha_k, \omega_{k+\rho} = -\omega_k \text{ and } \varphi_{k+\rho} = -\varphi_k$$

The signal can be written: $s(t) = \sum_{k=1}^{\rho} 2\alpha_k \cos(\omega_k t + \varphi_k) + e(t)$.

First, we have $R = APA^* + \sigma^2 I_m$ because even if the phases are not independant :

$$E[x(t)x^*(t)]_{j,k} = E[\alpha_j e^{i(\omega_j t + \varphi_j)} \alpha_k e^{-i(\omega_k t + \varphi_k)}] = \begin{cases} E[\alpha_j^2 e^{i(\omega_j t + \varphi_j) - i(\omega_j t + \varphi_j)}] = \alpha_j^2 & \text{if } j = k \\ E[\alpha_j^2 e^{i(\omega_j t + \varphi_j) + i(\omega_j t + \varphi_j)}] = 0 & \text{if } |j - k| = \rho \\ E[\alpha_j e^{i(\omega_j t + \varphi_j)}] E[\alpha_k e^{-i(\omega_k t + \varphi_k)}] = 0 & \text{else} \end{cases}$$

So : $E[x(t)x^*(t)]_{j,k} = \alpha_j \alpha_k \delta_j^k = P_{j,k}$ and $R = APA^* + \sigma^2 I_m$.

All the results on the spectral decomposition of R still hold and the MUSIC algorithm is still working.

The results on the variance are not modified even if $E[y(t)y^T(t-k)] = E[y(t)y^*(t-k)] = R_k$. Note that the matrix $y(t)$, R , \hat{R} , R_k , S , \hat{S} , G and \hat{G} are real.

We have :

$$\begin{aligned}
 E[\hat{R}u_i u_j^T \hat{R}]_{k,l} &= \frac{1}{M^2} \sum_{q=0}^{M-1} \sum_{r=0}^{M-1} E[y^T(q)s(q+k-1)u_i u_j^T y(r)s(r+l-1)] \\
 &= \frac{1}{M^2} \sum_{q=0}^{M-1} \sum_{r=0}^{M-1} E[y^T(q)u_i s(q+k-1)] E[u_j^T y(r)s(r+l-1)] \\
 &\quad + E[y^T(q)u_i u_j^T y(r)] E[s(q+k-1)s(r+l-1)] \\
 &\quad + E[y^T(q)u_i s(r+l-1)] E[s(q+k-1)u_j^T y(r)] \\
 &= \frac{1}{M^2} \sum_{q=0}^{M-1} \sum_{r=0}^{M-1} (R_0)_k^T u_i u_j^T (R_0)_l + u_j^T R_{r-q} u_i (R_{q-r})_{k,l} + (R_{r-q})_l u_i u_j^T (R_{r-q})_k \\
 &= (R u_i u_j^T R)_{k,l} + \frac{1}{M^2} \sum_{|t|<M} (M-|t|) [u_j^T R_{-t} u_i (R_t)_{k,l} + (R_{-t} u_i u_j^T R_{-t})_{l,k}] \\
 &= (R u_i u_j^T R)_{k,l} + \frac{1}{M} \sum_{|t|<M} [u_j^T R_{-t} u_i (R_t)_{k,l} + (R_t u_j u_i^T R_t)_{k,l}]
 \end{aligned}$$

So we have : $E[\hat{R}u_i u_j^T \hat{R}]_{k,l} = R u_i u_j^T R + \frac{1}{M} \sum_{|t|<M} u_j^T R_{-t} u_i R_t + R_t u_j u_i^T R_t$.

Hence : $E(B u_i u_j^T B) = \sum_{l=-m}^{m-1} (u_j^T R_{-l} u_i) R_l + R_l u_j u_i^T R_l$.

Then :

$$E[t_{1k}^i (t_{1l}^j)^\dagger] = E[t_{1k}^i t_{1l}^j] = E\left[\frac{u_k^T B u_i}{\lambda_i - \lambda_k} \frac{u_l^T B u_j}{\lambda_j - \lambda_l}\right] = \frac{1}{(\lambda_i - \lambda_k)(\lambda_j - \lambda_l)} \sum_{|t|<m} \underbrace{u_k^T (u_l^T R_{-t} u_i R_t + R_t u_l u_i^T R_t) u_j}_{(u_i^T R_t u_l)(u_k^T R_t u_j) + (u_k^T R_t u_l)(u_i^T R_t u_j)}$$

Let z_1, z_2 be two vectors in the column space of G (not necessarily real) and let v_k be

$$\begin{pmatrix} z_k^*(\hat{s}_1 - s_1) \\ \vdots \\ z_k^*(\hat{s}_p - s_p) \end{pmatrix}.$$

First :

$$\begin{aligned}
 (\lambda_i - \sigma^2)(\lambda_j - \sigma^2)E[v_1 v_2^*]_{i,j} &= (\lambda_i - \sigma^2)(\lambda_j - \sigma^2)E[z_1^*(\hat{s}_i - s_i)(\hat{s}_j - s_j)^T z_2] \\
 &= (\lambda_i - \sigma^2)(\lambda_j - \sigma^2)z_1^* E\left[\sum_{\substack{k=1 \\ k \neq i}}^m \varepsilon t_{1k}^i u_k \sum_{\substack{l=1 \\ l \neq j}}^m \varepsilon t_{1l}^j u_l^T\right] z_2 \\
 &= (\lambda_i - \sigma^2)(\lambda_j - \sigma^2) \frac{1}{M} \sum_{\substack{k=1 \\ k \neq i}}^m \sum_{\substack{l=1 \\ l \neq j}}^m z_1^* u_k \times \\
 &\quad \left[\frac{1}{(\lambda_i - \lambda_k)(\lambda_j - \lambda_l)} \sum_{|t| < m} (u_i^T R_t u_l u_k^T R_t u_j + u_k^T R_t u_l u_i^T R_t u_j) \right] u_l^T z_2 \\
 &= (\lambda_i - \sigma^2)(\lambda_j - \sigma^2) \frac{1}{M} \sum_{k=p+1}^m \sum_{l=p+1}^m z_1^* u_k \times \\
 &\quad \left[\frac{1}{(\lambda_i - \lambda_k)(\lambda_j - \lambda_l)} \sum_{|t| < m} (u_i^T R_t u_l u_k^T R_t u_j + u_k^T R_t u_l u_i^T R_t u_j) \right] u_l^T z_2 \\
 &= \frac{1}{M} \sum_{|t| < m} u_i^T R_t z_2 z_1^* R_t u_j + z_1^* R_t z_2 u_i^T R_t u_j \\
 &= \frac{1}{M} \sum_{|t| < m} u_i^T Q_t z_2 z_1^* Q_t u_j + z_1^* Q_t z_2 u_i^T R_t u_j
 \end{aligned}$$

So we have : $E[v_1 v_2^*] = \frac{1}{M} \sum_{k=-m}^{m-1} z_1^* Q_k z_2 \tilde{\Lambda}^{-1} S^T R_k S \tilde{\Lambda}^{-1} + \tilde{\Lambda}^{-1} S^T Q_k z_2 z_1^* Q_k S \tilde{\Lambda}^{-1}$.
 Then :

$$\begin{aligned}
 (\lambda_i - \sigma^2)(\lambda_j - \sigma^2)E[v_1 v_2^T]_{i,j} &= (\lambda_i - \sigma^2)(\lambda_j - \sigma^2)E[z_1^*(\hat{s}_i - s_i)(\hat{s}_j - s_j)^T z_2^+] \\
 &= (\lambda_i - \sigma^2)(\lambda_j - \sigma^2)z_1^* E\left[\sum_{\substack{k=1 \\ k \neq i}}^m \varepsilon t_{1k}^i u_k \sum_{\substack{l=1 \\ l \neq j}}^m \varepsilon t_{1l}^j u_l^T\right] z_2^+ \\
 &= (\lambda_i - \sigma^2)(\lambda_j - \sigma^2) \frac{1}{M} \sum_{k=p+1}^m \sum_{l=p+1}^m z_1^* u_k \times \\
 &\quad \left[\frac{1}{(\lambda_i - \lambda_k)(\lambda_j - \lambda_l)} \sum_{|t| < m} (u_i^T R_t u_l u_k^T R_t u_j + u_k^T R_t u_l u_i^T R_t u_j) \right] u_l^T z_2^+ \\
 &= \frac{1}{M} \sum_{|t| < m} u_i^T R_t z_2^+ z_1^* R_t u_j + z_1^* R_t z_2^+ u_i^T R_t u_j \\
 &= \frac{1}{M} \sum_{|t| < m} u_i^T Q_t z_2^+ z_1^* Q_t u_j + z_1^* Q_t z_2^+ u_i^T R_t u_j
 \end{aligned}$$

So we have : $E[v_1 v_2^T] = \frac{1}{M} \sum_{k=-m}^{m-1} \tilde{\Lambda}^{-1} S^T Q_k z_2^+ z_1^* Q_k S \tilde{\Lambda}^{-1} + z_1^* Q_k z_2^+ \tilde{\Lambda}^{-1} S^T R_k S \tilde{\Lambda}^{-1}$.

$$E[(\hat{\omega}_j - \omega_j)(\hat{\omega}_i - \omega_i)] = \frac{1}{h_i h_j} E[\operatorname{Re}(\beta_j^T v_j) \operatorname{Re}(\beta_i^T v_i)]$$

We have : $\operatorname{Re}(x) \operatorname{Re}(y) = \frac{1}{2}(\operatorname{Re}(xy) + \operatorname{Re}(xy^+))$.

So : $E[(\hat{\omega}_j - \omega_j)(\hat{\omega}_i - \omega_i)] = \frac{1}{2h_i h_j} E[\text{Re}(\beta_j^T v_j v_i^T \beta_i) + \text{Re}(\beta_j^T v_j v_i^* \beta_i^+)] = \frac{1}{2h_i h_j} [\text{Re}(\beta_j^T E[v_j v_i^T] \beta_i) + \text{Re}(\beta_j^T E[v_j v_i^*] \beta_i^+)]$.

$$\begin{aligned} M\beta_j^T E[v_j v_i^T] \beta_i &= \sum_{k=-(m-1)}^{m-1} \beta_j^T \tilde{\Lambda}^{-1} (\mu_j^* Q_k \mu_i^+ S^T R_k S + S^T Q_k \mu_i^+ \mu_j^* Q_k S) \tilde{\Lambda}^{-1} \beta_i \\ &= \sum_{k=-(m-1)}^{m-1} \mu_j^* Q_k \mu_i^+ x_j^T R_k x_i + x_j^T Q_k \mu_i^+ \mu_j^* Q_k x_i \end{aligned}$$

There exist l such that $1 \leq l \leq p$ and $|l - i| = \rho$. We have $x_i = S\tilde{\Lambda}^{-1}S^T a(\omega_i) = S\tilde{\Lambda}^{-1}S^T a(-\omega_l) = S\tilde{\Lambda}^{-1}S^T a^+(\omega_l) = x_l^+$. Then, $x_j^T R_k x_i = x_i^* R_k^* x_j = x_i^* (AD^{-k}PA^* + \sigma^2 J_{-k}) x_j = (P^{-1}D^{-k}PP^{-1})_{ij} + \sigma^2 x_i^* J_{-k} x_j = \frac{\delta_i^j e^{-ik\omega_j}}{\alpha_j^2} + \frac{\sigma^2}{\alpha_i^2 \alpha_j^2} b_i^* J_{-k} b_j = \frac{\sigma^2}{\alpha_i^2 \alpha_j^2} b_j^T J_k b_i$ because we have : $\sum_{k=-(m-1)}^{m-1} J_k e^{-ik\omega_j} = a(\omega_j) a^*(\omega_j)$ so : $\mu_j^* \sum_{k=-(m-1)}^{m-1} J_k e^{-ik\omega_j} \mu_i^+ = 0$ because $\mu_j^* A = 0$. We get :

$$\begin{aligned} M\beta_j^T E[v_j v_i^T] \beta_i &= \sigma^4 \sum_{k=-(m-1)}^{m-1} \mu_j^* J_k \mu_i^+ \frac{b_j^T}{\alpha_j} J_k \frac{b_i}{\alpha_i} + \frac{b_j^T}{\alpha_j} J_k \mu_i^+ \mu_j^* J_k \frac{b_i}{\alpha_i} \\ &= \frac{\sigma^4}{\alpha_i \alpha_j} \sum_{k=-(m-1)}^{m-1} \mu_j^* J_k \mu_i^+ b_j^T J_k b_i + b_j^T J_k \mu_i^+ \mu_j^* J_k b_i \end{aligned}$$

$$\begin{aligned} M\beta_j^T E[v_j v_i^*] \beta_i^+ &= \sum_{k=-(m-1)}^{m-1} \beta_j^T \tilde{\Lambda}^{-1} (\mu_j^* Q_k \mu_i S^T R_k S + S^T Q_k \mu_i \mu_j^* Q_k S) \tilde{\Lambda}^{-1} \beta_i^+ \\ &= \sum_{k=-(m-1)}^{m-1} \mu_j^* Q_k \mu_i \underbrace{x_j^T R_k x_i^+}_{=x_i^* R_k^* x_j} + x_j^T Q_k \mu_i \mu_j^* Q_k x_i^+ \\ &= x_i^* (AD^{-k}PA^* + \sigma^2 J_{-k}) x_j \\ &= (P^{-1}D^{-k}PP^{-1})_{ij} + \sigma^2 x_i^* J_{-k} x_j \\ &= \frac{\delta_i^j e^{-ik\omega_j}}{\alpha_j^2} + \frac{\sigma^2}{\alpha_i^2 \alpha_j^2} b_i^* J_{-k} b_j \\ &= \frac{\sigma^4}{\alpha_i^2 \alpha_j^2} \sum_{k=-(m-1)}^{m-1} \mu_j^* J_k \mu_i b_j^T J_k b_i^+ + b_j^T J_k \mu_i \mu_j^* J_k b_i^+ \end{aligned}$$

Finally : $E[(\hat{\omega}_j - \omega_j)(\hat{\omega}_i - \omega_i)] = \frac{\sigma^4}{2Mh_i h_j \alpha_i^2 \alpha_j^2} \text{Re}[\sum_{k=-(m-1)}^{m-1} \mu_j^* J_k \mu_i^+ b_j^T J_k b_i + b_j^T J_k \mu_i^+ \mu_j^* J_k b_i + \mu_j^* J_k \mu_i b_j^T J_k b_i^+ + b_j^T J_k \mu_i \mu_j^* J_k b_i^+]$

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