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Delay robust stabilization of an underactuated network of two interconnected PDE systems

Jean Auriol¹, Florent Di Meglio¹, and Federico Bribiesca-Argomedo²

Abstract—We detail in this article the development of a delay-robust stabilizing state feedback control law for an underactuated network of two systems of two heterodirectional linear first-order hyperbolic Partial Differential Equations interconnected through their boundaries. Only one of the two subsystems is actuated. The proposed approach combines successive backstepping transformations which allow to rewrite the original network system as a simple neutral system with distributed delays for which the control design is easy. The proposed feedback law is finally proved to be robust to small delays in the actuation.

I. INTRODUCTION

In this paper we develop a linear feedback control law that achieves delay-robust stabilization for an underactuated network of two systems of two heterodirectional linear first-order hyperbolic Partial Differential Equations (PDEs). More precisely, the two sub-systems are interconnected through their boundaries but only one of them is actuated.

The control of network of PDEs is an active research as this class of systems naturally arises in multiple industrial processes. Among them, we can cite electric power transmission across long distance using high voltage direct current transmission (HVDC) networks [1], [21], [22], where the electrical interconnection of the transmission lines at the network nodes leads to a coupling of the corresponding line PDEs at their boundaries. Similar kinds of networks can be found considering density-flow systems [6], [7], open canals [13], [14] or the design of a virtual flow meter pilot [17]. In most cases, only the PDE located at one extremity of the network can be actuated. These systems are consequently underactuated which makes their stabilization particularly challenging. To tackle this problem, different approaches have been proposed in the literature. PI boundary controllers have for instance been considered in [6], [7] for fully actuated networks (i.e. with one control per set of heterodirectional PDEs), establishing explicit stability conditions using appropriate quadratic Lyapunov functions. In [21], [22], the authors consider a flatness-based design of a feedforward control of tree-like transmission networks. Recently, the exact boundary controllability of nodal profile for quasilinear hyperbolic systems with interface conditions in a tree-like networks has been assessed in [14], [26] using

the method of characteristics. However, this last approach is non-constructive as it does not provide an explicit formulation of the corresponding feedback laws.

Considering linear hyperbolic systems, it has been possible using the backstepping approach [18] to design stabilizing control law for a large class of problems [4], [10], [9]. Recent contributions [2] have highlighted the necessity of a change of paradigm to achieve delay-robust stabilization, as it has been observed [11], [19] that for many feedback systems, the introduction of arbitrarily small time-delays in the loop may cause instability under linear feedback. The design of delay-robust control laws has been achieved using the backstepping approach to rewrite the considered system as neutral systems with distributed delays [2], [5] and adjusting the analysis methods developed for time delay systems [15], [16].

The main contribution of this paper is to provide an explicit control design that ensures the delay-robust stabilization of an **underactuated network** of two systems of two linear hyperbolic PDEs systems. This is done by means of successive backstepping transformations which allow to rewrite the original network system as a simple neutral system with distributed delays for which the control design is easy. Our approach is the following: (i) A Volterra transformation is constructed, removing some of the in-domain couplings present in the two subsystems. Due to the interconnection, some undesirable integral terms appear at the boundaries between the two subsystems. (ii) A second transformation is then used to get rid of these terms, moving them inside the actuated PDE-subsystem. (iii) Using a last Volterra transformation, it becomes possible to rewrite the system as a neutral difference system for which a state-feedback law is constructed. (iv) Finally, the delay-robustness properties of the system are studied by means of an algebraic analysis in the Laplace domain.

The paper is organized as follows. In Section 2, we introduce the model equations and the notations. In Section 3, we present the stabilization result: combining successive Volterra transformations, the original system can be rewritten as a difference delay system for which it is possible to derive a stabilizing control law. The corresponding feedback system is proved to be stable to small delays in Section 4.

II. PROBLEM DESCRIPTION

A. System under consideration

We consider the following 2-state linear hyperbolic system

$$\partial_t u_1(t, x) + \lambda_1 \partial_x u_1(t, x) = \sigma_1^+(x) v_1(t, x), \quad (1)$$

$$\partial_t v_1(t, x) - \mu_1 \partial_x v_1(t, x) = \sigma_1^-(x) u_1(t, x), \quad (2)$$

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evolving in $\{(t, x) \text{ s.t. } t > 0, x \in [0, 1]\}$, where $\lambda_1 > 0$ and $\mu_1 > 0$ are the constant transport velocities respectively associated to equations (1)-(2) and where the in-domain coupling terms σ_1^+ and σ_1^- are continuous functions defined on $[0, 1]$. The associated initial conditions are $u_1^0(\cdot) = u_1(\cdot, t)$ and $v_1^0(\cdot) = v_1(\cdot, t)$ and are defined in $L^2([0, 1], \mathbb{R})$. In addition, $u_1(t, x)$ and $v_1(t, x)$ satisfy the following boundary conditions

$$u_1(t, 0) = q_{11}v_1(t, 0) + q_{12}\bar{v}_2(t, 0), \quad (3)$$

$$v_1(t, 1) = \rho_1u_1(t, 1) + V(t), \quad (4)$$

where the coupling terms $q_{11} \neq 0$, q_{12} and ρ_1 are constant, where V is an input function (control law) that takes real values and where \bar{v}_2 corresponds to the evaluation at $x = 0$ of the state $\bar{v}_2(t, x)$ which is solution of the following linear hyperbolic system

$$\partial_t \bar{v}_2(t, x) + \mu_2 \partial_x \bar{v}_2(t, x) = \bar{\sigma}_2^-(x) \bar{u}_2(t, x), \quad (5)$$

$$\partial_t \bar{u}_2(t, x) - \lambda_2 \partial_x \bar{u}_2(t, x) = \bar{\sigma}_2^+(x) \bar{v}_2(t, x), \quad (6)$$

evolving in $\{(t, x) \text{ s.t. } t > 0, x \in [-1, 0]\}$, where $\mu_2 > 0$ and $\lambda_2 > 0$ are the constant transport velocities respectively associated to equations (5)-(6) and where the in-domain coupling terms $\bar{\sigma}_2^+$ and $\bar{\sigma}_2^-$ are continuous functions defined on $[-1, 0]$. The associated initial conditions are $\bar{u}_2^0(\cdot) = \bar{u}_2(\cdot, 0)$ and $\bar{v}_2^0(\cdot) = \bar{v}_2(\cdot, 0)$ and are defined in $L^2([-1, 0], \mathbb{R})$. We have in addition the following boundary conditions

$$\bar{u}_2(t, 0) = q_{22}\bar{v}_2(t, 0) + q_{21}v_1(t, 0), \quad (7)$$

$$\bar{v}_2(t, -1) = \rho_2u_1(t, -1), \quad (8)$$

where the coupling terms q_{22} , q_{21} and $\rho_2 \neq 0$ are constant. System (1)-(8) is depicted in Figure 1.

To simplify the computations in the next sections, we will rewrite the PDE system (5)-(8) in the space $[0, 1]$ instead of $[-1, 0]$. To this extent, we consider the new states $u_2(t, x)$ and $v_2(t, x)$ evolving in $\{(t, x) \text{ s.t. } t > 0, x \in [0, 1]\}$ and defined by $u_2(t, x) = \bar{u}_2(t, -x)$ and $v_2(t, x) = \bar{v}_2(t, -x)$. We also define on $[0, 1]$ the continuous functions $\sigma_2^+(x) = \bar{\sigma}_2^+(-x)$ and $\sigma_2^-(x) = \bar{\sigma}_2^-(-x)$. The interconnected system (1)-(4), (5)-(8) consequently rewrites on $[0, 1]$ as

$$\partial_t u_1(t, x) + \lambda_1 \partial_x u_1(t, x) = \sigma_1^+(x) v_1(t, x), \quad (9)$$

$$\partial_t v_1(t, x) - \mu_1 \partial_x v_1(t, x) = \sigma_1^-(x) u_1(t, x), \quad (10)$$

$$\partial_t u_2(t, x) + \lambda_2 \partial_x u_2(t, x) = \sigma_2^+(x) v_2(t, x), \quad (11)$$

$$\partial_t v_2(t, x) - \mu_2 \partial_x v_2(t, x) = \sigma_2^-(x) u_2(t, x), \quad (12)$$

along with the boundary condition

$$u_1(t, 0) = q_{11}v_1(t, 0) + q_{12}v_2(t, 0), \quad (13)$$

$$v_1(t, 1) = \rho_1u_1(t, 1) + V(t), \quad (14)$$

$$u_2(t, 0) = q_{22}v_2(t, 0) + q_{21}v_1(t, 0), \quad (15)$$

$$v_2(t, 1) = \rho_2u_1(t, 1). \quad (16)$$

The corresponding initial conditions are $L^2([0, 1], \mathbb{R})$ functions

Remark 1: The positive velocities λ_1 , λ_2 , μ_1 and μ_2 are assumed to be constant in this paper but the presented results can be extended to the case of spatially varying coefficients whose velocities are $C^1([0, 1], \mathbb{R})$ -functions.

B. Problem formulation

The goal of this paper is to design a state feedback control law $V = \mathcal{K}[(u_1, v_1, u_2, v_2)]$ where $\mathcal{K} : (L^2[0, 1])^4 \rightarrow \mathbb{R}$ is a linear operator, such that:

- the state (u_1, v_1, u_2, v_2) of the resulting feedback system (9)-(16) exponentially converges to its zero equilibrium (**stabilization problem**), i.e. there exists $\kappa_0 \geq 0$ and $\nu > 0$ such that for any initial condition $(u_1^0, v_1^0, u_2^0, v_2^0) \in (L^2[0, 1])^4$, for all $t \geq 0$

$$\|(u_1, v_1, u_2, v_2)\|_{L^2} \leq \kappa_0 e^{-\nu t} \|(u_1^0, v_1^0, u_2^0, v_2^0)\|_{L^2}. \quad (17)$$

- the resulting feedback system (9)-(16) is robustly stable with respect to small delays in the loop (**delay-robustness**), i.e. there exists $\delta^* > 0$ such that for any $\delta \in [0, \delta^*]$, the control law $V(t - \delta)$ stabilizes (9)-(16).

A control law that satisfies these two constraints is said to **delay-robustly stabilize** (in the sense of [19]) system (9)-(16). This notion is crucial in so far as it has been proven in [2] that depending on the boundary couplings, some systems of PDEs cannot be delay-robustly stabilized. The main difficulty in this paper, compared to the classical stabilization problem of linear PDEs [2], [4], [10], [25], is that the subsystem (u_2, v_2) can only be stabilized through the subsystem (u_1, v_1) . This induces the following natural assumption, without which it would not be possible to act on subsystem (u_2, v_2) using the control law V .

Assumption 1: The coefficient q_{21} is not equal to zero.

C. A necessary condition for delay-robust stabilization

It has been proved in [19] that to ensure the existence of a delay-stabilizing control law for system (9)-(16), it is necessary for its open-loop transfer functions to have at most a finite number of poles in the complex right half plane. The authors have proved in [5] that this condition is equivalent to requiring system (9)-(16) with zero in-domain couplings (i.e. $\sigma_1^+ \equiv \sigma_2^+ \equiv \sigma_1^- \equiv \sigma_2^- \equiv 0$) to be exponentially stable in open-loop. The stability properties of linear conservation laws have been broadly considered in [23], [24], [27] relying on a systematic use of direct estimates of the solutions and their derivatives along the characteristic curves. Necessary and sufficient stability conditions can be obtained by writing the corresponding complex characteristic equations [15], [16], [20]. However these conditions often require iterative optimization methods. In the case of four equations (which is the considered case here) a simpler condition can be found in [8]. More precisely, let us define, for $H \in \mathcal{M}_{p,p}(\mathbb{R})$ (where $\mathcal{M}_{r,s}(\mathbb{R})$ denotes the set of real matrices with r rows and s columns).

$$\text{Sp}_1(K) = \text{Inf}\{\|\Delta H \Delta^{-1}\|; \Delta \in \mathcal{D}_{p,+}\},$$

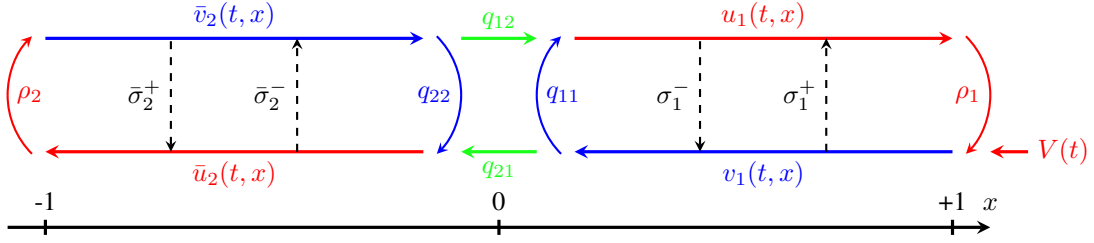


Fig. 1. Schematic representation of the interconnected system (1)-(8)

where $\mathcal{D}_{p,+}$ denotes the set of $p \times p$ real diagonal matrices with strictly positive diagonal elements. Then, for zero in-domain couplings the system (9)-(16) is exponentially stable if and only if the following assumption holds.

Assumption 2: We have $S p_1 \begin{pmatrix} 0 & 0 & q_{11} & q_{12} \\ 0 & 0 & q_{21} & q_{22} \\ \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \end{pmatrix} < 1$.

We consider that this assumption holds in all what follow. Let us denote $\tau_1 = \frac{1}{\lambda_1} + \frac{1}{\mu_1}$ and $\tau_2 = \frac{1}{\lambda_2} + \frac{1}{\mu_2}$, the total transport time of each subsystem. Using the method of characteristics it is straightforward to prove that the state $v_1(t, 1)$ and $v_2(t, 0)$ satisfy for $t \geq \tau_1 + \tau_2$

$$v_1(t, 1) = \rho_1 q_{11} v_1(t - \tau_1, 1) + \rho_1 q_{12} v_2(t - \frac{1}{\lambda_1}, 0) \quad (18)$$

$$v_2(t, 0) = \rho_2 q_{22} v_2(t - \tau_2, 0) + \rho_2 q_{21} v_1(t - \tau_2 - \frac{1}{\mu_1}, 1). \quad (19)$$

Then using the exponential stability of the considered open-loop uncoupled service stated by Assumption 2, we can conclude to the exponential stability of (18)-(19) (in the sense of [16]).

III. DESIGN OF THE CONTROL LAW

In this section we derive a control law that guarantees the stabilization of (9)-(16), using the following methodology. We start using a Volterra transformation to remove the in-domain coupling terms. Due to this transformation, new integral terms appear simultaneously in system (u_1, v_1) and in the boundary conditions (13)-(15). We then use a second integral transformation on subsystem (u_1, v_1) to remove these integral terms from the boundary conditions and from the v_1 -equation. A last Volterra transformation is introduced to remove undesired terms in the actuated PDE. Finally, using the method of characteristics, it becomes possible to rewrite the considered PDEs system as a neutral system for which the design of a stabilizing feedback law is straightforward.

A. First Volterra transformation: removing the local in-domain coupling terms

In this section, we remove the in-domain coupling terms σ_i^+ and σ_i^- from equations (9)-(12). This will transform equations (11)-(12) into a system of two transport equations. More precisely, inspired by [10], let us consider

the following Volterra transformations, defined for $i \in \{1, 2\}$ by

$$\begin{pmatrix} \alpha_i(t, x) \\ \beta_i(t, x) \end{pmatrix} = \begin{pmatrix} u_i(t, x) \\ v_i(t, x) \end{pmatrix} - \int_{a_i(x)}^{b_i(x)} \begin{pmatrix} K_i^{uu}(x, \xi) & K_i^{uv}(x, \xi) \\ K_i^{vu}(x, \xi) & K_i^{vv}(x, \xi) \end{pmatrix} \begin{pmatrix} u_i(t, \xi) \\ v_i(t, \xi) \end{pmatrix} d\xi \quad (20)$$

where $a_1(x) = 0$, $b_1(x) = a_2(x) = x$ and $b_2(x) = 1$ and where the kernels $K_i^{\cdot\cdot}$ are L^∞ -functions defined on the triangular domain $\mathcal{T}_1 = \{(x, \xi) \in [0, 1]^2, \xi \leq x\}$ while the kernels $K_2^{\cdot\cdot}$ are L^∞ -functions defined on the triangular domain $\mathcal{T}_2 = \{(x, \xi) \in [0, 1]^2, \xi \geq x\}$. They satisfy the set of PDEs on their corresponding domain of definition defined in [10] for $i \in \{1, 2\}$ by

$$\lambda_i \partial_x K_i^{uu}(x, \xi) + \lambda_i \partial_\xi K_i^{uu}(x, \xi) = -\sigma_i^-(\xi) K_i^{uv}(x, \xi), \quad (21)$$

$$\lambda_i \partial_x K_i^{uv}(x, \xi) - \mu_i \partial_\xi K_i^{uv}(x, \xi) = -\sigma_i^+(\xi) K_i^{uu}(x, \xi), \quad (22)$$

$$\mu_i \partial_x K_i^{vu}(x, \xi) - \lambda_i \partial_\xi K_i^{vu}(x, \xi) = \sigma_i^-(\xi) K_i^{vv}(x, \xi), \quad (23)$$

$$\mu_i \partial_x K_i^{vv}(x, \xi) + \mu_i \partial_\xi K_i^{vv}(x, \xi) = \sigma_i^+(\xi) K_i^{vu}(x, \xi), \quad (24)$$

along with the boundary conditions

$$K_2^{uv}(x, x) = -\frac{\sigma_2^+(x)}{\lambda_2 + \mu_2}, \quad K_2^{vu}(x, x) = -\frac{\sigma_2^-(x)}{\lambda_2 + \mu_2}, \quad (25)$$

$$K_2^{uu}(x, 1) = \frac{\mu_2}{\lambda_2} \rho_2 K_2^{uv}(x, 1), \quad (26)$$

$$K_2^{vv}(x, 1) = \frac{\lambda_2}{\mu_2 \rho_2} K_2^{vu}(x, 1), \quad (27)$$

$$K_1^{vu}(x, x) = -\frac{\sigma_1^-(x)}{\lambda_1 + \mu_1}, \quad K_1^{uv}(x, x) = \frac{\sigma_1^+(x)}{\lambda_1 + \mu_1}, \quad (28)$$

$$K_1^{vv}(x, 0) = \frac{q_{11} \lambda_1}{\mu_1} K_1^{vu}(x, 0), \quad (29)$$

$$K_1^{uu}(x, 0) = \frac{\mu_1}{q_{11} \lambda_1} K_1^{uv}(x, 0). \quad (30)$$

Note that equation (27) is only defined for $\rho_2 \neq 0$ while equation (30) is only defined for $q_{11} \neq 0$. We have the following lemma

Lemma 1: [10] Consider system (21)-(30). There exists a unique solution $K_i^{uu}, K_i^{uv}, K_i^{vu}, K_i^{vv}$ in $L^\infty(\mathcal{T}_i)$. Moreover,

there exists $L_2^{\alpha\alpha}, L_2^{\alpha\beta}, L_2^{\beta\alpha}, L_2^{\beta\beta}$ in $L^\infty(\mathcal{T}_2)$ such that

$$u_2(t, x) = \alpha_2(t, x) - \int_x^1 L_2^{\alpha\alpha}(x, \xi) \alpha_2(t, \xi) d\xi - \int_x^1 L_2^{\alpha\beta}(x, \xi) \beta_2(t, \xi) d\xi, \quad (31)$$

$$v_2(t, x) = \beta_2(t, x) - \int_x^1 L_2^{\beta\alpha}(x, \xi) \alpha_2(t, \xi) d\xi - \int_x^1 L_2^{\beta\beta}(x, \xi) \beta_2(t, \xi) d\xi. \quad (32)$$

Note that a similar result holds for the kernels K_1^{\cdot} . Applying the backstepping transformation (20) to (9)-(16) yields

$$\partial_t \alpha_1(t, x) + \lambda_1 \partial_x \alpha_1(t, x) = f_1(x) (\beta_2(t, 0) + I_1(t)) \quad (33)$$

$$\partial_t \beta_1(t, x) - \mu_1 \partial_x \beta_1(t, x) = f_2(x) (\beta_2(t, 0) + I_1(t)) \quad (34)$$

$$\partial_t \alpha_2(t, x) + \lambda_2 \partial_x \alpha_2(t, x) = 0 \quad (35)$$

$$\partial_t \beta_2(t, x) - \mu_2 \partial_x \beta_2(t, x) = 0, \quad (36)$$

with the boundary condition

$$\alpha_1(t, 0) = q_{11} \beta_1(t, 0) + q_{12} \beta_2(t, 0) + q_{12} I_1(t) \quad (37)$$

$$\beta_1(t, 1) = \rho_1 \alpha_1(t, 1) + \bar{V}(t) \quad (38)$$

$$\alpha_2(t, 0) = q_{21} \beta_1(t, 0) + q_{22} \beta_2(t, 0) + I_2(t) \quad (39)$$

$$\beta_2(t, 1) = \rho_2 \alpha_2(t, 1), \quad (40)$$

where

$$I_1(t) = - \int_0^1 L_2^{\beta\alpha}(0, \xi) \alpha_2(t, \xi) + L_2^{\beta\beta}(0, \xi) \beta_2(t, \xi) d\xi,$$

$$I_2(t) = \int_0^1 (L_2^{\alpha\alpha}(0, \xi) - q_{22} L_2^{\beta\alpha}(0, \xi)) \alpha_2(t, \xi) d\xi + \int_0^1 (L_2^{\alpha\beta}(0, \xi) - q_{22} L_2^{\beta\beta}(0, \xi)) \beta_2(t, \xi) d\xi,$$

$$f_1(x) = -\lambda_1 q_{12} K_1^{uu}(x, 0),$$

$$f_2(x) = -\lambda_1 q_{12} K_1^{vu}(x, 0),$$

$$\bar{V}(t) = V(t) + \int_0^1 (\rho_1 K_1^{uu}(1, \xi) - K_1^{vu}(1, \xi)) u_1(t, \xi) d\xi + \int_0^1 (\rho_1 K_1^{uv}(1, \xi) - K_1^{vv}(1, \xi)) v_1(t, \xi) d\xi.$$

The new control law $\bar{V}(\cdot)$ can obviously be rewritten in terms of $V(\cdot)$ and α_1 and β_1 , we choose not doing it due to space restrictions.

B. Second integral transformation: simplify the boundary conditions

In this section, we use an integral transformation on the subsystem (α_1, β_1) to remove the integral terms that appear in the boundary condition (39). More precisely, we consider the following **invertible** affine transformation

$$z_1(t, x) = \beta_1(t, x) - \int_0^1 F^\alpha(x, \xi) \alpha_2(t, \xi) d\xi - \int_0^1 F^\beta(x, \xi) \beta_2(t, \xi) d\xi, \quad (41)$$

where the kernels F^α and F^β are L^∞ -functions defined on $\mathcal{U} = \{(x, \xi) \in [0, 1]^2\}$ by the following set of PDEs

$$\mu_1 \partial_x F^\alpha(x, \xi) - \lambda_2 \partial_\xi F^\alpha(x, \xi) = f_2(x) L_2^{\beta\alpha}(0, \xi), \quad (42)$$

$$\mu_1 \partial_x F^\beta(x, \xi) + \mu_2 \partial_\xi F^\beta(x, \xi) = f_2(x) L_2^{\beta\beta}(0, \xi), \quad (43)$$

along with the boundary conditions

$$F^\alpha(0, \xi) = \frac{1}{q_{21}} (q_{22} L_2^{\beta\alpha}(0, \xi) - L_2^{\alpha\alpha}(0, \xi)), \quad (44)$$

$$F^\beta(0, \xi) = \frac{1}{q_{21}} (q_{22} L_2^{\beta\beta}(0, \xi) - L_2^{\alpha\beta}(0, \xi)), \quad (45)$$

$$F^\beta(x, 0) = \frac{\lambda_2 q_{22}}{\mu_2} F^\alpha(x, 0) - \frac{1}{\mu_2} f_2(x), \quad (46)$$

$$F^\alpha(x, 1) = \frac{\mu_2 \rho_2}{\lambda_2} F^\beta(x, 1). \quad (47)$$

The following lemma assesses the existence of a solution to equations (42)-(47).

Lemma 2: Consider system (42)-(47). There exists a unique solution F^α, F^β in $L^\infty(\mathcal{U})$.

Proof: The well-posedness of equations (42)-(47) will be proved in a recursive way. Let us consider the sequences x_n and y_n defined for all positive integer n by

$$x_0 = 0, \quad x_{n+1} = \min(1, y_n + \frac{\mu_1}{\mu_2}),$$

$$y_0 = 0, \quad y_{n+1} = \min(1, x_n + \frac{\mu_1}{\lambda_2}).$$

These sequences are strictly increasing and converge after a **finite** number of iterations to 1. We denote N the first integer for which $x_N = 1$ or $y_N = 1$. Let us consider the property $P(k)$ defined for all $0 \leq k \leq N$: “The PDE (42) along with the boundary conditions (44) and (47) admits a unique L^∞ -solution on the space $\mathcal{R}_n = \{(x, \xi) \in \mathcal{U}, \text{ s.t. } x \leq x_n - \frac{\mu_1}{\lambda_2}(\xi - 1)\}$. The PDE (43) along with the boundary conditions (45) and (46) admits a unique L^∞ -solution on the space $\mathcal{S}_n = \{(x, \xi) \in \mathcal{U}, \text{ s.t. } x \leq \frac{\mu_1}{\mu_2} \xi + y_n\}$.”

Initialization: Let us consider the domains \mathcal{R}_0 and \mathcal{S}_0 as represented in Figure 2. Let us consider the characteristic lines associated to (42)-(43) (and pictured in Figure 2). Using the method of characteristics with the boundary condition (44), we immediately have for $(x, \xi) \in \mathcal{R}_0$

$$F^\alpha(x, \xi) = \frac{1}{q_{21}} (q_{22} L_2^{\beta\alpha}(0, \xi + \frac{\lambda_2}{\mu_1} x) - L_2^{\alpha\alpha}(0, \xi + \frac{\lambda_2}{\mu_1} x)) + \int_0^{\frac{x}{\mu_1}} f_2(x - \mu_1 s) L_2^{\beta\alpha}(0, \xi + \lambda_2 s) ds,$$

Similarly, we have for $(x, \xi) \in \mathcal{S}_0$

$$F^\beta(x, \xi) = \frac{1}{q_{21}} (q_{22} L_2^{\beta\beta}(0, \xi - \frac{\mu_2}{\mu_1} x) - L_2^{\alpha\beta}(0, \xi - \frac{\mu_2}{\mu_1} x)) + \int_0^{\frac{x}{\mu_1}} f_2(x - \mu_1 s) L_2^{\beta\beta}(0, \xi - \mu_2 s) ds.$$

Since f_2 and L_2^{\cdot} are L^∞ functions, this immediately implies that $P(0)$ holds.

Induction: Let us assume that the property $P(k)$ ($0 \leq k \leq N-1$) is true. Let us consider $(x, \xi) \in \mathcal{R}_{k+1}$. If $(x, \xi) \in \mathcal{R}_0$, then we already know that F^α is well-defined. If $(x, \xi) \notin$

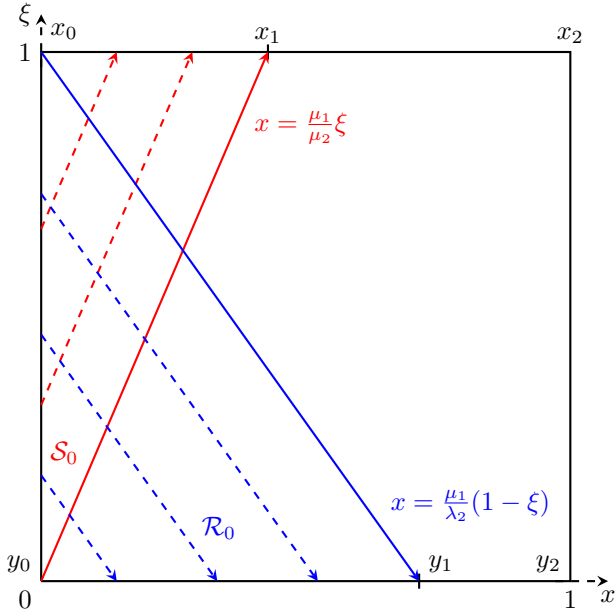


Fig. 2. Representation of the domains \mathcal{R}_0 and \mathcal{S}_0 and of the characteristic lines associated to equations (42)-(43)

\mathcal{R}_0 , then using the method of characteristics, we immediately have

$$F^\alpha(x, \xi) = \frac{\mu_2 \rho_2}{\lambda_2} F^\beta\left(x - \frac{\mu_1}{\lambda_2}(1 - \xi), 1\right) + \int_0^{\frac{1-\xi}{\lambda_2}} f_2(x - \mu_1 s) L_2^{\beta\alpha}(0, \xi + \lambda_2 s) ds.$$

Since $(x, \xi) \in \mathcal{R}_{k+1}$, we have $x - \frac{\mu_1}{\lambda_2}(1 - \xi) \leq x_{k+1}$. Thus, the point $(x - \frac{\mu_1}{\lambda_2}(1 - \xi), 1) \in \mathcal{S}_k$ and this immediately implies that F^α is a $L^\infty(\mathcal{R}_{k+1})$ function. Similar computations can be done to prove that F^β is a $L^\infty(\mathcal{S}_{k+1})$ function. This concludes the recursive proof of the property $P(k)$. The proof of the well-posedness on the rest of the unit square \mathcal{U} is now straightforward using the method of characteristics. ■

Transformation (41) maps the system (33)-(40) to the following target system

$$\partial_t \alpha_1(t, x) + \lambda_1 \partial_x \alpha_1(t, x) = f_1(x)(\beta_2(t, 0) + I_1(t)), \quad (48)$$

$$\partial_t z_1(t, x) - \mu_1 \partial_x z_1(t, x) = -\lambda_2 q_{21} F^\alpha(x, 0) z_1(t, 0), \quad (49)$$

$$\partial_t \alpha_2(t, x) + \lambda_2 \partial_x \alpha_2(t, x) = 0, \quad (50)$$

$$\partial_t \beta_2(t, x) - \mu_2 \partial_x \beta_2(t, x) = 0, \quad (51)$$

along with the boundary condition

$$\alpha_1(t, 0) = q_{11} z_1(t, 0) + q_{12} \beta_2(t, 0) + I_3(t), \quad (52)$$

$$z_1(t, 1) = \rho_1 \alpha_1(t, 1) + \tilde{V}(t), \quad (53)$$

$$\alpha_2(t, 0) = q_{21} z_1(t, 0) + q_{22} \beta_2(t, 0), \quad (54)$$

$$\beta_2(t, 1) = \rho_2 \alpha_2(t, 1), \quad (55)$$

where

$$I_3(t) = q_{12} I_2(t) + q_{11} \int_0^1 F^\alpha(0, \xi) \alpha_2(\xi) d\xi + q_{11} \int_0^1 F^\beta(0, \xi) \beta_2(\xi) d\xi, \quad (56)$$

$$\tilde{V}(t) = \bar{V}(t) - \int_0^1 F^\alpha(1, \xi) \alpha_2(t, \xi) d\xi - \int_0^1 F^\beta(1, \xi) \beta_2(t, \xi) d\xi. \quad (57)$$

Note that system (48)-(55) can be rewritten entirely in terms of $\alpha_1, \alpha_2, \beta_2, z_1$.

C. A last Volterra transformation

The last step consists in transforming equation (49) into a transport equation. Then, we consider the following (invertible) Volterra transformation

$$z_1(t, x) = \bar{\beta}_1(t, x) - \int_0^x R(x, \xi) \bar{\beta}_1(t, \xi) d\xi, \quad (58)$$

where the kernel R is a L^∞ -function defined on the triangular domain $\mathcal{T}_2 = \{(x, \xi) \in [0, 1]^2, \xi \geq x\}$ by

$$\partial_x R(x, \xi) + \partial_\xi R(x, \xi) = 0, \quad (59)$$

$$R(x, 0) = -\frac{\lambda_2 q_{21}}{\mu_1} F^\alpha(x, 0). \quad (60)$$

The system (59)-(60) obviously admits a unique solution defined for $(x, \xi) \in \mathcal{T}_2$ by $R(x, \xi) = -\frac{\lambda_2 q_{21}}{\mu_1} F^\alpha(x - \xi, 0)$. This yields the following target system

$$\partial_t \alpha_1(t, x) + \lambda_1 \partial_x \alpha_1(t, x) = f_1(x)(\beta_2(t, 0) + I_1(t)), \quad (61)$$

$$\partial_t \bar{\beta}_1(t, x) - \mu_1 \partial_x \bar{\beta}_1(t, x) = 0, \quad (62)$$

$$\partial_t \alpha_2(t, x) + \lambda_2 \partial_x \alpha_2(t, x) = 0, \quad (63)$$

$$\partial_t \beta_2(t, x) - \mu_2 \partial_x \beta_2(t, x) = 0, \quad (64)$$

along with the boundary conditions

$$\alpha_1(t, 0) = q_{11} \bar{\beta}_1(t, 0) + q_{12} \beta_2(t, 0) + I_3(t), \quad (65)$$

$$\bar{\beta}_1(t, 1) = \rho_1 \alpha_1(t, 1) + V_1(t), \quad (66)$$

$$\alpha_2(t, 0) = q_{21} \bar{\beta}_1(t, 0) + q_{22} \beta_2(t, 0), \quad (67)$$

$$\beta_2(t, 1) = \rho_2 \alpha_2(t, 1), \quad (68)$$

where

$$V_1(t) = \int_0^1 R(1, \xi) \bar{\beta}_1(t, \xi) d\xi + \tilde{V}(t). \quad (69)$$

In what follows we use the following notations: for any $t > 0$ and any $x \in [0, 1]$ we have

$$w(t, x) = \begin{pmatrix} u_1(t, x) \\ v_1(t, x) \\ u_2(t, x) \\ v_2(t, x) \end{pmatrix}, \quad \gamma(t, x) = \begin{pmatrix} \alpha_1(t, x) \\ \bar{\beta}_1(t, x) \\ \alpha_2(t, x) \\ \beta_2(t, x) \end{pmatrix}. \quad (70)$$

The corresponding initial conditions will be denoted w^0 and γ^0 . We can sum up the multiple change of coordinates we have done by the following theorem

Theorem 1: There exists an invertible bounded linear map $\mathcal{F} : (L^2([0, 1]))^4 \rightarrow (L^2([0, 1]))^4$ such that for every initial condition $w^0 \in (L^2([0, 1]))^4$, if γ denotes the solution to (61)-(68) satisfying the initial data $\gamma(0, \cdot) = \mathcal{F}^{-1}w^0$, then, $w(t, \cdot) = \mathcal{F}(\gamma(t, \cdot))$ is the solution to (9)-(16) satisfying $w(0, \cdot) = w^0$. Moreover, there exists four bounded linear operators $\mathcal{K}_i : (L^2([0, 1]))^4 \rightarrow \mathbb{R}$ ($i \in \{0, 1, 2, 3\}$) such that for any control law $V(t)$, we have for all $j \in \{1, 2, 3\}$

$$V_1(t) = \mathcal{K}_0(w(t, \cdot)) + V(t). \quad (71)$$

$$I_j(t) = \mathcal{K}_j(w(t, \cdot)) \quad (72)$$

Proof: The proof is straightforward and is not detailed here. It simply consists in rewriting the successive backstepping transformations as linear operators and to compose them. The invertibility of the operator \mathcal{F} is a consequence of the invertibility of all the transformations. Additional details can be found in [9]. ■

D. Design of the stabilizing control law

In this section we use the specific transport structure of system (61)-(68) to rewrite it as a system of difference equations. Using the method of characteristics, system (61)-(68) can easily be rewritten for $t \geq \tau_1 + \tau_2$ as

$$\beta_2(t, 0) = \rho_2 q_{22} \beta_2(t - \tau_2, 0) + \rho_2 q_{21} \bar{\beta}_1(t - \tau_2 - \frac{1}{\mu_1}, 1) \quad (73)$$

$$\begin{aligned} \bar{\beta}_1(t, 1) &= V_1(t) + \rho_1 q_{11} \bar{\beta}_1(t - \tau_1, 1) + \rho_1 q_{12} \beta_2(t - \frac{1}{\lambda_1}, 0) \\ &+ \rho_1 \int_0^{\frac{1}{\lambda_1}} (\beta_2(t - s, 0) + I_1(t - s)) f_1(1 - \lambda_1 s) ds \\ &+ \rho_1 I_3(t - \frac{1}{\lambda_1}). \end{aligned} \quad (74)$$

For almost every $t < \tau_1 + \tau_2$, $\beta_2(t, 0)$ and $\bar{\beta}_1(t, 1)$ remain bounded and can be expressed (see Theorem 1) as functions of w^0 , the initial condition of the original PDE system (9)-(16). We have the following theorem

Theorem 2: Consider the following feedback law defined for $t > \tau_1 + \tau_2$ by

$$\begin{aligned} V(t) &= -\rho_1 \mathcal{K}_3(w(t - \frac{1}{\lambda_1}, \cdot)) - \mathcal{K}_0(w(t, \cdot)) \\ &- \rho_1 \int_0^{\frac{1}{\lambda_1}} f_1(1 - \lambda_1 s) (\mathcal{F}(\begin{pmatrix} 0 & 0 & 0 & v_2(t - s, 0) \end{pmatrix}^T) \\ &+ \mathcal{K}_1(w(t - s, \cdot))) ds, \end{aligned} \quad (75)$$

where the operators \mathcal{K}_i are defined in Theorem 1. Consider the PDEs system (9)-(16) along with the feedback control law (75). Then, for any initial condition $w^0 \in (L^2([0, 1]))^4$, the state $w(t, \cdot)$ exponentially converges to zero in the sense of equation (17).

Proof: Using the definition of the operators \mathcal{K}_i and the backstepping transformations introduced in the previous

sections, the control law $V(t)$ rewrites

$$\begin{aligned} V(t) &= -\mathcal{K}_0(w(t, \cdot)) - \rho_1 I_3(t - \frac{1}{\lambda_1}) \\ &- \rho_1 \int_0^{\frac{1}{\lambda_1}} (\beta_2(t - s, 0) + I_1(t - s)) f_1(1 - \lambda_1 s) ds. \end{aligned}$$

Thus, system (73)-(74) rewrites

$$\begin{aligned} \beta_2(t, 0) &= \rho_2 q_{22} \beta_2(t - \tau_2, 0) + \rho_2 q_{21} \bar{\beta}_1(t - \tau_2 - \frac{1}{\mu_1}, 1) \\ \bar{\beta}_1(t, 1) &= \rho_1 q_{11} \bar{\beta}_1(t - \tau_1, 1) + \rho_1 q_{12} \beta_2(t - \frac{1}{\lambda_1}, 0). \end{aligned}$$

Due to Assumption 2, this system is exponentially stable and $\beta_2(t, 0)$ and $\bar{\beta}_1(t, 1)$ exponentially converge to zero. This in turn implies from (62) and (64) that $\beta_2(t, \cdot)$ and $\bar{\beta}_1(t, \cdot)$ converge L^2 -exponentially to zero. This implies, from the boundary condition (68) and from (63) that $\alpha_2(t, \cdot)$ converges also L^2 -exponentially to zero. Finally, the same holds for $\alpha_1(t, \cdot)$. Using the invertibility of the operator \mathcal{F} , it implies the convergence L^2 -exponentially to zero of $w(t, \cdot)$. This yields the existence of $\kappa_0 > 0$ and $\eta_0 > 0$ such that $\|w\|_{L^2} \leq \kappa_0 e^{-\nu t} \|w^0\|_{L^2}$. This concludes the proof. ■

IV. DELAY ROBUST STABILIZATION

In this section, we prove the delay-robustness of the control law designed in the previous section. We have the following preliminary result on the structure of the control law $V(t)$.

Lemma 3: There exists two functions g and $h \in (L^\infty([0, \tau_1 + \tau_2]))^2$ such that for all $t > \tau_1 + \tau_2$

$$V(t) = \int_0^{\tau_1 + \tau_2} g(\nu) \bar{\beta}_1(t - \nu, 1) + h(\nu) \beta_2(t - \nu, 0) d\nu.$$

Proof: The proof is straightforward and is not detailed there due to page limitation. It simply consists in using the method of characteristics on the different integral terms that appear in the expression of the control law $V(\cdot)$ (see [5] for details). ■

Let us consider a small positive delay $\delta > 0$ on the actuation input $V(\cdot)$. Due to Lemma 3, system (73)-(74) rewrites

$$\begin{aligned} \beta_2(t, 0) &= \rho_2 q_{22} \beta_2(t - \tau_2, 0) + \rho_2 q_{21} \bar{\beta}_1(t - \tau_2 - \frac{1}{\mu_1}, 1) \\ \bar{\beta}_1(t, 1) &= \rho_1 q_{11} \bar{\beta}_1(t - \tau_1, 1) + \rho_1 q_{12} \beta_2(t - \frac{1}{\lambda_1}, 0) \\ &+ \int_0^{\tau_1 + \tau_2} g(\nu) \bar{\beta}_1(t - \nu - \delta, 1) + h(\nu) \beta_2(t - \nu - \delta, 0) d\nu \\ &- \int_0^{\tau_1 + \tau_2} g(\nu) \bar{\beta}_1(t - \nu, 1) + h(\nu) \beta_2(t - \nu, 0) d\nu. \end{aligned}$$

Taking the Laplace transform of this system, we obtain

$$\beta_2(s, 0) = \rho_2 q_{22} e^{-\tau_2 s} \beta_2(s, 0) + \rho_2 q_{21} e^{-(\tau_2 + \frac{1}{\mu_1})s} \bar{\beta}_1(s, 1) \quad (76)$$

$$\begin{aligned} \bar{\beta}_1(t, s) &= \rho_1 q_{11} e^{-\tau_1 s} \bar{\beta}_1(s, 1) + \rho_1 q_{12} e^{-\frac{1}{\lambda_1} s} \beta_2(s, 0) \\ &- (1 - e^{-\delta s}) \int_0^{\tau_1 + \tau_2} (g(\nu) \bar{\beta}_1(s, 1) + h(\nu) \beta_2(s, 0)) e^{-\nu s} d\nu. \end{aligned} \quad (77)$$

We consequently get the following characteristic quasipolynomial $P(s)$

$$P(s) = \det(F(s) - (1 - e^{-\delta s})H(s)),$$

where

$$F(s) = \begin{pmatrix} 1 - \rho_2 q_{22} e^{-\tau_2 s} & -\rho_2 q_{21} e^{-(\tau_1 + \frac{1}{\mu_1})s} \\ -\rho_1 q_{12} e^{-\frac{1}{\lambda_1} s} & 1 - \rho_1 q_{11} e^{-\tau_1 s} \end{pmatrix}$$

$$H(s) = \int_0^{\tau_1 + \tau_2} (g(\nu)\bar{\beta}_1(s, 1) + h(\nu)\beta_2(s, 0))e^{-\nu s} d\nu \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Consequently, the characteristic equation associated to (76)-(77) can be rewritten $P(s) = 0$. It is straightforward to prove that $H(s)$ is bounded in the complex right-half plane. We are now finally able to prove that the control law $V(t)$ as defined in (75) delay-robustly stabilizes the system (9)-(16).

Theorem 3: The control law $V(t)$ as defined in (75) delay-robustly stabilizes the system (9)-(16). That is, there exists $\delta^* > 0$ such that, for all $\delta \in [0, \delta^*]$, $V(t - \delta)$ exponentially stabilizes the system (9)-(16).

Proof: As the proof is equivalent to the one of [3, Theorem 2], we just recall the main steps. Due to Assumption 2, the function $F(s)$ is positively bounded on the right-half plane. The function $P(s)$ has only a finite number of zeros in the right half plane, where the zeros have finite module [15]. For each of this point, we can prove that there exists $\delta_m > 0$ such that for $\delta < \delta_m$, $P(s)$ does not cancel in the right half plane. It is then sufficient to choose $\delta^* = \min_s \delta_m(s)$. Furthermore, since the principal term of $P(s)$ is precisely the function $F(s)$ which is stable by assumption, the asymptotic vertical chain of zeros of $P(s)$ cannot be the imaginary axis. This implies delay robust stability since all zeros of $P(s)$ are in the open left-half complex plane. ■

V. CONCLUDING REMARKS

In this paper, a delay-robust stabilizing feedback control law has been developed for a network of two systems of two coupled hyperbolic PDEs actuated at only one boundary of the network. The proposed method combines successive backstepping transformations that reduce the PDEs network to a neutral system with distributed delay. The control design becomes then straightforward. The robustness to small delays (in the actuation) of the proposed feedback law is ensured due to the integral structure of the control law. The delay-robustness properties of the output-feedback controller (crucial for application on an industrial problem) remains to be considered. The extension to a network with a larger number of interconnected systems should also be considered. Finally, this raises the general question of the stabilization of underactuated systems. A possible application of these underactuated system is the stabilization of a Rijke tube, as this problem has been formulated in [12] as a network of two PDEs with an ODE sandwiched between them.

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