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Partitioning of the Free Space-Time for On-Road Navigation of Autonomous Ground Vehicles

Florent Altché[2,1] and Arnaud de La Fortelle[1]

***Abstract*— In this article, we consider the problem of trajectory planning and control for on-road driving of an autonomous ground vehicle (AGV) in presence of static or moving obstacles. We propose a systematic approach to partition the collision-free portion of the space-time into convex sub-regions that can be interpreted in terms of relative positions with respect to a set of fixed or mobile obstacles. We show that this partitioning allows decomposing the NP-hard problem of computing an optimal collision-free trajectory, as a path-finding problem in a well-designed graph followed by a simple (polynomial time) optimization phase for any quadratic convex cost function. Moreover, robustness criteria such as margin of error while executing the trajectory can easily be taken into account at the graph-exploration phase, thus reducing the number of paths to explore.**

I. Introduction

In order to drive on public roads, autonomous ground vehicles (AGVs) will be required to navigate efficiently inside a potentially dense flow of other vehicles with uncertain behaviors. For this reason, planning safe, efficient and dynamically feasible trajectories that can be safely followed by a low-level controller is a particularly important problem.

One of the difficulties of "optimal" trajectory planning for AGVs is that the presence of obstacles renders the search space non-convex, and multiple possible maneuver variants (for which there is at least one locally optimum trajectory [1]) exist, such as illustrated in Figure 1. At control level, tracking the computed trajectory may involve highly nonlinear vehicle dynamics when nearing the handling limits (see, *e.g.*, [2] for a review); in less demanding (low-slip) scenarios, simpler dynamic models (some of which exhibit the interesting flatness property [3], [4]) can be used.

Because they allow simultaneous trajectory generation with obstacle avoidance and control computation, model predictive control (MPC) approaches have been very popular for AGVs (see, *e.g.*, [5], [6]). However, real-time constraints usually force authors considering very precise dynamic models to choose a short (sub-second) prediction horizon, which may in turn cause the MPC problem to become infeasible, for instance when a new obstacle is detected with not enough time to stop. Even with simpler dynamic models, the non-convexity of the state-space renders continuous optimization techniques inefficient. For this reason, hierarchical frameworks [7], [8] have been proposed, in which a medium-term (up to a dozen seconds) planner generates a rough trajectory

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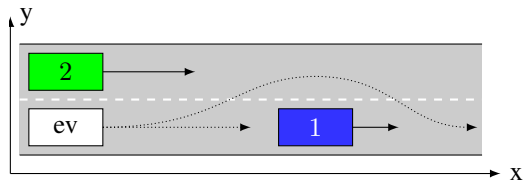


Fig. 1. Example driving situation involving multiple maneuver choices for an AGV (denoted ev): overtake the slower (blue, denoted 1) vehicle before the green vehicle (2) passes or wait behind the blue vehicle, possibly overtaking after the green vehicle has passed. Solid arrows represent the velocity of each vehicle, dotted arrows represent possible AGV trajectories.

which is then refined by a short-term (sub-second to a few seconds) controller. Mixed-integer programming (MIP) methods are often used in medium-term trajectory planning to encode the discrete decisions arising from multiple maneuver choices [9], [10], generalized as *logical constraints* in [11]. However, MIP problems are known to be NP-hard [12] and are therefore difficult to solve in real-time.

In this article, we propose a different approach for maneuver selection, inspired by the use of graph-based coordination of robots [13] and the decomposition of the collision-free space presented in [14] for 2D path-planning. First, we introduce a systematic algorithm to partition the collision-free space-time into 3D regions with geometrical adjacency relations; the structure of on-road driving allows to assign a semantic interpretation to each partition subset. Using time discretization, we further divide these regions into convex polyhedrons, and design a *transition graph* in which any path corresponds to a collision-free trajectory (that may however be dynamically infeasible). The main advantage of our approach is to reduce the entire combinatorial decision-making process (choosing from which side to avoid each obstacle) to the selection of a path in a graph. Once a path in the graph has been selected, we show that computing a corresponding optimal trajectory (for a quadratic convex cost function) in an MPC fashion is widely simplified and can be performed in polynomial time. Note, however, that the number of edges in the transition graph still does grow exponentially with the number of obstacles; future work will focus on exploration heuristics that can be used to avoid exploring inefficient paths.

A second advantageous property of our decomposition approach is to simplify the use of risk metrics, which can be directly taken into account at the graph exploration phase; in [15], the authors used a similar partitioning technique to design a "space margin" metric for 2D path planning.

In this article, we introduce a complementary *time margin* metric, corresponding to a temporal tolerance to execute a particular maneuver, that can be easily computed from our graph representation. This measure is related to the notion of "gap acceptance", commonly used in stochastic decision-making (see, *e.g.*, [16]). We believe that combining a temporal margin (notably accounting for uncertainty in predicting the future trajectory of moving obstacles) as well as a spatial margin (accounting for perception and control errors) is key for trajectory planning and tracking in real-world situations, for instance coupled with MPC or Linear Quadratic Gaussian motion planning and control [17].

Our transition graph approach generalizes state-machine-based techniques [18], [19] which rely on a predefined set of maneuvers (such as *track lane* or *change lane*) that needs to be manually adapted to the driving situation. By contrast, our method can be applied in many scenarios (including highway and urban driving, for instance crossing an intersection) with the same formalism. Although spatio-temporal graphs have already been used for the control of AGVs [20], [21], no existing approach provides the same desirable properties, and notably to easily account for margins in planning.

The rest of this article is structured as follows: in Section II, we present intuitions of our main ideas using the example scenario of Figure 1. In Section III, we formalize these intuitions mathematically, and we present applications of our results to planning and control for autonomous ground vehicles in Section IV. Finally, Section V concludes the study.

II. A GUIDING EXAMPLE

The goal of this section is to give an intuition of our main mathematical results using the example scenario shown in Figure 1; the formal mathematical theory is developed in the next section. In our example, we consider an autonomous ground vehicle (called ego-vehicle in the remainder of this article) navigating on a road with two other vehicles (obstacles). Intuitively, the ego-vehicle has three classes of maneuvers to choose from: either it can remain behind vehicle 1, overtake it before vehicle 2 passes, or overtake it after vehicle 2 has passed; in [1], these maneuver choices are linked to the notion of homotopy classes of trajectories. Assuming that the future trajectory of the obstacles is known in advance, it is possible to compute the *obstacle set* χ_o of (x, y, t) positions of the ego-vehicle for which a collision exists at time t; the complement of this set is the *collision-free* region of the space-time (or free space-time), denoted by χ_f. Any collision-free trajectory for the ego-vehicle corresponds to a path in χ_f; Figure 2 provides an illustration of the free space-time in our example.

Due to the complex structure of the free space-time, notably its non-convexity, this abstraction is difficult to use directly to compute optimal collision-free trajectories. Inspired by the work in [13] and [14], we propose a decomposition of χ_f in convex subregions with adjacency relations. First, we partition horizontal planes (corresponding to fixed time instants) using relative positions with respect to

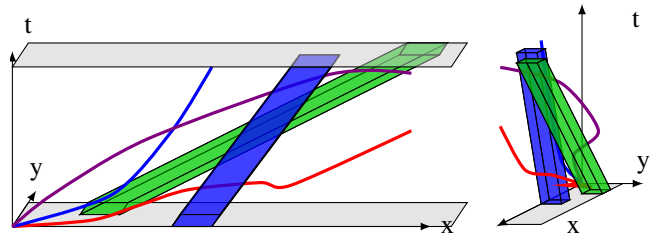
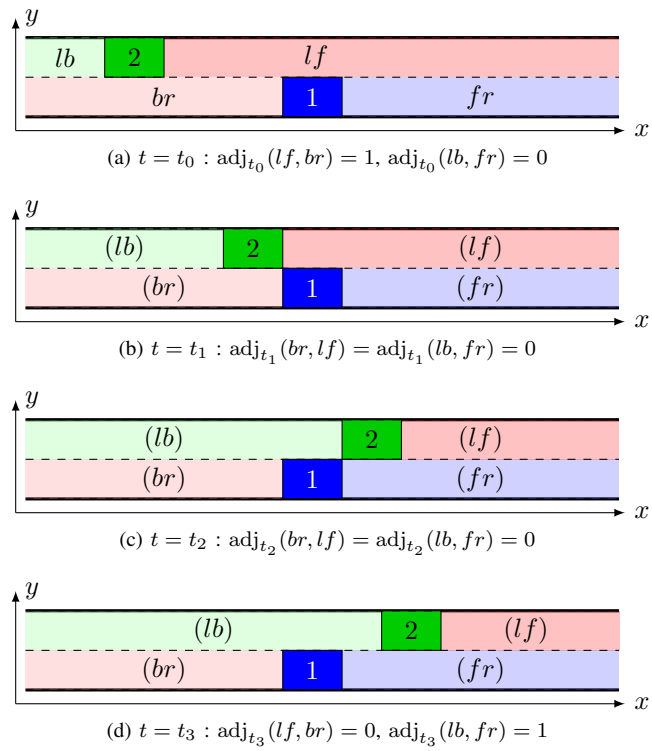


Fig. 2. Free space-time χ_f (in white) corresponding to the situation of Figure 1. Obstacles are pictured in the color of the corresponding vehicle. Light-gray planes represent the road extent in the y direction. The thick curves represent possible collision-free trajectories for the ego-vehicle.



(a) $t = t_0$: $\mathrm{adj}_{t_0}(lf, br) = 1$, $\mathrm{adj}_{t_0}(lb, fr) = 0$

(b) $t = t_1$: $\mathrm{adj}_{t_1}(br, lf) = \mathrm{adj}_{t_1}(lb, fr) = 0$

(c) $t = t_2$: $\mathrm{adj}_{t_2}(br, lf) = \mathrm{adj}_{t_2}(lb, fr) = 0$

(d) $t = t_3$: $\mathrm{adj}_{t_3}(lf, br) = 0$, $\mathrm{adj}_{t_3}(lb, fr) = 1$

Fig. 3. Partitioning of the 2D space at different times in our example scenario, and adjacency relations adj. In this example, $\mathrm{adj}_t(lb, br) = \mathrm{adj}_t(lf, fr) = 1$ and adj_t is symmetrical at all times.

each obstacle as illustrated in Figure 3. Each subset of the partition corresponds to positions where the ego-vehicle is either located in front (f), to the left (l), behind (b) or to the right (r) of each obstacle. Using the additional information given by road boundaries, this partitioning technique yields four subsets denoted by (lb), (lf), (br) and (fr), indicating the relative position of the ego-vehicle from obstacle 1 and 2 in this order. We call these labels *signature* of each subset. Additionally, for two such subsets A and B at a given time t, we can define an adjacency relation adj_t (related to that of [14]), such that $\mathrm{adj}_t(A, B) = 1$ if the intersection of their closures is not empty, *i.e.* $\bar{A} \cap \bar{B} \neq \emptyset$.

This partitioning method can be generalized to the three-dimensional space-time by using unions of regions sharing the same signature, as shown in Figure 4. The notion of adjacency described above can be extended, and we let

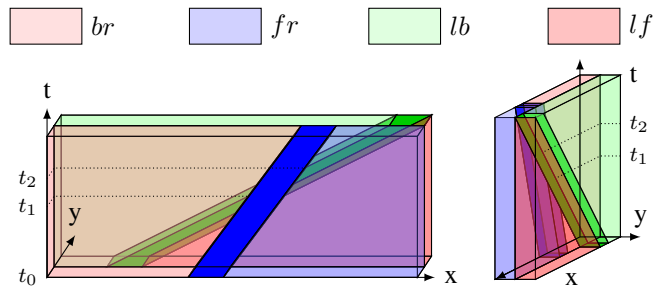


Fig. 4. Partitioning of the free space-time of Figure 2 into four cells. The legend gives the signature of each cell, with blue obstacle first.

TABLE I

VALIDITY SETS $\mathrm{Adj}(A, B)$

	br	fr	lb	lf
br	$[t_0, +\infty)$	$\emptyset$	$[t_0, +\infty)$	$[t_0, t_1)$
fr	$\emptyset$	$[t_0, +\infty)$	$(t_2, +\infty)$	$[t_0, +\infty)$
lb	$[t_0, +\infty)$	$(t_2, +\infty)$	$[t_0, +\infty)$	$\emptyset$
lf	$[t_0, t_1)$	$[t_0, +\infty)$	$\emptyset$	$[t_0, +\infty)$

$\mathrm{Adj}(A, B)$ be the set of times t such that $\mathrm{adj}_t(A, B) = 1$. We call the set $\mathrm{Adj}(A, B)$ the *validity set* of the transition from A to B, corresponding to time periods for which a collision-free trajectory from A to B exists. The validity sets in this example are given in Table I, with initial time t_0.

Using Table I, we can build a directed graph (that we call *transition graph*) representing all the possible transitions between cells of the partition as shown in Figure 5: each vertex of this graph corresponds to a partition cell, and we add the edge $A \to B$ if $\mathrm{Adj}(A, B) \neq \emptyset$. Additionally, we associate to each edge of the graph the corresponding validity set. A path in this graph is given as a succession of edges and associated transition times within the validity set of each edge, for instance $((br \to lb, t_1), (lb \to fr, t_3))$ corresponding to the maneuver of waiting for the green vehicle (2) to pass before overtaking the blue one (1). Between these explicit transition times, the ego-vehicle is supposed to remain inside the last reached cell.

TABLE II

TIME MARGINS OF EXAMPLE PATHS

Path	Most constr. trans.	Margin
$((br \to br, t_0))$	$br \to br$	$+\infty$
$((br \to lb, t_1), (lb \to fr, t_3))$	$lb \to fr$	$+\infty$
$((br \to lf, t_0), (lf \to fr, t_1))$	$br \to lf$	$t_1 - t_0$

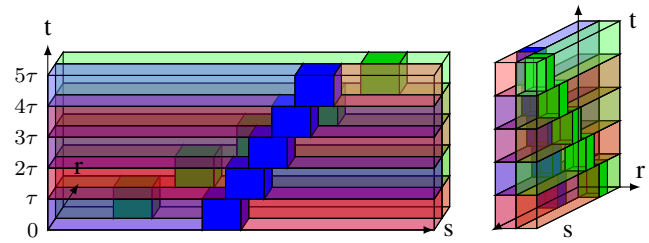


Fig. 6. Discrete partitioning of the free space-time of Figure 2, with $t_0 = 0$.

Using this graph-based representation also allows to compute a risk metric associated to a maneuver, called *time margin*. This measure is defined as the time which remains to the ego-vehicle to perform a particular maneuver, before the most constrained transition becomes impossible. To illustrate this notion (which is formally defined in Section III), we present example time margins for a selection of paths in Table II.

Although this continuous approach is interesting mathematically, it is not necessarily suited for practical computer implementation, which is generally based on time sampling. For this reason, we also propose a discrete partitioning as shown in Figure 6: for a discretization time step $\tau > 0$, we approximate the free space as a union of disjointed cylinders of the form $A \times [t_0 + k\tau, t_0 + (k+1)\tau)$ where A is a subset in the partition at time $t_0 + k\tau$. Using this time-discretized partition, we can adapt the notion of adjacency to design a time-discretized transition graph, as shown in Figure 7. In this graph, a path can be simply given as a list of successive

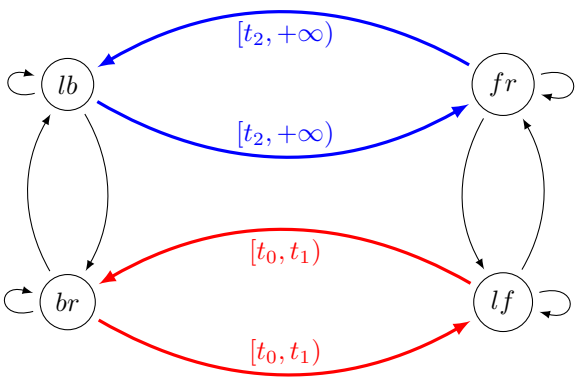


Fig. 5. Transition graph corresponding to Figure 4, with validity set of each edge. Thinner edges shown in black have a validity set $[t_0, +\infty)$ (omitted for readability).

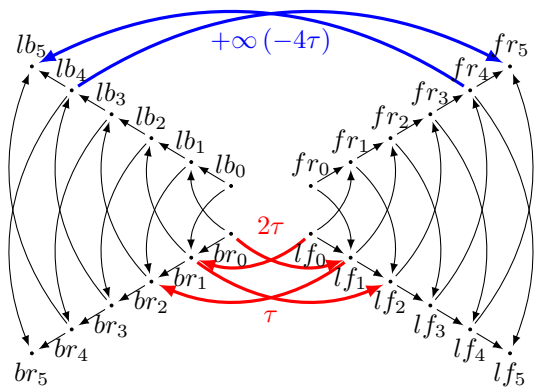


Fig. 7. Discrete-time transition graph and time margins corresponding to the partition of Figure 6. Vertex A_k corresponds to the ego-vehicle being in set A at time $t_0 + k\tau$.

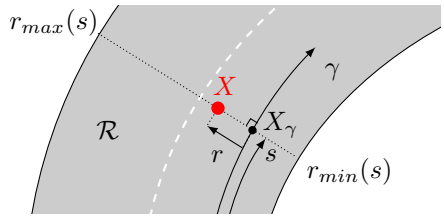


Fig. 8. Frenet coordinates of a point on the road.

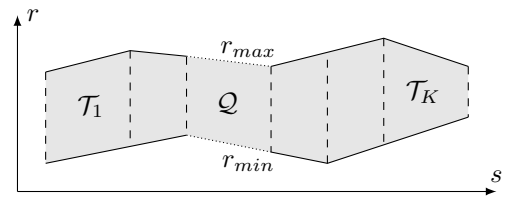


Fig. 9. Decomposition of the road (in grey) in trapezes.

vertices, thus allowing to use classic exploration algorithms. The time margin of any edge in the graph can also be easily computed (as shown in Section III).

We believe that this graph-based representation has two main advantages. First, the combinatorial part of the trajectory planning problem, consisting in choosing a feasible maneuver around the obstacles, is reduced to selecting a path in a transition graph. We will show in Section IV that, once such a path is given, computing a corresponding optimal trajectory becomes extremely simple for a large class of cost functions. Second, the graph approach makes it easy to take into account safety margins by avoiding exploration of time-constrained edges, which can be useful to handle uncertainty in trajectory estimation. Additional metrics can also be computed (see, *e.g.*, [15]) for spatial constraints, in order to account for control or positioning error.

III. Mathematical results

A. Modeling

We now proceed to theorize and generalize the intuitions exposed in the previous section. We consider an autonomous ego ground vehicle, driving on a road in presence of obstacles, which can either be fixed or mobile. We assume that the ego-vehicle remains parallel to the local direction of the road, as it usually is the case in normal (non-crash) situations, so that the configuration of the ego-vehicle is fully given by the position of its center of mass, denoted by (x, y) in ground coordinates.

We assume that the ego-vehicle has knowledge of the road geometry, for instance through cartography, as a $\mathcal{C}^2$ reference path γ and bounds on the lateral deviation from γ, as shown in Figure 8; we let $\mathcal{R} \subset \mathbb{R}^2$ be such that the ego-vehicle is on the road if, and only if, $(x, y) \in \mathcal{R}$. Finally, we assume that the road curvature and width are such that, for all $X = (x, y) \in \mathcal{R}$, there exists a unique point $X_\gamma \in \gamma$ which is closest to X.

According to Figure 8, we define the Frenet coordinates of X as $(s(X), r(X))$, where $s(X)$ is the curvilinear position of the corresponding point X_γ along γ, and $r(X) = (X - X_\gamma) \cdot \mathbf{N}$ with $(\mathbf{T}, \mathbf{N})$ the Frenet frame of γ at point X_γ. With these notations, we let r_{min} and r_{max} be such that $X \in \mathcal{R}$ if, and only if, $r_{min}(s(X)) \leq s(X) \leq r_{max}(s(X))$, and we let $\mathcal{Q} = \left\{(s, r) \in \mathbb{R}^2 \ : r_{min}(s) \leq r \leq r_{max}(s)\right\}$ denote the extent of the road in Frenet coordinates. In what follows, we only consider the Frenet coordinates of the ego-vehicle, and we drop the dependence of s and r in X. We assume that the ego-vehicle only moves forward along the road, in the direction of increasing s.

We denote by $\mathcal{O}$ the set of obstacles existing on the road around the ego-vehicle, and by $N = |\mathcal{O}|$ the number of obstacles. At a given time t_0, we consider a time horizon T and we assume that an estimation of the trajectory of each obstacle $o \in \mathcal{O}$ is available over $[t_0, t_0 + T]$. We let $\chi = \mathcal{Q} \times [t_0, t_0 + T]$ the set of space-time points for which the vehicle is on the road. With the previous assumptions and knowing the shapes of all vehicles, it is possible to compute the (collision-)free space-time $\chi_f \subset \chi$ such that $(s, r, t) \in \chi_f$ if, and only if, the vehicle is on the road and does not collide with any obstacle at time t; we call obstacle space-time χ_o the complement of χ_f in χ. Note that the free space-time is similar to the notion of configuration space(-time), which is widely used in robotics [22], and can be computed efficiently [23]. In this article, we also suppose perfect knowledge of these future trajectories over $[t_0, t_0 + T]$; however, probabilistic trajectory estimates can also be taken into account, for instance by defining χ_f^p as the set of points of χ which are free with probability p.

B. Semantic partitioning

We consider that for all $t_1 \in [t_0, t_0+T]$, the intersection of the obstacle space-time χ_o is a union of (potentially rotated) rectangles; due to the roughly rectangular shape of classical vehicles, this assumption does not excessively sacrifice precision. Moreover, we assume that the road boundary functions r_{min} and r_{max} are piecewise-linear and continuous.

Many algorithms can be used to partition the free space-time. However, we believe that partitioning χ_f in a semantically meaningful way, *i.e.* that can be easily understood by humans, is preferable to purely arbitrary partitions that could be achieved, for instance, by tetrahedral meshing. Moreover, it is demonstrated in [14] that partitioning the free (2D) space with a minimal set of edges provides interesting topological properties. In what follows, we systematize the approach presented in Section II in a two-step partitioning algorithm: first, we partition planes corresponding to a fixed time $t_1 \in [t_0, t_0 + T]$; second, we deduce a partition of χ_f. We let $\mathcal{Q}_f^{t_1} = \Pi_{\mathcal{Q}} \left(\chi_f \cap \{t = t_1\}\right)$ be the free (2D) space at time t_1, with $\Pi_{\mathcal{Q}}$ the projection operator on $\mathcal{Q}$.

1) Partitioning at fixed t_1: First, note that we can use trapeze decomposition to partition the road in convex regions using r_{min} and r_{max} as shown in Figure 9; since the road profile does not depend on time, this decomposition allows to fully partition χ by using cylinders with trapezoidal base. Each trapeze $\mathcal{T}_k$ (with $k \in \{1 \,..\, K\}$) can be defined by a

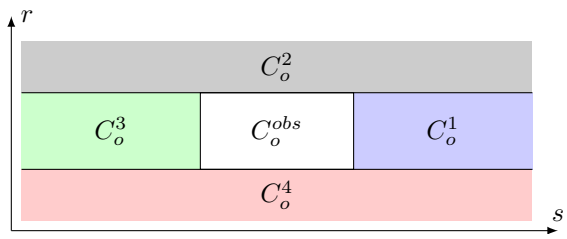


Fig. 10. Partitioning of the 2D space around a single obstacle (C_o^{obs}) into four collision-free regions C_o^i.

set of linear constraints[1], in the form $A_k X \leq b_k$ with A_k a 4-by-2 matrix, $X = [s, r]^T$ and b_k a vector of $\mathbb{R}^4$.

For a single rectangular obstacle o at time t_1, we define four regions $C_o^i \subset \mathbb{R}^2$ ($i \in \{1..4\}$) as illustrated in Figure 10; as in Section II, these regions can be identified as positions where the ego-vehicle is located in front, to the left, behind or to the right of the obstacle. Similarly, we let C_o^{obs} be the obstacle region corresponding to o. Since all obstacles are assumed rectangular, each of the C_o^i regions is defined by a set of linear constraints[1] in the form $A_o^i X \leq b_o^i$, with A_o^i a two-column matrix, $X = [s, r]^T$ and b_o^i a vector having the same number of lines as A_o^i. The partition of $\mathcal{Q}_f^{t_1}$ is built recursively according to Algorithm 1.

Algorithm 1 Partitioning of $\mathcal{Q}_f^{t_1}$

```
P_0 ← {T_k}_{k=1..K}                ▷ Initialize P_0 as a partition of Q
N ← |O|
for n = 1..N do                     ▷ Loop over all obstacles o_n
  P_n ← {}
  for all C ∈ P_{n-1} do            ▷ Loop over cells C in P_{n-1}
    for j = 1..4 do
      if C^j_{o_n} ∩ C ≠ ∅ then     ▷ Partition C \ C^{obs}_{o_n}
        P_n ← P_n ∪ {C^j_{o_n} ∩ C}
      end if
    end for
  end for
end for
P^{t_1} ← P_N
```

Theorem 1: $\mathcal{P}^{t_1}$ is a partition of $\mathcal{Q}_f^{t_1}$.

Proof: We will prove that, for all $0 \leq n \leq N$, $\mathcal{P}_n$ is a partition of $\mathcal{Q}_n = \mathcal{Q} \setminus \bigcup_{i=1}^n C_{o_i}^{obs}$. First, this property is verified for $\mathcal{P}_0$ which is a partition of $\mathcal{Q}$. Second, the loop preserves the following invariants for all $n \geq 1$ and any element $e \in \mathcal{P}_n$:

- $e \neq \emptyset$ and $\exists e' \in \mathcal{P}_{n-1}$ such that $e \subset e'$;
- for all $1 \leq i \leq n$, $\exists j \in \{1..4\}$ such that $e \subset C_{o_i}^j$.

Thus, $\mathcal{P}_n = \left\{ e \cap C_{o_n}^j \middle| e \in \mathcal{P}_{n-1}, j \in \{1..4\}, e \cap C_{o_n}^j \neq \emptyset \right\}$. Since the sets $\left(\mathcal{C}_{o_n}^j\right)_{j=1..4}$ define a partition of $\mathbb{R}^2 \setminus C_{o_n}^{obs}$ and since all $e \in \mathcal{P}_0$ is a subset of $\mathcal{Q}$, we deduce by induction that all elements of $\mathcal{P}_n$ are nonempty subsets of $\mathcal{Q}_n$.

[1]To ensure the sets are disjoint, some of the inequalities should be strict. In practice, we use non-strict inequalities with a small tolerance ε.

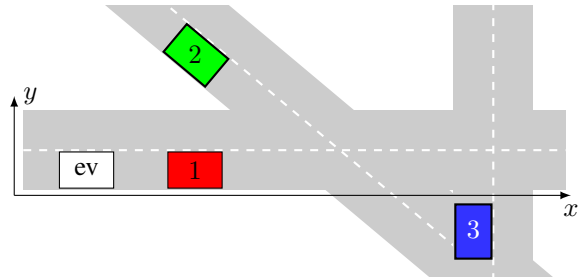


Fig. 11. A more complicated example with 3 obstacles.

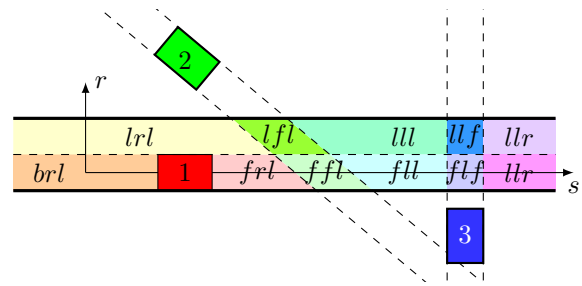


Fig. 12. Partitioning in the example of Figure 11, with subsets signature; for instance, *brl* means that the ego-vehicle is behind the first vehicle, to the right of the second and left of the third. The thick black lines correspond to the unique trapeze encoding road boundaries; its index is omitted for clarity.

Reciprocally, for all any $q \in \mathcal{Q}_n = \mathcal{Q}_{n-1} \setminus C_{o_n}^{obs}$ there exists $j \in \{1..4\}$ such that $q \in \mathcal{Q}_{n-1} \cap C_{o_n}^j$. Since $\mathcal{P}_0$ is a partition of $\mathcal{Q}$, inductive reasoning yields $\mathcal{Q}_n \subset \bigcup_{e \in \mathcal{P}_n} e$. ■

From the previous proof, we deduce the corollary:

Corollary 1 (Semantic partitioning): For all $e \in \mathcal{P}^{t_1}$, there exists a unique tuple $\sigma_{t_1}(e) = (k, j_1, \ldots, j_N) \in \{1..K\} \times \{1..4\}^N$ such that $e = \mathcal{T}_k \cap \bigcap_{n=1..N} C_{o_n}^{j_n}$. We call $\sigma_{t_1}(e)$ the *signature* of subset e, and we let $\Sigma = \{1..K\} \times \{1..4\}^N$; σ_{t_1} is a bijection from Σ to $\mathcal{P}^{t_1} \cup \emptyset$.

Therefore, our partitioning of $\mathcal{Q}_f^{t_1}$ bijectively corresponds to relative positions from all N obstacles in the free space at time t_1, and there is a finite number of elements in the partition which is bounded by $K4^N$. Thus, each element in the partition can be uniquely defined by its signature. Moreover, all elements $e \in \mathcal{P}^{t_1}$ also are convex polygons, which can be fully described using a single (matrix, vector) pair that can easily be stored in computer memory. Figures 11 and 12 illustrate our partitioning in a more complex scenario[2] with 3 vehicles; note that, for clarity purposes, we respectively used f, l, b, r instead of $1, 2, 3, 4$ as defined in Corollary 1. Also remark that, although Figure 11 is shown in world coordinates (x, y), Figure 12 uses Frenet coordinates (s, r).

In order to encode the relation between elements of the partition, we introduce the notion of adjacency as follows:

Definition 1 (Adjacency): For $e_1, e_2 \in \mathcal{P}^{t_1}$, we say that e_1 and e_2 are adjacent if, and only if the intersection of their closures is not empty, *i.e.* $\overline{e_1} \cap \overline{e_2} \neq \emptyset$.

[2]Obstacle regions 1, 2, 3 in Figure 12 are computed for a point-mass ego-vehicle, in order to match the vehicle shapes shown in Figure 11.

Note that this property can be verified in polynomial time using the matrix inequality representation and linear programming; for $\sigma, \sigma' \in \Sigma$, we let $\mathrm{adj}_{t_1}(\sigma, \sigma') = 1$ if $\sigma_{t_1}^{-1}(\sigma)$ and $\sigma_{t_1}^{-1}(\sigma')$ are adjacent, and 0 otherwise.

2) Continuous-time partitioning: We now define a partition of the free space-time χ_f as follows: for $\sigma \in \Sigma$, we let $E_\sigma = \bigcup_{t_1 \in [t_0, t_0+T]} \sigma_{t_1}^{-1}(\sigma) \times \{t_1\}$ be the set of all points in the free space-time sharing the same signature σ. The set $\mathcal{P} = \{E_\sigma \,|\, \sigma \in \Sigma, E_\sigma \neq \emptyset\}$ then defines a partition of χ_f (see Figure 4). For σ and $\sigma' \in \Sigma$, we define the validity set $\mathrm{Adj}(\sigma, \sigma') = \{t \in [t_0, t_0 + T] \,|\, \mathrm{adj}_t(\sigma, \sigma') = 1\}$.

We can now define the transition graph (see Figure 5):

Definition 2: The continuous transition graph is the directed graph $\mathcal{G}^c = (\mathcal{V}^c, \mathcal{E}^c, \mathrm{Adj})$ with vertex set $\mathcal{V}^c = \{\sigma \in \Sigma \,|\, E_\sigma \neq \emptyset\}$, edges set $\mathcal{E}^c = \big\{(\sigma_1, \sigma_2) \in \Sigma^2 \,\big|\, \mathrm{Adj}(E_{\sigma_1}, E_{\sigma_2}) \neq \emptyset\big\}$ and associated validity set Adj.

Definition 3: A path in $\mathcal{G}^c$ is given by a list of vertices $(\sigma_1, \ldots, \sigma_{m+1}) \in \mathcal{V}^c$ so that for all $i \leq m$, $(\sigma_i, \sigma_{i+1}) \in \mathcal{E}^c$, and a list of strictly increasing transition times $(t_1, \ldots, t_m)$ such that for all $i \in \{1 \,..\, m\}$, $t_i \in \mathrm{Adj}(\sigma_i, \sigma_{i+1})$ and $[t_i, t_{i+1}) \subset \mathrm{Adj}(\sigma_i, \sigma_i)$.

Therefore, a path in $\mathcal{G}^c$ corresponds to a sequence of cells that are adjacent at each transition time, and for which the ego-vehicle can remain in a given cell between two consecutive transitions. We can now define:

Definition 4: Let $\pi^c = \big((\sigma_1, \ldots, \sigma_{m+1}), (t_1, \ldots, t_m)\big)$ be a path in $\mathcal{G}^c$. The time margin of π^c is $V(\pi^c) = \min_{i=1\ldots m} \Big(\sup \Big\{t - t_i \;\Big|\; [t_i, t) \subset \mathrm{Adj}(\sigma_i, \sigma_{i+1})\Big\}\Big)$.

3) Discrete time: Due to the potentially complex trajectories followed by the obstacles, there is no guarantee regarding the topology of the subsets E_σ, which can for instance have multiple connected components; similarly, $\mathrm{Adj}(E_{\sigma_1}, E_{\sigma_2})$ is in general a union of disjointed intervals. To make practical applications easier, we also propose a temporal discretization of the free space-time with a time step duration τ (with $T = P\tau$), and we let $\theta_p = t_0 + p\tau$. Instead of using the exact shape of E_σ, we define $E_\sigma^p = \sigma_{\theta_p}^{-1}(\sigma) \times [\theta_p, \theta_{p+1})$ and we assume that $E_\sigma^p \subset \chi_f$ for all $\sigma \in \Sigma$ and $p \in \{0 \,..\, P\}$. Since $\sigma_{\theta_p}^{-1}(\sigma)$ is a convex (or empty) set, E_σ^p is either empty or convex, and fully defined by a set of linear inequalities in the form $A[X, t]^T \leq b$ (the comments of footnote 1 also apply here). Finally, we define a partition of the free space-time χ_f in convex box-shaped cells as $\mathcal{P}^\tau = \{E_\sigma^p | p \in \{0 \,..\, P\}, \sigma \in \Sigma, E_\sigma^p \neq \emptyset\}$ (see Figure 6).

Definition 5: The discrete transition graph is the directed graph $\mathcal{G}^d = (\mathcal{V}^d, \mathcal{E}^d)$ with vertex set $\mathcal{V}^d = \mathcal{P}^\tau$ and edges set $\mathcal{E}^d = \Big\{(E_{\sigma_1}^p, E_{\sigma_2}^{p+1}) \Big| E_{\sigma_1}^p, E_{\sigma_2}^{p+1} \in \mathcal{V}^d, \mathrm{adj}_{\theta_p}(\sigma_1, \sigma_2) = 1\Big\}$.

Therefore, each vertex of $\mathcal{G}$ corresponds to a certain cell of the partition $\mathcal{P}^\tau$ at a given time θ_p, and the edge $v_1 \to v_2$ exists if v_1 and v_2 represent two adjacent cells (possibly twice the same) at two consecutive time steps. Paths in $\mathcal{G}^d$ comply with the usual definitions of graph theory and can be given as a set of vertices.

Finally, we define the notion of time margin in $\mathcal{G}^d$:

Definition 6: For a path $\pi^d = \big(E_{\sigma_0}^0, \ldots, E_{\sigma_{m+1}}^{m+1}\big)$ in $\mathcal{G}^d$, the time margin is $v(\pi^d) = \min\limits_{i=0\ldots m} \max\limits_{p=i\ldots m} \tilde{v}(i, p)$ with $\tilde{v}(i, p) = \big\{\tau(p - i + 1) \,\big|\, \forall q \in \{i \,..\, p\}, \mathrm{adj}_{\theta_q}(\sigma_i, \sigma_{i+1}) = 1\big\}$.

IV. Application to Planning and Control

Before presenting the applications of our approach to planning and control, let us formulate the following definition:

Definition 7: Let $\pi_0^c = \big((\sigma_1, \ldots, \sigma_{m+1}), (t_1, \ldots, t_m)\big)$ be a path in $\mathcal{G}^c$ and $x(t)$ be a collision-free trajectory for the ego-vehicle. We say that π_0 corresponds to x if, for all $i \in \{1 \,..\, m\}$, $x\big([t_{i-1}, t_i)\big) \subset E_{\sigma_i}$ and $x\big([t_m, T)\big) \subset E_{\sigma_{m+1}}$.

From this definition, we deduce that for a given collision-free trajectory $x(t)$ there exists a unique corresponding path in $\mathcal{G}^c$ denoted by $\pi(x)$. Reciprocally, for a given path π_0 in $\mathcal{G}^c$, there exists a set of corresponding trajectories denoted by $\pi^{-1}(\pi_0)$. We obtain the following theorem:

Theorem 2: Let $J(x)$ be a cost function for a given trajectory $x(t)$, X the set of collision-free trajectories, and Π the set of paths in $\mathcal{G}^c$. We have:

$$\min_{x \in X} J(x) = \min_{\pi_0 \in \Pi} \left(\min_{x \in \pi^{-1}(\pi_0)} J(x) \right) \tag{1}$$

Proof: From the previous definition, for any collision-free trajectory $x \in X$ there exists $\pi_0 \in \Pi$ such that $\pi(x) = \pi_0$. Therefore, $\min_{x \in X} J(x) \leq \min_{\pi_0 \in \Pi} \big(\min_{x \in \pi^{-1}(\pi_0)} J(x)\big)$. Reciprocally, for all $\pi_0 \in \Pi$, any $x \in \pi^{-1}(\pi_0)$ is guaranteed to be collision-free, leading to the reciprocal inequality. ■

In other words, it is equivalent to find an optimal trajectory for the ego-vehicle, and to find an optimal path in the transition graph $\mathcal{G}^c$ and then the optimal trajectory corresponding to this path. These results can be extended to paths in the discrete graph $\mathcal{G}^d$; due to space limitations, details are not presented here and we only provide the following definition:

Definition 8: Let $\tau > 0$ be a discretization time step, $\pi_0^d = (E_{\sigma_0}^0, \ldots, E_{\sigma_{m+1}}^{m+1})$ be a path in $\mathcal{G}^d$ and $x(t)$ be a collision-free trajectory for the ego-vehicle. We say that π_0^d corresponds to x if, for all $p \in \{0 \,..\, m\}$, $x(\theta_p) \in E_{\sigma_p}^p$ and $x([\theta_p, \theta_{p+1})) \subset E_{\sigma_p}^p \cup E_{\sigma_{p+1}}^p$.

An interesting feature of this decomposition of the trajectory planning problem is that we effectively separate the discrete choice of a maneuver variant, and the search for an optimal control corresponding to this maneuver as was obtained in [14] for path-planning. This second problem can be solved efficiently under certain assumptions on the vehicle dynamics. We consider an AGV with linear discrete dynamics (using, *e.g.*, the flatness property [3]) $x_{p+1} = Ax_p + Bu_p$ for a state x_p and a control $u_p \in U$ (with U a convex polyhedron) with a discretization time step τ, where A and B have constant coefficient. For a positive semi-definite matrix Q, a line vector L and defining $X_p = [x_p^T, u_p^T]^T$, we consider a generic quadratic cost function $J(x, u) = \frac{1}{2} X_p^T Q X_p + L^T X_p$. In this case:

Theorem 3: Let π_0 be a path in $\mathcal{G}^d$ and x_0 an initial AGV state. The optimal trajectory (and associated control sequence) (X_p) starting from x_0 realizing $\min\limits_{(x_p) \in \pi^{-1}(\pi_0), u_p \in U} J(x, u)$ can be computed in polynomial time in the number of obstacles and time steps.

Proof: We will show that this problem is an instance of convex quadratic programming (QP), which has a complexity $\mathcal{O}(n^3)$ where n is the number of constraints [24]. First, the cost function J is quadratic and convex. Second, vehicle dynamics and control bounds can be encoded as linear constraints. Moreover, the condition $(x_p) \in \pi^{-1}(\pi_0)$ corresponds to a set of $\mathcal{O}(PN)$ linear constraints, leading to a QP problem with complexity $\mathcal{O}\left((PN)^3\right)$ for N obstacles over P time steps,thus proving the announced result. ■

Note, however, that the graphs $\mathcal{G}^c$ and $\mathcal{G}^d$ do have a number of vertices scaling exponentially with the number of obstacles. An advantage of our approach is that exhaustive exploration is not required, especially when considering safety margins, for instance on minimum required time or space margin (see [15]). Direct optimization techniques such as those based on mixed-integer programming [11] do not allow similar pruning of the decision tree using practical considerations (although branch-and-bound heuristics do allow pruning). We believe that the use of such safety metrics coupled with tailored heuristics for graph exploration can prove very interesting for safe and efficient trajectory planning under uncertainty.

V. Conclusion

This article generalized the divide-and-conquer approach used in [14] in two dimensions, to the 3D space-time for on-road navigation of autonomous ground vehicles in presence of moving obstacles. We described a systematic method to partition the collision-free space-time in the presence of fixed or moving obstacles, and we provided a graph representation of all possible collision-free trajectories. This approach allows to treat the combinatorial problem of optimal trajectory planning in two steps: first, a path-finding problem in a graph, and then a simple optimization that can be performed in polynomial time in the number of obstacles for any quadratic cost function. Moreover, we introduced a notion of time margin and showed that our graph-based approach can easily take into account margin of error in the execution for a particular maneuver. Coupled with additional similar metrics, we believe that our approach can have useful applications for planning under prediction and control uncertainty, notably in the frame of stochastic decision-making.

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