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Linear observed systems on groups

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Abstract

We propose a unifying and versatile framework for a class of discrete time systems whose state is an element of a general group G, that we call linear observed systems on groups. Those systems strictly mimic linear systems in the sense that $+$ is replaced with group multiplication, and linear maps with automorphisms. We argue they are the true generalization of linear systems of the form $X_{n+1} = F_n X_n + B_n u_n$ in the context of state estimation, since 1- when $G = \mathbb{R}^N$ the latter systems are recovered, 2- they are proved to possess the "preintegration" property, a characteristic property of linear systems that relates continuous time to discrete time, and has recently proved extremely useful in robotics applications, and 3- we can build observers that ensure the evolution between the true state and estimated state does not depend on the followed trajectory, a characteristic feature of Luenberger (and invariant) observers. The theory is applied to a 3D inertial navigation example. Interestingly, this example cannot be put in the form of an invariant system and the proposed generalization is required.

1 Introduction

Geometric observer design has been long researched, see e.g. [16,18,20]. Symmetry-preserving (also called invariant) observers [7], or more generally nonlinear observers on Lie groups, e.g. [6,5,12,10,11] have drawn attention over the past decade, both for their theoretical convergence properties, and for their simplicity in applications.

In [6] it is advocated that invariant systems on Lie groups with equivariant output maps yield autonomous error equations, that is, the discrepancy between the estimate and the true state does not explicitly depends on the followed trajectory. This fact was also noticed and exploited in various other works, see [13,19] amongst others. In [3], the class of systems such that the invariant error between two trajectories of the system is autonomous was extended, and referred to as group affine systems.

In the present article we consider the discrete-time case, and we study in Sec. 3 the class of systems obtained by mimicking the linear systems but replacing the addition with group law and linear maps with automorphisms. Although this sounds simple, we do not know of previous work following this route in the context of state estimation. We prove the obtained class of systems share two desirable properties of linear systems. First, in Sec. 4, the preintegration property which does generally not hold for nonlinear systems, and has recently been proved for inertial navigation [15,9], a very popular result in robotics. Then, we prove in Sec. 5 the autonomous error equation property of linear and invariant systems, see e.g. [6], actually carries over to the entire proposed class. Proceeding further, we provide in Sec. 6 a characterization explicitly based on automorphisms of all systems that satisfy the implicit form used in [4]; and prove those systems are in fact the *only ones* such that the invariant error satisfies an autonomous equation. The theory is applied to a 3D inertial navigation example.

2 Linear observed systems in $\mathbb{R}^N$

In this section we consider a classical discrete-time linear system as defined below:

Definition 1 (Linear system) *For all $n \in \mathbb{N}$, let $F_n \in \mathbb{R}^{N\times N}$, $H_n \in \mathbb{R}^{P\times N}$, $B_n \in \mathbb{R}^{N\times M}$ and $u_n \in \mathbb{R}^M$. A discrete time linear observed system with state $x_n \in \mathbb{R}^N$ is defined through:*

$$x_{n+1} = F_n x_n + B_n u_n \tag{1}$$
$$y_n = H_n x_n + D_n u_n \tag{2}$$

where $y_n \in \mathbb{R}^P$ is the observed output.

We dedicate sections 2.1 and 2.2 to recalling two classical properties of linear systems which are central in the

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results presented in the rest of the paper, as what we propose is generalizing them to a class of nonlinear systems. Note that, all the following properties do *not* hold for nonlinear systems in general.

2.1 Desirable feature n° 1: the preintegration property

The continuous-time counterpart of (1) writes:

$$\frac{d}{dt}x(t) = F_t x(t) + B_t u_t. \tag{3}$$

In estimation problems, we are usually interested in the values taken by $x(t)$ only at discrete instants $t_0, t_1, \ldots$. Thus, the system we work on is:

$$x(t_{n+1}) = \psi_{t_n}^{t_{n+1}}(x(t_n)). \tag{4}$$

with $\psi_{t_n}^{t_{n+1}}$ the flow of Eq. (3) between times t_n and t_{n+1}. Now, if we want to compute $x_{n+1} := x(t_{n+1})$ from $x_n := x(t_n)$ for *various values* of x_n, do we have to store the values of the processes F_t, B_t, u_t between t_n and t_{n+1} and integrate Eq. (3) for each initial condition x_n, or can we obtain a finite set of parameters once and for all, allowing then computing $x(t_{n+1})$ from $x(t_n)$ in closed form? In robotics, such a procedure is called preintegration and is extremely useful to apply modern state estimation techniques. Preintegration of linear systems proves easy:

Proposition 2 (preintegration of linear systems) *Given two instants $t_n < t_{n+1}$, there exist a matrix F_n and a vector v_n such that the flow $\psi_{t_n}^{t_{n+1}}$ of Eq. (3) reads:*

$$\psi_{t_n}^{t_{n+1}}(x) = F_n x + v_n, \quad \forall x \in \mathbb{R}^N$$

PROOF. Let $M_t \in \mathbb{R}^{N\times N}$ and v_t be defined as:

$$\begin{aligned} M_{t_n} &= I_N, \quad \frac{d}{dt}M_t = F_t M_t, \\ v_{t_n} &= 0, \quad \frac{d}{dt}v_t = F_t v_t + B_t u_t, \end{aligned} \qquad (\text{“Preintegration”}) \tag{5}$$

Then, for any value $x(t_n) := x_n$, the solution at arbitrary time $t_{n+1} > t_n$ to (3) writes $x(t_{n+1}) = M_{t_{n+1}} x_n + v_{t_{n+1}}$ as may be immediately verified by differentiation.

Note that, the preintegration property does not spare the practitioner the implemention of a numerical integration scheme for (5). But instead of running it to solve Eq. (3) for each desired values of x_n, numerical integration, which may be costly numerically for small time steps, is required only once to obtain F_n and v_n. Then, computing the new value of x_{n+1} from a different initial value x_n does not require a new numerical integration.

2.2 Desirable feature n° 2 : autonomous error

Given a partially observed dynamical system, an observer is another dynamical system fed with observations coming from the first system and designed to provide an estimation of the state, without ever differentiating the measured signals (which are always noisy in practice).

Definition 3 (Linear observer) *A linear observer, known as Luenberger observer [14], of the system* (1), (2) *is an observer of the form:*

$$\hat{x}_{n+1|n} = F_n \hat{x}_{n|n} + B_n u_n, \tag{6}$$

$$\hat{x}_{n+1|n+1} = \hat{x}_{n+1|n} + L_{n+1} z_{n+1} \tag{7}$$

where $z_{n+1} = \left(y_{n+1} - H_{n+1}\hat{x}_{n+1|n} - D_{n+1}u_{n+1}\right)$ is called innovation, and L_{n+1} a tunable matrix called gain.

The property making the study of linear observers for linear systems particularly easy, is related to the evolution of the estimation error e supposed to measure discrepancy between true state and estimated state: $e := x - \hat{x}$. The evolution of e writes:

$$e_{n+1|n} := x_{n+1} - \hat{x}_{n+1|n} = F_n x_n - F_n \hat{x}_{n|n} = F_n e_{n|n}, \tag{8}$$

$$e_{n+1|n+1} := x_{n+1} - \hat{x}_{n+1|n+1} = \left(I - L_{n+1}H_{n+1}\right) e_{n+1|n}. \tag{9}$$

At both propagation (8) and update step (9), the error variable follows an autonomous equation, i.e., independent from the actual value of $\hat{x}$. Note that the equation explicitly depends on the time step n, as such the term “autonomous” is abusive, but is used here to insist on the absence of explicit dependency on the estimated state $\hat{x}$. This means matrices L_{n+1} can be tuned without considering any specific trajectory of the system and the tuning will then be satisfactory for *all* trajectories.

3 Linear observed systems on groups

In this section we use basic notions of the group theory to build a generalization of linear observed systems, by replacing vector addition with a generic operation.

3.1 Preliminaries

Definition 4 (Group) *A group is a set G endowed with a group composition law, i.e. a map $G \times G \to G$ denoted $a \cdot b$, and referred to as “dot”, that verifies:*

- *Associativity: $\forall x, y, z \in G, \quad x \cdot (y \cdot z) = (x \cdot y) \cdot z$.*
- *Neutral element: There exists an element $Id \in G$ such that $\forall x \in G, \quad x \cdot Id = Id \cdot x = x$.*
- *Inversion: For any $x \in G$ there exists an element $x^{-1} \in G$ such that $x \cdot x^{-1} = x^{-1} \cdot x = Id$.*

The neutral element is the equivalent of 0 in a vector space. Note that $Id^{-1} = Id$ and $(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$. To generalize linearity to any group law in view of designing observers, recall the useful property of linearity was factorizing the addition in Section 2.2. The counterpart of this property for a general group law is as follows.

Definition 5 (Group automorphism) *An automorphism of G is an invertible map $\phi : G \to G$ such that:*

$$\forall a, b \in G, \qquad \phi(a \cdot b) = \phi(a) \cdot \phi(b)$$

It can be easily shown that this assumption implies $\phi(Id) = Id$ and $\forall x \in G, \phi(x^{-1}) = \phi(x)^{-1}$. The set of all automorphisms of G is denoted by $Aut(G)$.

Let us now consider a second class of maps which can be considered as a generalization of linear maps suited to the generalization of linear observations.

Definition 6 (Left group action, see e.g. [17]) *Let G be a group and Y an arbitrary set. A left group action of G on Y is an operation we will denote by star $\star$:*

$$G \times Y \to Y, \qquad (x, b) \mapsto x \star b$$

and which satisfies:

$$\forall x_1, x_2 \in G, \forall b \in Y, \quad x_1 \star (x_2 \star b) = (x_1 \cdot x_2) \star b, \quad (10)$$

which is a sort of associativity property. In this paper, the argument $b \in Y$ will be referred to as the "target".

Note that this operation is closely related to the compatibility of the output map introduced in [1,7].

3.2 Linear observed systems on groups

For linear dynamics the propagated state (1) is of the form "linear mapping of the state + vector". The counterpart we expect here is thus: "automorphism of the state dot group element". Linear observation (2) have the form "linear mapping + vector", which is a specific case of group action as shown in Sect. 3.3. The natural generalization of linear systems to groups thus writes:

Definition 7 (Left linear observed system) *Let G be a group, and for all $n \in \mathbb{N}$, let $u_n \in G$. We define a linear observed system with state $x_n \in G$ through:*

$$x_{n+1} = \phi_n(x_n) \cdot u_n \quad (11)$$
$$y_n = x_n \star b_n \quad (12)$$

with ϕ_n a group automorphism (possibly varying with n) and star $\star$ a group action of x_n on a "target" b_n of a set Y. Note that the group action can also depend on n, but denoting it by $\star$ is no source of ambiguity.

3.3 Relation to linear systems in $\mathbb{R}^N$

To support our claim that linear observed systems on groups of Def. 7 are the true generalization of linear observed systems in $\mathbb{R}^N$ in the context of deterministic state estimation, we start off noticing that when $G = \mathbb{R}^N$ endowed with $+$ as composition law, (11) boils down to (1), as the only automorphisms of $\mathbb{R}^N$ are linear maps.

As concerns (12), the operation $\star : (G, Y) \to Y$ defined by $x \star b = H_n x + b$ is an action of $\mathbb{R}^N$ on $\mathbb{R}^P$ indeed, as $(x_1 + x_2) \star b = H_n(x_1 + x_2) + b = H_n x_1 + H_n x_2 + b = H_n x_1 + x_2 \star b = x_1 \star (x_2 \star b)$. Thus, linear outputs $y_n = H_n x + D_n u_n$ can be written as $y_n = x \star b_n$ with $\star$ the action of $\mathbb{R}^N$ on $\mathbb{R}^P$ just defined and $b_n = D_n u_n$.

In the remainder of this paper, we will prove that more importantly the desirable features n°1 and n°2 of Section 2 both carry over to linear observed systems on groups.

3.4 Relation to invariant systems on groups

Let us comment on the relation with discrete-time left-invariant systems on G, i.e. dynamics of the form $x_{n+1} = x_n \cdot v_n$. In the case where $G = \mathbb{R}^N$ is endowed with addition, these dynamics boil down to $x_{n+1} = x_n + v_n$. Obviously, this encompasses systems of the form $x_{n+1} = x_n + B_n u_n$ but not (1). Invariant systems may thus be viewed as pure integrators, while using automorphisms $\phi_n(x_n)$ in (11) allows recovering linear maps $F_n x_n$ in $G = \mathbb{R}^N$. Thus the proposed generalization (11) encompasses the *much* wider class of linear systems (1) on $\mathbb{R}^N$.

4 Desirable feature n°1: preintegration

In this section we assume G is a Lie group and remove the "$\cdot$" as for matrix Lie groups. In [3] we named "group-affine systems" the dynamics defined for $x_t \in G$ by $\frac{d}{dt} x_t = f_t(x_t)$ and where f_t satisfies the condition:

$$f_t(x_1 x_2) = f_t(x_1) x_2 + x_1 f_t(x_2) - x_1 f_t(Id) x_2. \quad (13)$$

Note that a local version also appeared in the context of control in [2]. Let us denote by ψ the flow of such dynamics. Then, Proposition 2 generalizes as:

Proposition 8 (preintegration of group-affine systems) *Given two instants t_n and t_{n+1}, there exists $\phi_n \in Aut(G)$ and an element $v_n \in G$ such that the flow ψ satisfies:*

$$\psi_{t_n}^{t_{n+1}}(x) = \phi_n(x) \cdot u_n.$$

The dimension of $Aut(G)$ being upper bounded by the square of the dimension of G, $\psi_{t_n}^{t_{n+1}}$ can always be encoded by a finite number of parameters, as in the linear case.

PROOF. We first recall $\psi_{t_n}^t$ denotes the flow of f. For $t > t_n$ let $\phi_{t_n}^t$ denote the flow of the vector field $x \to f_t(x) - xf_t(I_d)$ from time t_n. For $a, b \in G$, differentiating $\alpha_t := \phi_{t_n}^t(a)\phi_{t_n}^t(b)$ and using (13) we obtain that α_t is solution of $\frac{d}{dt}\alpha_t = f_t(\alpha_t) - \alpha_t f_t(I_d)$ with initial condition $\alpha_{t_n} = ab$. By definition of the flow we have thus $\alpha_t = \phi_{t_n}^t(ab)$. We proved $\phi_{t_n}^t(ab) = \phi_{t_n}^t(a)\phi_{t_n}^t(b)$, i.e., $\phi_{t_n}^t \in Aut(G)$. Now, letting $u_t := \psi_{t_n}^t(Id)$, for $x \in G$ we have $\frac{d}{dt}\left[\phi_{t_n}^t(x)u_t\right] = \left[\frac{d}{dt}\phi_{t_n}^t(x)\right]u_t + \phi_{t_n}^t(x)\frac{d}{dt}u_t = \left[f(\phi_{t_n}^t(x)) - \phi_{t_n}^t(x)f(Id)\right]u_t + \phi_{t_n}^t(x)f_t(u_t) = f_t(\phi_{t_n}^t(x)u_t)$ using (13). As ψ is the flow of f this proves $\phi_{t_n}^t(x)u_t = \psi_{t_n}^t(x)$, and it suffices to set $\phi_n = \phi_{t_n}^{t_{n+1}}$ and $u_n := u_{t_{n+1}}$.

The latter theoretical result is the exact counterpart of preintegration of linear systems. We have the following result owing to non-commutativity of multiplication.

Corollary 9 (Left-right preintegration) *Consider*

$$\frac{d}{dt}x_t = f_t(x_t) := w_t x_t + \tilde{f}_t(x_t) + x_t u_t \tag{14}$$

where $\tilde{f}_t$ verifies $\tilde{f}_t(x_1x_2) = \tilde{f}_t(x_1)x_2 + x_1\tilde{f}_t(x_2)$ and denote by $\tilde{\phi}$ the flow of $\frac{d}{dt}x_t = \tilde{f}_t(x_t)$. Then $\tilde{\phi}(\cdot)$ is an automorphism and the flow ψ of f_t writes:

$$\psi_{t_n}^{t_{n+1}}(x) = \gamma_n \tilde{\phi}_{t_n}^{t_{n+1}}(x)\upsilon_n, \quad t_{n+1} > t_n,\ x \in G, \tag{15}$$

with $\gamma_n \in G$, $\upsilon_n \in G$ the solutions at t_{n+1} of:

$$\frac{d}{dt}\gamma_t = w_t\gamma_t + \tilde{f}_t(\gamma_t), \quad \frac{d}{dt}\upsilon_t = \upsilon_t u_t + \tilde{f}_t(\upsilon_t), \tag{16}$$

with initial condition $\gamma_{t_n} = \upsilon_{t_n} = Id$.

Indeed, it is easily proved f_t defined by (14) satisfies (13), and thus Prop. 8 applies. By looking into the proof of the latter proposition manipulations prove that $u_n = \gamma_n\upsilon_n$ and $\tilde{\phi}_n(x) = \gamma_n\phi_n(x)\gamma_n^{-1}$ which can in turn easily be proved to be an automorphism.

The expression for γ, ϕ, υ are obtained independently from x. This means quantities $w_t, \tilde{f}_t, u_t$ may be first "preintegrated" and then applied to *any* initial condition, truly generalizing Prop 2. As in the linear time-varying case, this does not mean the preintegrated factors γ, υ can be computed in closed form: an integration scheme shall still be used, but only once and for all.

5 Desirable feature n°2 : autonomous error

5.1 Definition of generalized linear observers

We now build nonlinear observers sharing some of the properties of linear observers:

Definition 10 (Left generalized linear pre-observer) *For the linear observed system* (11), (12) *a generalized pre-observer on the group G is defined by a sequence of estimates $(\hat{x}_{n|n}), (\hat{x}_{n+1|n})$ of the following form:*

$$\hat{x}_{n+1|n} = \phi_n(\hat{x}_{n|n}) \cdot u_n \tag{17}$$

$$\hat{x}_{n+1|n+1} = \hat{x}_{n+1|n} \cdot L_{n+1}\left(\hat{x}_{n+1|n}^{-1} \star y_n\right), \tag{18}$$

with $L_{n+1}()$ any operator from Y to G.

In the *particular case* where $\phi_n(x) = x$, (11) is said left-invariant, since if $(x_n)_{n\geq 0}$ is a solution then so is $(g \cdot x_n)_{n\geq 0}$ for $g \in G$. And then, (17), (18) is called a left-invariant observer, or invariant observer [1,6].

5.2 Properties

Given a group law, Def. 4 gives all the tools we need to transpose the definition of the error variable $e = x - \hat{x}$:

Definition 11 (Left-invariant error variable) *Let x and $\hat{x}$ be two elements of a group G. We define the left-invariant error e between $\hat{x}$ and x as:*

$$e = \hat{x}^{-1} \cdot x \tag{19}$$

This error is called left-invariant, since it is unchanged by the transformation $(\hat{x}, x) \mapsto (g \cdot \hat{x}, g \cdot x)$, for arbitrary $g \in G$: $(g \cdot \hat{x})^{-1} \cdot (g \cdot x) = \hat{x}^{-1} \cdot g^{-1} \cdot g \cdot x = \hat{x}^{-1} \cdot x$. Note that if the dot is addition, we obtain $e = -\hat{x} + x = x - \hat{x}$, the classical error variable.

Just like linear systems, linear systems on groups yield autonomous error evolution being analog to (8), (9). From (11) and (17), and using a left-invariant error (19) to measure discrepancy between true state and estimate,

$$\begin{aligned} e_{n+1|n} := \hat{x}_{n+1|n}^{-1} \cdot x_{n+1} &= \left[\phi_n(\hat{x}_{n|n}) \cdot u_n\right]^{-1} \cdot \left[\phi_n(x_n) \cdot u_n\right] \\ &= u_n^{-1} \cdot \phi_n(\hat{x}_{n|n})^{-1} \cdot \phi(x_n) \cdot u_n \\ &= u_n^{-1} \cdot \phi_n(\hat{x}_{n|n}^{-1}) \cdot \phi(x_n) \cdot u_n \\ &= u_n^{-1} \cdot \phi_n(\hat{x}_{n|n}^{-1} \cdot x_n) \cdot u_n = I_{u_n^{-1}}(\phi_n(e_n)) \end{aligned} \tag{20}$$

where we let I_g denote the map (called the inner automorphism): $I_g : x \mapsto g \cdot x \cdot g^{-1}$. We see that $\hat{x}_n$ and x_n collapse to e_n at the last line and disappear from the equation, as in the linear case. As well, innovations are only dependent on the error variable, if defined as:

$$z_n = \hat{x}_n^{-1} \star y_n = \hat{x}_n^{-1} \star (x_n b) = (\hat{x}_n^{-1} \cdot x_n) \star b = e_n \star b_n \tag{21}$$

The update step also yields autonomy, as:

$$\begin{aligned} e_{n+1|n+1} &= \hat{x}_{n+1|n+1}^{-1} \cdot x_{n+1} \\ &= \left[\hat{x}_{n+1|n} \cdot L_{n+1}(e_{n+1|n} \star b_n)\right]^{-1} \cdot x_{n+1} \\ &= L_{n+1}\left(e_{n+1|n} \star b_n\right)^{-1} \cdot \hat{x}_{n+1|x_n}^{-1} \cdot x_{n+1} \\ &= L_{n+1}\left(e_{n+1|n} \star b_n\right)^{-1} \cdot e_{n+1|n}. \end{aligned} \tag{22}$$

Proposition 12 *Consider a linear observed system* (11),(12) *as introduced in Definition 7, and a linear pre-observer* (17),(18) *as introduced in Definition 10. Then, the error evolution (including propagation step and update step) is autonomous. Namely, it writes:*

$$e_{n+1|n} = u_n^{-1} \cdot \phi_n(e_{n|n}) \cdot u_n \tag{23}$$

$$e_{n+1|n+1} = \left[L_{n+1}\left(e_{n+1|n} \star b_n\right)\right]^{-1} \cdot e_{n+1|n} \tag{24}$$

Of course, this boils down to (8), (9), if the group is a vector space with addition as group law.

5.3 Generalized linear observers for right group actions

When the output is a so-called right group action, generalized linear observers have a slightly different form.

Definition 13 (Right group action [17]) *Let G be a group and Y a set. A right group action of G on Y is an operation $Y \times G \to Y$ we will denote by a star $\star$ and define as $(b, x) \mapsto b \star x$, which verifies:*

$$\forall x, y \in G, \forall b \in Y, \quad (b \star x) \star y = b \star (x \cdot y), \tag{25}$$

Considering this second type of observations we obtain a second family of systems:

Definition 14 (Right linear observed system) *A right generalized linear system with state $x_n \in G$ is a system defined by a group automorphism ϕ_n, and observed through right group actions on a set Y:*

$$x_{n+1} = \phi_n(x_n) \cdot u_n \tag{26}$$

$$y_n = b_n \star x_n \tag{27}$$

Another natural transposition of the linear difference $e = x - \hat{x}$ would be $e = x \cdot \hat{x}^{-1}$, called right-invariant error since it is unchanged by transformation $(x, \hat{x}) \mapsto (x \cdot g, \hat{x} \cdot g)$. This error turns out to be suited to observations defined through right group actions. In this case, linear pre-observers read:

$$\hat{x}_{n+1|n} = \phi_n(\hat{x}_{n|n}) \cdot u_n \tag{28}$$

$$\hat{x}_{n+1|n+1} = L_{n+1}\left(y_n \star \hat{x}_{n+1|n}^{-1}\right) \cdot \hat{x}_{n+1|n}, \tag{29}$$

For errors $e_{n+1|n} = x_n \cdot \hat{x}_{n+1|n}^{-1}$ and $e_{n|n} = x_n \cdot \hat{x}_{n|n}^{-1}$, we have indeed $e_{n+1|n} = \phi_n(e_{n|n})$, $e_{n+1|n+1} = e_{n+1|n} \cdot L_{n+1}\left(y_n \star e_{n+1|n}\right)^{-1}$ ensuring error autonomy again.

5.4 Convergence properties

In the present paragraph we show that some properties of the continuous time case proved in [3] carry over to our discrete time framework. To linearize, we assume G is a N-dimensional matrix Lie group $G \subset \mathbb{R}^{D \times D}$ and define the "linearized error" ξ in the Lie algebra $\mathfrak{g}$ through:

$$e_{n|n} = \exp(\xi_{n|n}), \quad e_{n+1|n} = \exp(\xi_{n+1|n}). \tag{30}$$

A key property of linear observed systems is that the linearized error variable $\xi_{n|n}$ evolves linearly during the propagation step, i.e., there exists a matrix F_n such that:

$$\xi_{n+1|n} = F_n \xi_{n|n} \tag{31}$$

This is a consequence of the Lie group - Lie algebra homomorphism correspondance, a classical result.

Theorem 15 (Lie gr. - Lie alg. correspondance) *Let G be a Lie group and $\phi : G \to G$ an automorphism of G. Then, there exists a linear map $f : \mathfrak{g} \to \mathfrak{g}$ such that: $\phi \circ \exp = \exp \circ f$.*

The map f appearing in Theorem 15 being linear, it can be represented under a classical matrix form: $f(\xi) = F\xi$ if $\mathfrak{g}$ is identified to $\mathbb{R}^N$. As $I_{u_n^{-1}} \circ \phi_n$ of (20) is an automorphism by automorphism composition, Thm. 15 applies: there exists a matrix F_n such that $I_{u_n^{-1}}(\phi_n(\exp(\xi))) = \exp(F_n \xi)$ for any $\xi \in \mathfrak{g}$, yielding (31). Let us see how to use this property to build an observer in the case where $Y = \mathbb{R}^P$. Consider the Luenberger-like observer:

$$L_n(z_n) = \exp\left(K_n(\hat{x}_n^{-1} \star y_n - b)\right), \tag{32}$$

with $\exp() : \mathbb{R}^N \to G$ the exponential map, $K_n \in \mathbb{R}^{N \times P}$ a gain matrix and b the target of the action, appearing in the observation. A consequence of Equation (31) is that ξ_n associated to our errors has a very specific form:

Proposition 16 *Let e be the left-invariant error variable of a linear system on G observed via left actions. Consider a left linear pre-observer with $L_n(\cdot)$ defined by* (32) *and let ξ denote the linearized errors* (30)*. Then*

$$\begin{aligned} \xi_{n+1|n} &= F_n \xi_{n|n}, \\ \xi_{n+1|n+1} &= BCH\left[-K_n\left(\exp(\xi_{n+1|n}) \star b - b\right), \xi_{n+1|n}\right], \end{aligned} \tag{33}$$

where $BCH : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ denotes the Baker-Campbell-Hausdorff formula. A first-order approximation of the second equation readily writes $\xi_{n+1|n+1} = (I - K_n H_n)\xi_{n+1|n}$ where H_n is defined through the first-order Taylor expansion $\exp(\xi) \star b - b = H_n \xi + O(||\xi||^2)$.

Obtaining local convergence around *any* trajectory is known to be difficult for non-linear observers, owing to the dependency of the linearized system on the estimated trajectory. Owing to the latter proposition we obtain:

Proposition 17 *Consider invertible matrices $Q \in \mathbb{R}^{N\times N}$, $R \in \mathbb{R}^{P\times P}$, and $P_0 \in \mathbb{R}^{N\times N}$. Let K_n be defined through the recursion $P_{n+1|n} = F_n P_n F_n^T + Q$, $K_{n+1} = P_{n+1|n} H_n^T (H_n P_{n+1|n} H_n^T + R)^{-1}$, $P_{n+1} = P_{n+1|n} - K_{n+1} P_{n+1|n}$. Then, the linear observed observer defined through* (32) *converges locally around* any *trajectory if the pair (F_n, H_n) is uniformly observable, owing to standard convergence results on the Kalman filter [8].*

6 Characterizing linear systems on groups

In Sect. 4 we linked the class of systems we derived in Sect. 3 by analogy with the linear case to our previous work on continuous-discrete estimation [3]. In this section we relate the explicit characterization (11) to the more less constructive discrete-time form we proposed in [4], i.e. $x_{n+1} = f(x_n)$ where f satisfies $f(a \cdot b) = f(a)(f(Id))^{-1} f(b)$. As a byproduct of showing the equivalence of the two approaches we provide additional characterization of the same class of systems. In particular, we prove linear observed systems are in fact the only ones that ensure error autonomy on groups that notably led to Prop. 17.Consider general dynamics:

$$x_{n+1} = g_n(x_n), \tag{34}$$

with $g_n : G \to G$ a general map, *not* assumed to be of the form (11) at this stage. Consider then the propagation step of a pre-observer, which is a copy of the dynamics:

$$\hat{x}_{n+1|n} = g_n(\hat{x}_{n|n}), \tag{35}$$

For an analog of (8) to hold, all we need is the propagated error to be a function of the error before propagation, i.e., we need the existence of a map μ_n such that:

$$\hat{x}_{n+1|n}^{-1} \cdot x_{n+1} = g_n(\hat{x}_{n|n})^{-1} \cdot g_n(x_n) = \mu_n(\hat{x}_{n|n}^{-1} \cdot x_n).$$

The following result proves the only dynamics ensuring this, and thus autonomous error, have the form (11).

Theorem 18 (Characterization of autonomy) *Let G be a group and $g : G \to G$. Then there exists a function μ on G such that $\forall x_1, x_2 \in G, g(x_1)^{-1} \cdot g(x_2) = \mu(x_1^{-1} \cdot x_2)$ if and only if there exists $\phi \in Aut(G)$ and $u \in G$ such that $g(x) = \phi(x) \cdot u$ for all $x \in G$.*

To prove implication $\Rightarrow$, let us show $\mu(x_1 \cdot x_2) = \mu(x_1) \cdot \mu(x_2)$. Setting $z_1 = Id, z_2 = x_1, z_3 = x_1 \cdot x_2$ we have $\mu(z_1^{-1} \cdot z_2) \cdot \mu(z_2^{-1} \cdot z_3) = g(z_1)^{-1} \cdot g(z_2) \cdot g(z_2)^{-1} \cdot g(z_3) = g(z_1)^{-1} \cdot g(z_3) = \mu(z_1^{-1} \cdot z_3)$. Replacing now z_1, z_2, z_3 with their values we get $\mu(x_2) \cdot \mu(x_2) = \mu(x_1 \cdot x_2)$. Letting $x_1 = Id$, $x_2 = x$ we get $g(x) = g(Id) \cdot \mu(x)$ and we have proved $\mu \in Aut(G)$. The result is proved with $u := g(Id)$. Regarding the converse $\Leftarrow$, set $\mu = I_{u^{-1}} \circ \phi$.

In Thm. 18 we considered a left-invariant error, so we shall wonder what happens when using right-invariant errors. Note that in (11) we also chose to put the factor u_n on the right, and may wonder if it shall be on the left. The following corollary addresses the question, and also relates (11) to the symmetric definition proposed in [4].

Corollary 19 *The following propositions are equivalent*

(1) There exists $\phi_l \in Aut(G)$ and $u \in G$ such that $g(x) = \phi_l(x) \cdot u$ for all $x \in G$
(2) There exists $\gamma, \upsilon \in G$ and $\phi \in Aut(G)$ such that $g(x) = \gamma \cdot \phi(x) \cdot \upsilon$
(3) There exists $u \in G$ and $\phi_r \in Aut(G)$ such that $g(x) = u \cdot \phi_r(x)$
(4) A function μ_l on G exists, such that $\forall x_1, x_2 \in G, g(x_1)^{-1} \cdot g(x_2) = \mu_l(x_1^{-1} \cdot x_2)$
(5) A function μ_r on G exists, such that $\forall x_1, x_2 \in G, g(x_1) \cdot g(x_2)^{-1} = \mu_r(x_1 \cdot x_2^{-1})$
(6) For all $a, b \in G$ we have $g(ab) = g(a) \cdot g(Id)^{-1} \cdot g(b)$; which is the definition proposed in [4].

The proposition may be proved along the lines of the proof of Theorem 18.

7 Application to 3D inertial navigation

Consider a navigating vehicle characterized by its orientation $R_t \in SO(3)$ (the rotation matrix mapping the vehicle-fixed frame to a reference static frame), its velocity $V_t \in \mathbb{R}^3$ and position $X_t \in \mathbb{R}^3$. Its gyroscopes measure angular rates $\omega_t \in \mathbb{R}^3$ in continuous-time t, and its accelerometers the "specific force" $a_t \in \mathbb{R}^3$, i.e. vehicle acceleration minus gravity vector $\bar{g}$. Both sensors are part of an Inertial Measurement Unit (IMU) attached to the vehicle. In continuous time the dynamics write:

$$\frac{d}{dt} R_t = R_t (\omega_t)_\times, \quad \frac{d}{dt} V_t = R_t a_t + \bar{g}, \quad \frac{d}{dt} X_t = V_t. \tag{36}$$

where $(\omega_t)_\times$ is the skew-symmetric matrix defined by $(\omega_t)_\times b = \omega \times b \; \forall b \in \mathbb{R}^3$. We consider the observations:

$$y_n = X_{t_n}, \tag{37}$$
$$\textbf{and/or} \quad y_n = V_{t_n}, \tag{38}$$
$$\textbf{and/or} \quad y_n = X_{t_n} + R_{t_n} b_0, \tag{39}$$
$$\textbf{and/or} \quad y_n = R_{t_n}^T (b - \beta X_{t_n}). \tag{40}$$

(37) and (38) are position and velocity measurements. (39) represents position measurement of a point with

known lever arm b_0, typically GNSS antenna having a lever arm with respect to the IMU. Note that observations (37), (38), (39) can be used simultaneously while still fitting the framework of the present paper. Regrading Eq. (40), $\beta = 1$ represents measurement of a known landmark $b \in \mathbb{R}^3$ in the vehicle's frame whereas $\beta = 0$ corresponds to vector measurement (e.g., magnetic field).

7.1 Group $SE_2(3)$

We will make use of the group $SE_2(3)$ introduced in [3].

Definition 20 *The special orthogonal group $SO(3)$ is defined as: $SO(3) = \{R \in \mathbb{R}^{3\times 3},\ R^T R = Id,\ \det(R) = 1.\}$ The group $SE_2(3)$ is in turn defined as the set:*

$$SE_2(3) = \{(R, V, X), R \in SO(3), V, X \in \mathbb{R}^3\},$$

endowed with the following group law: $(R, V, X) \cdot \left(\tilde{R}, \tilde{V}, \tilde{X}\right) = \left(R\tilde{R}, R\tilde{V} + V, R\tilde{X} + X\right)$. Neutral element is $(Id, 0, 0)$ and $x^{-1} = \left(R^T, -R^T V, -R^T X\right)$.

We introduce a novel family of group actions of $SE(3)$:

Definition 21 (Actions of the group $SE_2(3)$) *We call vector action of $SE_2(3)$ on $\mathbb{R}^3$ with parameters $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ the action of $x = (R, V, X) \in SE_2(3)$ on $b \in \mathbb{R}^3$ defined by: $x \star b = Rb + \alpha_1 V + \alpha_2 X$.*

It can be checked that this defines an action of $SE_2(3)$.

7.2 Discretization of navigation equations

Embedding the state x_t in $G = SE_2(3)$, we noticed in [3] that (36) possesses the group affine property (13). Applying our preintegration Proposition 9 yields:

Proposition 22 (Preintegration of IMU outputs) *Define $R_t^\upsilon, V_t^\upsilon, X_t^\upsilon$ as the solutions to $\frac{d}{dt} R_t^\upsilon = R_t^\upsilon (\omega_t)_\times$, $\frac{d}{dt} V_t^\upsilon = R_t^\upsilon a_t$ and $\frac{d}{dt} X_t^\upsilon = V_t^\upsilon$, where $R_0^\upsilon = I_3$, $V_0^\upsilon = 0_3$ and $X_0^\upsilon = 0_3$. Then for arbitrary initial condition (R_0, V_0, X_0), the solution of (36) at* all *time $t \geq 0$ writes:*

$$R_t = R_0 R_t^\upsilon, \tag{41}$$
$$V_t = V_0 + t\bar{g} + R_0 V_t^\upsilon, \tag{42}$$
$$X_t = X_0 + tV_0 + \frac{1}{2}\bar{g}t^2 + R_0 X_t^\upsilon. \tag{43}$$

PROOF. The Lie algebra $\mathfrak{g}$ of $SE_2(3)$ consists of elements of the form $((\omega)_\times, a, b)$ with $\omega, a, b \in \mathbb{R}^3$, see [3]. Letting $w_t = \left(0_3, \bar{g}, 0_{3,1}\right)$, $u_t = \left((\omega_t)_\times, a_t, 0_{3,1}\right)$, $\tilde{f}_t(x_t) = \left(0_3\ 0_{3,1}\ V_t\right)$ it is easily checked (36) becomes $\dot{x}_t = w_t x_t + \tilde{f}_t(x_t) + x_t u_t$. The computation of γ_t, υ_t following Proposition 9 yields γ_t as $R_t^\gamma = I_3, V_t^\gamma = t\bar{g}, X_t^\gamma = \frac{1}{2}\bar{g}t^2$, and υ_t as $\frac{d}{dt}(R_t^\upsilon, V_t^\upsilon, X_t^\upsilon) = (R_t^\upsilon(\omega_t)_\times, R_t^\upsilon a_t, V_t^\upsilon)$ with $(R_0^\upsilon, V_0^\upsilon, X_0^\upsilon) = (I_3, 0_3, 0_3)$. As concerns $\tilde{\phi}_t(x)$ let us decompose it as $\tilde{\phi}_t(x) = \left(R_t^\phi(x), V_t^\phi(x), X_t^\phi(x)\right)$. By definition we have $\frac{d}{dt}(\tilde{\phi}_t(x)) = (\frac{d}{dt}\tilde{\phi}_t)(x) = \tilde{f}_t(\tilde{\phi}_t(x))$. The equation transposes to $R_t^\phi(x), V_t^\phi(x), X_t^\phi(x)$ as: $R_t^\phi(x) = R^{x_0}$, $\frac{d}{dt} R_t^\phi(x) = 0_3$, $V_t^\phi(x) = v^{x_0}$, $\frac{d}{dt} V_t^\phi(x) = 0_3$, $X_t^\phi(x) = X^{x_0}$, $\frac{d}{dt} X_t^\phi(x) = V_t^\phi(x)$, which gives immediately $R_t^{\phi(x)} = R^{x_0}, V_t^{\phi(x)} = V^{x_0}, X_t^{\phi(x)} = X^{x_0} + tV^{x_0}$. This being true for any x_0, we have: $\phi_t : \left(R_t^x, V_t^x, X_t^x\right) \to \left(R_t^x, V_t^x, X_t^x + tV_t^x\right)$. The result is then shown injecting the obtained values in (15) and then back to original variables.

Of course the computation of $R_t^\nu, V_t^\nu, x_t^\nu$ relies on an integration scheme on respectively $SO(3)$ and $\mathbb{R}^6$, and any scheme may be used. The important point, though, is that those quantities are computed based solely on the IMU measurements a_t, ω_t. Thus, the solution of the system at time t can be computed for any initialization without having to re-integrate when starting from a different initial condition. Actually, this was pioneered for inertial navigation by [15] using Euler angles, and recently shown using rotation matrices in [9] using smart linear algebra tricks without suspecting the result is actually grounded in group-theoretic properties.

The preintegration method of inertial measurements currently enjoys much popularity in robotics, see [9] and related papers. Indeed, it allows to apply modern optimization based techniques to localize robots that use IMUs. At each optimization step, the linearization point x_{t_n} changes, and $x_{t_{n+1}}$ may be computed without having to re-integrate. Given the high frequency of IMUs, real time is only made possible by preintegration.

The result now appears as a direct consequence of Proposition 9, that holds for all group affine systems.

7.3 Discrete-time inertial navigation as a linear system

Consider the navigation equations (36) discretized at times $t_1, t_2, \cdots$ with $t_{n+1} - t_n = \Delta t_n$. Using Prop. 22 they write $R_{t_{n+1}} = R_{t_n} R_{\Delta t_n}^\upsilon$ $V_{t_{n+1}} = V_{t_n} + \Delta t_n \bar{g} + R_{t_n} V_{\Delta t_n}^\upsilon, X_{t_{n+1}} = X_{t_n} + \Delta t_n V_{t_n} + \frac{1}{2}\bar{g}\Delta {t_n}^2 + R_{t_n} X_{\Delta t_n}^\upsilon$.

Using embedding in $G = SE_2(3)$ those equations combined with observations being either (37) or (38) or (39) or a combination of them may be re-written as

$$x_{n+1} = \gamma_{\Delta t_n} \cdot \phi_n(x_n) \cdot \upsilon_{\Delta t_n} \tag{44}$$
$$y_n = x_n \star b_n \tag{45}$$

with $\gamma_{\Delta t_n}, \upsilon_{\Delta t_n} \in SE_2(3)$ and $\phi_n = (R, V, X + \Delta t_n V)$.

As $\phi_n(x) \neq x$, (44) is *not* a combination of left and right invariance and the system is not invariant.

Proposition 23 *Discrete-time navigation equations define a linear observed system on $G = SE_2(3)$.*

As concerns (44), this stems from Corollary 19 since $\phi_n(R, V, X)$ is obviously an automorphism of $SE_2(3)$. As concerns (45), it can easily be verified that $(37), (38), (39)$ are defined using the group action of Def. 21. Finally, as concerns (40), it is easily shown to write $x_n^{-1} \star b$, e.g., a right group action [17]. As a direct application of the proposed framework, we have thus the following result about error autonomy:

Proposition 24 *Letting the error be $e = \hat{x}^{-1} \cdot x = (\hat{R}^T R, \hat{R}^T(V - \hat{V}), \hat{R}^T(X - \hat{X}))$, any pre-observer*

$$\hat{x}_{n+1|n} = \gamma_{\Delta t_n} \cdot \phi_n(\hat{x}_n) \cdot \upsilon_{\Delta t_n} \tag{46}$$

$$\hat{x}_{n+1|n+1} = \hat{x}_{n+1|n} \cdot L_{n+1}(\hat{x}_n^{-1} \star y_n) \tag{47}$$

leads to the autonomous error equation $e_{n+1|n} = I_{\gamma_{\Delta t_n}^{-1}} \circ \phi_n(e_{n|n})$, $e_{n+1|n+1} = L_{n+1}\left(e_{n+1|n} \star b_n\right)^{-1} \cdot e_{n+1|n}$.

Using the results of Section 5.4, we have immediately:

Proposition 25 *For navigation equations with position only measurements* (37)*, consider the observer*

$$\hat{R}_{n+1|n} = \hat{R}_{n|n} R^{\upsilon}_{\Delta t_n} \; \hat{V}_{n+1|n} = \hat{V}_{n|n} + \Delta t_n \bar{g} + \hat{R}_{n|n} V^{\upsilon}_{\Delta t_n},$$
$$\hat{x}_{n+1|n} = \hat{x}_{n|n} + \Delta t_n \hat{V}_{n|n} + \frac{1}{2}\bar{g}\Delta t_n{}^2 + \hat{R}_{n|n} x^{\upsilon}_{\Delta t_n}$$
$$(\hat{R}_{n+1|n+1}, \hat{V}_{n+1|n+1}, \hat{X}_{n+1|n+1}) = \exp_{SE_2(3)}(K_{n+1} z_{n+1})$$
$$z_{n+1} := (\hat{R}_{n+1|n} y_n - \hat{R}_{n+1|n} \hat{X}_{n+1|n})$$

Let $P_0 \in \mathbb{R}^{N \times N}$, and K_n be defined through the recursion $P_{n+1|n} = F_n P_n F_n^T + Q$, $K_{n+1} = P_{n+1|n} H^T (H P_{n+1|n} H^T + R)^{-1}$, $P_{n+1} = P_{n+1|n} - K_{n+1} P_{n+1|n}$, where $Q \in \mathbb{R}^{9 \times 9}$, $R \in \mathbb{R}^{3 \times 3}$ are invertible tuning matrices. Then, this observer is locally convergent around any *trajectory ensuring uniform observability of the linearized system.*

Recalling $e_{n+1|n} = I_{\gamma_{\Delta t_n}^{-1}} \circ \phi_n(e_{n|n})$, the matrix F_n is obtained from Theorem 15 and manipulations show:

$$F_n = \begin{pmatrix} (R^{\upsilon}_{\Delta t_n})^T & 0 & 0 \\ -(R^{\upsilon}_{\Delta t_n})^T (V^{\upsilon}_{\Delta t_n})_\times & (R^{\upsilon}_{\Delta t_n})^T & 0 \\ -(R^{\upsilon}_{\Delta t_n})^T (X^{\upsilon}_{\Delta t_n})_\times & (R^{\upsilon}_{\Delta t_n})^T \Delta t_n & (R^{\upsilon}_{\Delta t_n})^T \end{pmatrix}$$

and $H_n = (0_{3\times3}, 0_{3\times3}, I_3)$. Besides we recall that $\exp_{SE_2(3)}(\xi_1, \xi_2, \xi_3) = (\exp_{SO(3)}(\xi_1), V\xi_2, V\xi_3)$ with $V = I_3 + \frac{1 - \cos||\xi_1||}{||\xi_1||^2}(\xi_1)_\times + \frac{||\xi_1|| - \sin||\xi_1||}{||\xi_1||^3}(\xi_1)_\times$.

8 Conclusion

In this paper we derive by analogy to linear systems in $\mathbb{R}^N$ an explicit characterization of "linear" systems on groups based on group automorphisms and group actions. This characterization is shown to encompass classical linear systems, invariant systems [6,13,19] (note invariant systems are more restrictive as they don't encompass all linear systems when $G = \mathbb{R}^N$), and is proved to be related to the continuous time formluation of [3] through a group theoretic novel preintegration property. It is applied to a 3D inertial navigation example that is neither left nor right invariant nor a combination of both. In the future, we would like to apply the theoretical results to address the preintegration of inertial measurements on round earth with Coriolis acceleration.

References

[1] N. Aghannan, and P. Rouchon. On invariant asymptotic observers. In *Decision and Control (CDC), IEEE 41st Conference on*, pages 1479-1484, 2002.

[2] V. Ayala and J. Tirao. Linear control systems on lie groups and controllability. In *Proceedings of symposia in pure mathematics*, volume 64, pages 47–64, 1999.

[3] A. Barrau and S. Bonnabel. The Invariant Extended Kalman Filter as a Stable Observer. *IEEE Transactions on Automatic Control*, 62(4):1797–1812, 2017.

[4] A. Barrau and S. Bonnabel. Invariant Kalman Filtering. *Annual Review of Control, Robotics, and Autonomous Systems*, 2018.

[5] P. Batista, C. Silvestre, and P. Oliveira. A GES attitude observer with single vector observations. *Automatica*, 48(2):388–395, 2012.

[6] S. Bonnabel, P. Martin, and P. Rouchon. Non-Linear Symmetry-Preserving Observers on Lie Groups. *IEEE Transactions on Automatic Control*, 54(7):1709–1713, 2009.

[7] S. Bonnabel, P. Martin, and P. Rouchon. Symmetry-Preserving Observers. *IEEE Transactions on Automatic Control*, 53(11):2514–2526, 2008.

[8] J. Deyst and C. Price Conditions for asymptotic stability of the discrete minimum-variance linear estimator. *IEEE Transactions on Automatic Control*, 13(6):702–705, 1968.

[9] C. Forster, L. Carlone, F. Dellaert, and D. Scaramuzza. On-Manifold Preintegration for Real-Time Visual-Inertial Odometry. *IEEE Transactions on Robotics*, 33(1):1–21, 2017.

[10] M.-D. Hua, T. Hamel, R. Mahony, and G. Allibert. Explicit complementary observer design on special linear group sl (3) for homography estimation using conic correspondences. In *56th IEEE Conference on Decision and Control (CDC)*, 2017.

[11] M. Izadi and A. K Sanyal. Rigid body attitude estimation based on the lagrange–d'alembert principle. *Automatica*, 50(10):2570–2577, 2014.

[12] A. Khosravian, J. Trumpf, R. Mahony, and C. Lageman. Observers for invariant systems on Lie groups with biased input measurements and homogeneous outputs. *Automatica*, 55:19–26, 2015.

[13] C. Lageman, J. Trumpf, and R. Mahony. Gradient-like observers for invariant dynamics on a lie group. *IEEE Transactions on Automatic Control*, 55(2):367–377, 2010.

[14] D. Luenberger. An introduction to observers. *IEEE Transactions on automatic control*, 16(6):596–602, 1971.

[15] T. Lupton and S. Sukkarieh. Efficient integration of inertial observations into visual slam without initialization. In *IEEE Conference on Intelligent Robot and Systems*, 2009.

[16] S. K. Ng and P. E. Caines. Nonlinear filtering in Riemannian manifolds. *IMA JMCI*, 2(1):25–36, 1985.

[17] P. J. Olver. *Classical Invariant Theory*. Cambridge University Press, 1999.

[18] S. Said and J. H. Manton. On filtering with observation in a manifold: Reduction to a classical filtering problem. *SIAM Journal on Control and Optimization*, 51(1):767–783, 2013.

[19] J. F. Vasconcelos, R. Cunha, C. Silvestre, and P Oliveira. A nonlinear position and attitude observer on SE(3) using landmark measurements. *Systems & Control Letters*, 59(3):155–166, 2010.

[20] A. S. Willsky and S. I. Marcus. Estimation for bilinear stochastic systems. In *Variable Structure Systems with Application to Economics and Biology*, pages 116–137. 1975.