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# STEIN'S METHOD FOR ROUGH PATHS

L. COUTIN AND L. DECREUSEFOND

**ABSTRACT.** The original Donsker theorem says that a standard random walk converges in distribution to a Brownian motion in the space of continuous functions. It has recently been extended to enriched random walks and enriched Brownian motion. We use the Stein-Dirichlet method to precise the rate of this convergence in the topology of fractional Sobolev spaces.

## 1. INTRODUCTION

The Donsker theorem says that a random walk

$$X^m(t) = \frac{1}{\sqrt{m}} \sum_{k=1}^{[mt]} X_k$$

where the  $X_k$ 's are independent, identically distributed random variables with mean 0 and variance 1, converges in a functional space to the Brownian motion  $B$ . In the original version (see [8]), the convergence was proved to hold in the space of continuous functions. The first evolution was the paper of Lamperti [14], which proved the convergence in Hölder spaces. Namely, he stated that if the increments of the  $X_k$ 's are  $p$ -integrable then the random walk  $X^m$  converges to  $B$  in  $\text{Hol}(1/2 - 1/p)$ . The higher the integrability, the stronger the topology. There are numerous other extensions which can be made to the Donsker theorem. In the 90s, Barbour [2] estimated the rate of convergence in the space  $\mathcal{C}$  of continuous functions on  $[0, 1]$  equipped with a stronger topology than the usual sup-norm topology. He proved that, provided that  $\mathbf{E}[|X_k|^3]$  is finite, then

$$\sup_{|F|_M \leq 1} \mathbf{E}[F(X^m)] - \mathbf{E}[F(B)] \leq c \frac{\log m}{\sqrt{m}},$$

where  $M$  is roughly speaking, the set of thrice Fréchet differentiable functions on  $\mathcal{C}$  with bounded derivatives and  $|F|_M$  is a function of the supremum of  $D^i F$ ,  $i = 0, \dots, 3$  over  $\mathcal{C}$ . The strategy is to compare  $X^m$  with  $B^m$ , the affine interpolation, of mesh  $1/m$ , of the Brownian motion and then to compare  $B^m$  with  $B$ . The latter comparison is a sample-path comparison since the two processes live in the same probability space. This is the part which yields the  $\log m$  factor. The former comparison is done via the Stein's method in finite dimension. The rate of this convergence is as usual (see [3, 15]) for Gaussian limits, of the order of  $m^{-1/2}$ .

In [5], we quantified the rate of convergence of  $X^m$  towards  $B$  in Besov-Liouville spaces (see (2), which are one scale of fractional Sobolev spaces.

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The spaces we considered were not included in Hölder spaces but the method could likely be adapted to obtain convergence rate in Hölder spaces. However, it would not fit to the present context where we are considering enriched-paths (see definition below).

Note that in both [2] and [5], an higher integrability of the  $X_k$ 's would not improve the convergence rates but would give more flexibility on the choice of the topology in which the convergence holds: The higher the integrability, the higher the Hölder exponent may be chosen.

In [12, Theorem 13.3.3], Friz and Victoir essentially showed that a Lamperti's like result holds for the convergence of the enriched random walk in the sense of rough path to the enriched Brownian motion.

The motivation of this paper is to quantify the rate of this convergence in rough-paths sense. The first difficulty is that the limiting process is no longer a Gaussian process: The Lévy area of a Brownian motion is not Gaussian. Hence we cannot expect to have a direct application of the Stein's method. However, there is no more randomness in the Lévy area than there is in the Brownian motion itself: The Lévy area is adapted to the filtration generated by the underlying Brownian motions. Saying that has two consequences. First, that the probability space we have to consider depends only on the Brownian motion. Moreover, we have to find functional spaces for which the map which sends a Brownian motion to its Lévy area is not only continuous but also Lipschitz. As mentioned in [13], the Besov-Liouville spaces (we used in [5]) are not well fitted to deal with the iterated integral processes we encounter in rough-paths theory. It is much better to work with the Slobodetsky scale of fractional Sobolev spaces.

Once the functional framework is set up, in order to avoid some complicated calculations in infinite dimensional spaces, the idea is to go back to the approach of [2]: Comparing the random walk with the affine interpolation of the Brownian motion in the Slobodetsky scale of fractional Sobolev spaces. This can be done by an application of the Stein's method in finite dimension. The novelty comes from the treatment of the iterated integrals whose existence hugely complexifies the computations of the remainders. The final result is obtained by considering the known distance between the enriched affine interpolation and the enriched Brownian motion in fractional Sobolev spaces.

Besides the definition of the convenient functional spaces, one of our contribution, interesting in its own, is an improvement of the Stein's method (SM for short). Actually, SM for Gaussian limits in dimension 1 can be applied to Lipschitz functions but when the limit is in dimension 2 or more, for technical reasons, the standard procedure works only for functions twice differentiable with bounded derivatives. We show here that we can work with Lipschitz functions in any dimension by splitting the integral of the semi-group into two parts. To be more precise but still informal (all the notations will be rigorously defined below), the Stein's equation is solved by the semi-group method:

$$f_h(x) = \int_0^\infty LP_t h(x) \, dt.$$

The whole scheme of the SM implies that at some step, we control uniformly the increments of  $f_h''$  or roughly speaking, the supremum norm of  $f_h^{(3)}$ . Applying twice an integration by parts formula, the third derivative of  $P_t h$  gives rise to a factor  $e^{2t}(1 - e^{-2t})^{-1}$  which is not integrable on  $\mathbf{R}^+$  hence  $f_h^{(3)}$  cannot be easily represented. To circumvent this problem, we split the former integral from 0 to  $\tau$  and then from  $\tau$  to infinity. The first term is easily handled using the continuity of the semi-group. The second requires a deeper scrutiny but can be controlled in terms of the Lipschitz norm of  $h$  only. Then, the optimization of the sum of the two terms with respect to  $\tau$  yields a logarithmic factor as in [2] but not for the same reason at all.

Rough paths theory is essentially a deterministic theory, it is therefore tempting to make the estimate we need in a deterministic setting and then to take the expectation of these bounds. This turns out to be a misleading approach. For instance, consider a sequence of centered, independent identically distributed random variables  $(X_n, n \geq 1)$  and let  $S_n = \sum_{i=1}^n X_i$ . If we evaluate the  $p$ -th moment of  $S_n$  with Hölder inequality, we get that this moment is bounded by a constant times  $m^p$ . But if we use, as in the sequel, the Burkholder-Davis-Gundy inequality for discrete time martingale, we get an upper-bound proportional to  $m^{p/2}$ . This martingale argument which is implicitly used in [14] is the key to our work. For the sake of simplicity, this implies to separate the treatment of the symmetric and anti-symmetric parts of the signature. However, this is of no real importance since, as detailed below, the symmetric part of the signature can be handled as a classical  $\mathbf{R}^d$ -valued process.

This paper is organized as follows: In Section 2, we give the necessary notions about fractional Sobolev spaces and rough-paths theory. We also give a detailed proof of Lamperti's result in the fractional Sobolev spaces scale for further use and comparison. In Section 3, we then define the Kolmogorov-Rubinstein distance and show that this distance between the random walk and the affine interpolation of the Brownian motion can be reduced to a problem in finite dimension, should we consider a special set of Lipschitz functions. In Section 4, we then present our development of the Stein-Dirichlet method to estimate this distance.

## 2. PRELIMINARIES

**2.1. Fractional Sobolev spaces.** As in [6, 12], we consider the fractional Sobolev spaces  $W_{\eta,p}$  defined for  $\eta \in (0, 1)$  and  $p \geq 1$  as the closure of  $\mathcal{C}^1$  functions with respect to the norm

$$|f|_{\eta,p}^p = \int_0^1 |f(t)|^p dt + \iint_{[0,1]^2} \frac{|f(t) - f(s)|^p}{|t - s|^{1+p\eta}} dt ds.$$

For  $\eta = 1$ ,  $W_{1,p}$  is the completion of  $\mathcal{C}^1$  for the norm:

$$|f|_{1,p}^p = \int_0^1 |f(t)|^p dt + \int_0^1 |f'(t)|^p dt.$$

They are known to be Banach spaces and to satisfy the Sobolev embeddings [1, 11]:

$$W_{\eta,p} \subset \text{Hol}(\eta - 1/p) \text{ for } \eta - 1/p > 0$$

and

$$W_{\eta,p} \subset W_{\gamma,q} \text{ for } 1 \geq \eta \geq \gamma \text{ and } \eta - 1/p \geq \gamma - 1/q.$$

As a consequence, since  $W_{1,p}$  is separable (see [4]), so does  $W_{\eta,p}$ .

We also need to introduce the Riemann-Liouville fractional spaces for Lemma 4.3 as we were not able to prove it in the sole framework of fractional Sobolev spaces. For  $f \in L^1([0, 1]; dt)$ , (denoted by  $L^1$  for short) the left and right fractional integrals of  $f$  are defined by :

$$\begin{aligned} (I_{0+}^\gamma f)(x) &= \frac{1}{\Gamma(\gamma)} \int_0^x f(t)(x-t)^{\gamma-1} dt, \quad x \geq 0, \\ (I_{1-}^\gamma f)(x) &= \frac{1}{\Gamma(\gamma)} \int_x^1 f(t)(t-x)^{\gamma-1} dt, \quad x \leq 1, \end{aligned}$$

where  $\gamma > 0$  and  $I_{0+}^0 = I_{1-}^0 = \text{Id}$ . For any  $\gamma \geq 0$ ,  $p, q \geq 1$ , any  $f \in L^p$  and  $g \in L^q$  where  $p^{-1} + q^{-1} \leq \gamma$ , we have :

$$(1) \quad \int_0^1 f(s)(I_{0+}^\gamma g)(s) ds = \int_0^1 (I_{1-}^\gamma f)(s)g(s) ds.$$

The Besov-Liouville space  $I_{0+}^\gamma(L^p) := \mathcal{I}_{\gamma,p}^+$  is usually equipped with the norm :

$$(2) \quad \|I_{0+}^\gamma f\|_{\mathcal{I}_{\gamma,p}^+} = \|f\|_{L^p}.$$

Analogously, the Besov-Liouville space  $I_{1-}^\gamma(L^p) := \mathcal{I}_{\gamma,p}^-$  is usually equipped with the norm :

$$\|I_{1-}^\gamma f\|_{\mathcal{I}_{\gamma,p}^-} = \|f\|_{L^p}.$$

We now define the Besov-Liouville spaces of negative order which turn to be in duality with Besov-Liouville of positive order. Denote by  $\mathcal{D}_+$  the space of  $\mathcal{C}^\infty$  functions defined on  $[0, 1]$  and such that  $\phi^{(k)}(0) = 0$ . Analogously, set  $\mathcal{D}_-$  the space of  $\mathcal{C}^\infty$  functions defined on  $[0, 1]$  and such that  $\phi^{(k)}(1) = 0$ . They are both equipped with the projective topology induced by the seminorms  $p_k(\phi) = \sum_{j \leq k} \|\phi^{(j)}\|_\infty$ . Let  $\mathcal{D}'_+$ , resp.  $\mathcal{D}'_-$ , be their strong topological dual. It is straightforward that  $\mathcal{D}_+$  is stable by  $I_{0+}^\gamma$  and  $\mathcal{D}_-$  is stable  $I_{1-}^\gamma$ , for any  $\gamma \in \mathbf{R}$ . Hence, guided by (1), we can define the fractional integral of any distribution (i.e., an element of  $\mathcal{D}'_-$  or  $\mathcal{D}'_+$ ):

$$\begin{aligned} \text{For } T \in \mathcal{D}'_-; I_{0+}^\gamma T : \phi \in \mathcal{D}_- \mapsto \langle T, I_{1-}^\gamma \phi \rangle_{\mathcal{D}'_-, \mathcal{D}_-}, \\ \text{For } T \in \mathcal{D}'_+; I_{1-}^\gamma T : \phi \in \mathcal{D}_+ \mapsto \langle T, I_{0+}^\gamma \phi \rangle_{\mathcal{D}'_+, \mathcal{D}_+}. \end{aligned}$$

**Definition 2.1.** For  $\gamma > 0$  and  $r > 1$ ,  $\mathcal{I}_{-\gamma,r}^+$  (resp.  $\mathcal{I}_{-\gamma,r}^-$ ) is the space of distributions such that  $I_{0+}^\gamma T$  (resp.  $I_{1-}^\gamma T$ ) belongs to  $L^r$ . The norm of an element  $T$  in this space is the norm of  $I_{0+}^\gamma T$  in  $L^r$  (resp. of  $I_{1-}^\gamma T$ ).

The following theorem has been proved in [6].

**Theorem 2.1.** For  $\gamma > 0$  and  $r > 1$ , the dual space of  $\mathcal{I}_{\gamma,r}^+$  (resp.  $\mathcal{I}_{\gamma,r}^-$ ) is canonically isometrically isomorphic to  $\mathcal{I}_{1-\gamma}^-(L^{r^*})$  (resp.  $\mathcal{I}_{0+}^-(L^{r^*})$ ), where  $r^* = r(r-1)^{-1}$ .

It is proved in [10] that for  $1 \geq a > b > c > 0$  that the following embeddings are continuous (even compact)

$$W_{a,p} \subset \mathcal{I}_{b,p}^+ \subset W_{c,p}.$$

It follows that

$$(3) \quad W_{c,p}^* \subset \mathcal{I}_{-b,p}^- \subset W_{a,p}^*.$$

**2.2. Rough paths.** We give a quick introduction to the rough-paths theory. For details, we refer to the monograph [12]. Consider  $T^2(\mathbf{R}^d)$ , the graded algebra of step two:

$$T^2(\mathbf{R}^d) = \mathbf{R} \oplus \mathbf{R}^d \oplus (\mathbf{R}^d \otimes \mathbf{R}^d).$$

We endow  $T^2(\mathbf{R}^d)$  with an algebra structure  $(+, \cdot, \otimes)$  where for all  $(w_0, w_1, w_2), (z_0, z_1, z_2) \in T^2(\mathbf{R}^d), \lambda \in \mathbf{R}$

$$(w_0, w_1, w_2) + (z_0, z_1, z_2) = (w_0 + z_0, w_1 + z_1, w_2 + z_2)$$

$$\lambda \cdot (w_0, w_1, w_2) = (\lambda w_0, \lambda w_1, \lambda w_2)$$

$$(w_0, w_1, w_2) \otimes (z_0, z_1, z_2) = (w_0 z_0, w_0 z_1 + z_0 w_1, w_0 z_2 + z_0 w_2 + w_1 \otimes z_1).$$

Introduce the projection maps: For  $i = 0, 1, 2$

$$\begin{aligned} \pi_i : T^2(\mathbf{R}^d) &\longrightarrow (\mathbf{R}^d)^{\otimes i} \\ (w_0, w_1, w_2) &\longmapsto w_i, \end{aligned}$$

The set

$$\begin{aligned} 1 + t^2(\mathbf{R}^d) &= \{w \in T^2(\mathbf{R}^d), \pi_0(w) = 1\} \\ &= \left\{ g = (1, w_1, w_2), (w_1, w_2) \in \mathbf{R}^d \oplus (\mathbf{R}^d \otimes \mathbf{R}^d) \right\} \end{aligned}$$

is a Lie group with respect to the tensor multiplication  $\otimes$ , [12, Prop. 7.17]. Note that

$$(1, w_1, w_2)^{-1} = (1, -w_1, -w_2 + w_1 \otimes w_1).$$

As usual, a Lie group, like  $1 + t^2(\mathbf{R}^d)$ , leads to a Lie algebra when equipped with notions of product and commutator. Here, the Lie algebra is  $(t^2(\mathbf{R}^d), +, \cdot)$  with product  $\otimes$  and commutator

$$\begin{aligned} [g, w] &= g \otimes w - w \otimes g = 0 \oplus (\pi_1(g) \otimes \pi_1(w) - \pi_1(w) \otimes \pi_1(g)), \\ &\text{for any } w, g \in t^2(\mathbf{R}^d). \end{aligned}$$

Denote by  $(e_i, 1 \leq i \leq d)$  the canonical basis of  $\mathbf{R}^d$ , so that  $(e_i \otimes e_j, 1 \leq i, j \leq d)$  is the canonical basis of  $\mathbf{R}^d \otimes \mathbf{R}^d$ . Then

$$\begin{aligned} \left[ \sum_{i=1}^d a_i e_i + \sum_{i,j=1}^d c_{i,j} e_i \otimes e_j, \sum_{i=1}^d b_i e_i + \sum_{i,j=1}^d f_{i,j} e_i \otimes e_j \right] \\ = \sum_{i < j} (a_i b_j - a_j b_i) [e_i, e_j]. \end{aligned}$$

The exponential and logarithm maps are useful to go back and forth between  $t^2(\mathbf{R}^d)$  and  $1 + t^2(\mathbf{R}^d)$ :

$$\begin{aligned} \exp : t^2(\mathbf{R}^d) &\longrightarrow 1 + t^2(\mathbf{R}^d) \\ w &\longmapsto 1 + w + \frac{1}{2}(\pi_1 w)^{\otimes 2} \end{aligned}$$

and

$$\begin{aligned} \log : 1 + t^2(\mathbf{R}^d) &\longrightarrow t^2(\mathbf{R}^d) \\ 1 + w &\longmapsto w - \frac{1}{2}(\pi_1 w)^{\otimes 2}. \end{aligned}$$

We denote by  $\Sigma$  the set of finite partitions  $\sigma = \{t_1, \dots, t_n\}$  of  $[0, 1]$ . A continuous path  $z$  from  $[0, 1]$  into  $\mathbf{R}^d$  is said to have 1-finite variation whenever

$$\sup_{\sigma = \{t_1, \dots, t_n\} \in \Sigma} \sum_{i=1}^{n-1} |z_{t_{i+1}} - z_{t_i}| < \infty.$$

The set of such functions equipped with this quantity as a norm is denoted by  $C^{1-\text{var}}$ .

**Definition 2.2.** The step-2 signature of  $z \in C^{1-\text{var}}$  is given by:

$$\begin{aligned} S_2(z) : [0, 1] &\longrightarrow 1 + t^2(\mathbf{R}^d) \\ t &\longmapsto \left( 1, z_t - z_0, \int_0^t (z_s - z_0) \otimes dz_s \right). \end{aligned}$$

The free nilpotent group of order 2,  $G^2(\mathbf{R}^d)$ , is the closed subgroup of  $1 + t^2(\mathbf{R}^d)$  defined by

$$G^2(\mathbf{R}^d) = \{S_2(z), z \in C^{1-\text{var}}\}.$$

We also consider  $\check{G}_2(\mathbf{R}^d)$  ( $\check{G}_2$  for short since  $d$  is fixed), the image of  $G_2(\mathbf{R}^d)$  by the logarithm map. For  $z \in C^{1-\text{var}}$ , this corresponds to consider only the anti-symmetric part of  $\pi_2(S_2(z))$ :

$$\begin{aligned} \check{S}_2(z) : [0, 1] &\longrightarrow t^2(\mathbf{R}^d) \\ t &\longmapsto \left( z_t - z_0, \int_0^t [(z_s - z_0), dz_s] \right). \end{aligned}$$

*Remark 1.* If

$$z(t) = \sum_{i=1}^d \sum_{k=1}^m z_{ik} h_k(t) e_i$$

where  $(h_1, \dots, h_m)$  are elements of  $C^{1-\text{var}}$ , we have

$$(4) \quad \log S_2(z)(t) = \left( z_t - z_0, \sum_{1 \leq i < j \leq d} \sum_{1 \leq k < l \leq m} (z_{ik} z_{jl} - z_{il} z_{jk}) \left( \int_0^t h_k(s) dh_l(s) - \int_0^t h_l(s) dh_k(s) \right) [e_i, e_j] \right).$$

For the sake of notations, we set  $\mathcal{A} = \{1, \dots, d\} \times \{1, \dots, m\}$  and define the  $\prec$  relation by:

$$a = (a_1, a_2) \prec b = (b_1, b_2) \iff (a_1 < b_1) \text{ and } (a_2 < b_2).$$

With these notations, Eqn. (4) then becomes

$$\log S_2(z)(t) = \left( z_t - z_0, \sum_{a \prec b} [z_a, z_b] \left( \int_0^t h_{a_2}(s) dh_{b_2}(s) - \int_0^t h_{b_2}(s) dh_{a_2}(s) \right) [e_{a_1}, e_{b_1}] \right).$$

The group  $G^2(\mathbf{R}^d)$  has the structure of a sub-Riemannian manifold. We will not dwell into the meanders of this very rich but intricate structure. It suffices to say that we can proceed equivalently by considering usual norms as follows.

For  $\alpha \in (0, 1)$ , a path  $w = 1 \oplus w_1 \oplus w_2$  is said to be  $\alpha$ -Hölder whenever

$$\rho_\alpha(w) = \max \left( \sup_{s \neq t} \frac{|w_1(t) - w_1(s)|}{|t - s|^\alpha}, \sup_{s \neq t} \frac{|\pi_2(w(s)^{-1} \otimes w(t))|^{1/2}}{|t - s|^\alpha} \right) < \infty.$$

Note that for  $z \in C^{1-\text{var}}$ ,

$$\rho_\alpha(S_2(z)) = \max \left( \sup_{s \neq t} \frac{|z(t) - z(s)|}{|t - s|^\alpha}, \sup_{s \neq t} \frac{\left| \int_s^t (z_r - z_s) \otimes dz_r \right|^{1/2}}{|t - s|^\alpha} \right).$$

**Definition 2.3.** We denote by  $H_\alpha(t^2(\mathbf{R}^d))$ , the vector space of paths  $w$  from  $[0, 1]$  into  $t^2(\mathbf{R}^d)$  such that  $\rho_\alpha(w)$  is finite. It is equipped with the homogeneous norm: For  $w$  and  $v$  in  $H_\alpha(t^2(\mathbf{R}^d))$

$$|w - v|_{H_\alpha(t^2(\mathbf{R}^d))} = \rho_\alpha(w - v) = \rho_\alpha((w_1 - v_1) \oplus (w_2 - v_2)).$$

Unfortunately, as mentioned in [12, Chapter 8.3], this metric space is complete but not separable, which is unacceptable for our purpose (see Definition 3.1 and the remark below). We thus introduce fractional Sobolev spaces as in [13].

**Definition 2.4.** For any  $\eta \in (0, 1)$ , any  $p \geq 2$ ,  $\check{G}_2 W_{\eta, p}$  is the vector space of paths  $w$  from  $[0, 1]$  into  $t^2(\mathbf{R}^d)$  such that

$$|w_1|_{\eta, p}^p + \iint_{[0, 1]^2} \frac{|\pi_2[w_s^{-1}, w_t]|^{p/2}}{|t - s|^{1+\eta p}} ds dt < \infty.$$

The distance on  $\check{G}_2 W_{\eta, p}$  is defined by

$$|w - v|_{\check{G}_2 W_{\eta, p}} = |\pi_1(w) - \pi_1(v)|_{W_{\eta, p}} + \left( \iint_{[0, 1]^2} \frac{|\pi_2[w_s^{-1}, w_t] - \pi_2[v_s^{-1}, v_t]|^{p/2}}{|t - s|^{1+\eta p}} ds dt \right)^{1/p}.$$

Following [12], we know that  $\check{G}_2 W_{\eta, p}$  is a Banach space included into  $H_\alpha(t^2(\mathbf{R}^d))$  for  $\alpha = \eta - 1/p$ , provided  $\alpha > 0$ .

**Lemma 2.2.** For any  $\eta \in (0, 1)$ , any  $p \geq 2$ ,  $\check{G}_2 W_{\eta, p}$  is separable.



*Proof.* Consider the map  $\kappa$  defined as

$$\begin{aligned} \kappa : ([0, 1] \rightarrow t_2(\mathbf{R}^d)) &\longrightarrow ([0, 1]^2 \rightarrow t_2(\mathbf{R}^d)) \\ w &\longmapsto \left( (s, t) \mapsto \left( \pi_1(w_t) - \pi_1(w_s), \pi_2[w_s^{-1}, w_t] \right) \right). \end{aligned}$$

Consider the measure  $d\mu_{\eta,p}(s, t) = |t - s|^{-1-\eta p} ds dt$ . Then,

$$(5) \quad |w|_{\check{G}_2 W_{\eta,p}} = |\pi_1 \circ \kappa(w)|_{L^p(\mu_{\eta,p})} + |\pi_2 \circ \kappa(w)|_{L^{p/2}(\mu_{2\eta,p/2})}^{1/2}.$$

For any  $p \geq 1$  and  $\eta \in (0, 1)$ ,  $L^p(\mu_{\eta,p})$  is isometrically isomorphic to  $L^p(ds \otimes dt)$  hence it is separable. This entails that  $E = L^p(\mu_{\eta,p}) \times L^{p/2}(\mu_{2\eta,p/2})$  is separable. Equation (5) means that the application  $T$  which maps  $w \in \check{G}_2 W_{\eta,p}$  to the couple  $(\pi_1 \circ \kappa(w), \pi_2 \circ \kappa(w)) \in E$ , is an isometry. Thus  $\check{G}_2 W_{\eta,p}$  is isometrically isomorphic to a closed subspace of the separable space  $E$ , hence it is separable.  $\square$

**2.3. Donsker-Lamperti theorem.** For the sake of completeness and for further comparison, we give the proof of the Donsker-Lamperti theorem in the scale of fractional Sobolev spaces, which induces the convergence in Hölder spaces.

**Definition 2.5.** The random walk associated to the sequence  $(X_k, k \geq 1)$  is defined by

$$X^m(t) = \sqrt{m} \sum_{k=1}^m X_k r_k^m(t) = \sum_{k=1}^m X_k h_k^m(t)$$

where

$$(6) \quad r_k^m(t) = \int_0^t \mathbf{1}_{((k-1)/m, k/m]}(s) ds \text{ and } h_k^m = \sqrt{m} r_k^m.$$

**Theorem 2.3.** *If for any  $k \geq 1$ ,  $X_k$  belongs to  $L^p$  for some  $p \geq 2$ , then there exists  $c > 0$  such that*

$$(7) \quad \sup_{m \geq 1} \frac{\mathbf{E} \left[ \left| \sum_{k=1}^m X_k \sqrt{m} (r_k^m(t) - r_k^m(s)) \right|^p \right]}{|t - s|^{p/2}} < c \mathbf{E} [|X_1|^p]$$

*Proof.* For  $0 \leq s < t \leq 1$  fixed, the discrete time process

$$Y^{st} : m \longmapsto Y_m^{st} = \sum_{k=1}^m X_k (r_k^m(t) - r_k^m(s))$$

is a martingale with respect to the filtration  $\mathcal{F}_n = \sigma(X_k, 1 \leq k \leq n)$ . The Burkholder-Davis-Gundy [18] entails that

$$\begin{aligned} m^p \mathbf{E} \left[ \left| \sum_{k=1}^m X_k (r_k^m(t) - r_k^m(s)) \right|^p \right] \\ \leq c m^{p/2} \mathbf{E} \left[ \left| \sum_{k=1}^m X_k^2 (r_k^m(t) - r_k^m(s)) \right|^{p/2} \right]. \end{aligned}$$

If  $|t - s| \leq 1/m$ , there is at most two values of  $k$  such that  $r_k^m(t) - r_k^m(s)$  is not zero. Furthermore,

$$|r_k^m(t) - r_k^m(s)| \leq |t - s|$$

hence

$$|r_k^m(t) - r_k^m(s)|^2 \leq m^{-1}|t - s|.$$

In this situation,

$$\mathbf{E} \left[ \left| \sum_{k=1}^m X_k^2 \left( r_k^m(t) - r_k^m(s) \right)^2 \right|^{p/2} \right] \leq c m^{-p/2} \mathbf{E} [|X_1|^p] |t - s|^{p/2},$$

so that (7) holds true for  $|t - s| \leq 1/m$ . For  $|t - s| > 1/m$ , we remark that  $r_k^m(t) - r_k^m(s)$  is not null for at most  $[m(t - s)] + 2$  values of  $k$  and since  $r_k^m$  is Lipschitz continuous,  $|r_k^m(t) - r_k^m(s)| \leq 1/m$  for such value of  $k$ . Hence, by convexity inequality,

$$\begin{aligned} \mathbf{E} \left[ \left| \sum_{k=1}^m X_k^2 \left( r_k^m(t) - r_k^m(s) \right)^2 \right|^{p/2} \right] &\leq c (m|t - s| + 2)^{p/2} m^{-p} \mathbf{E} [|X_1|^p] \\ &\leq c m^{-p/2} \mathbf{E} [|X_1|^p] |t - s|^{p/2}. \end{aligned}$$

Hence, (7) is true for  $|t - s| \geq 1/m$ .  $\square$

It is then straightforward that we have:

**Corollary 2.4.** *Assume that for any  $k \geq 1$ ,  $X_k$  belongs to  $L^p$  for some  $p \geq 2$ . Then, for any  $\eta < 1/2$ ,*

$$\sup_{m \geq 1} \mathbf{E} \left[ \|X^m\|_{W_{\eta,p}}^p \right] < \infty.$$

*Proof.* Actually,  $(s, t) \mapsto |t - s|^{p/2}$  is  $\mu_{\eta,p}$ -integrable provided that  $p(1/2 - \eta) > 0$ , i.e.  $\eta < 1/2$ .  $\square$

**Corollary 2.5** (Lamperti). *Assume that for any  $k \geq 1$ ,  $X_k$  belongs to  $L^p$  for some  $p \geq 2$ . Then, for any  $1/p < \eta < 1/2$ , the sequence  $(X^m, m \geq 1)$  converges in distribution in  $\text{Hol}(\eta - 1/p)$  to  $B$ .*

*Proof.* It is well-known that the finite dimensional distributions of  $X^m$  converge to that of  $B$ . From the previous lemma, we know that for any  $0 < \eta < \zeta < 1/2$ , there exists  $K$  such that

$$\sup_{m \geq 1} \mathbf{P}(\|X^m\|_{W_{\zeta,p}} \geq K) \leq \eta.$$

The embedding of  $W_{\zeta,p}$  into  $W_{\eta,p}$  is compact, hence the sequence  $(X^m, m \geq 1)$  is tight in  $W_{\eta,p}$  hence convergent. The result follows by the continuous embedding of  $W_{\eta,p}$  into  $\text{Hol}(\eta - 1/p)$ .  $\square$

**2.4. Gelfand triplet.** The construction of the Gaussian measure on a Banach space is a delicate question, we refer to [17, 19] for details. For  $H$  a Hilbert space, a cylindrical set is a set of the form

$$Z = \{x \in H, (\langle x, h_1 \rangle_H, \dots, \langle x, h_n \rangle) \in B\}$$

for some integer  $n$ , where  $(h_1, \dots, h_n)$  is an orthonormal family of  $H$  and  $B$  a Borelean subset of  $\mathbf{R}^n$ . Let  $m_n$  be the standard Gaussian measure on  $\mathbf{R}^n$ . By setting,  $\mu_H(Z) = m_n(B)$ , we get a cylindrical standard Gaussian measure on  $H$ . A radonifying operator is a linear map which transforms a cylindrical measure into a true regular Radon measure. We will not dwell into the details of this theory, it suffices to say that an Hilbert-Schmidt map between two Hilbert spaces is automatically radonifying (see [19, Exposé 8]).

**Lemma 2.6.** *For any  $\eta < 1/2$ , for any  $p > 1/\eta$ , the embedding*

$$W_{1,2} \xrightarrow{\iota_{\eta,p}} W_{\eta,p} \subset \text{Hol}(\eta - 1/p)$$

*is radonifying for any  $\epsilon > 0$ .*

*Proof.* It is well known (see [19]) that, for any  $\eta' < 1/2$ , the embedding

$$W_{1,2} \longrightarrow W_{\eta',2}$$

is Hilbert-Schmidt hence radonifying. For  $p > 2$ , as mentioned in section 2.1, the space  $W_{\eta',2}$  is continuously embedded in  $W_{\eta,p}$  for any  $\eta < \eta'$ . Since the composition of a radonifying map with a continuous map is always radonifying, the result follows.  $\square$

**Definition 2.6.** An abstract Wiener space is a triple  $(\iota, H, W)$  where  $H$  is a separable Hilbert space,  $W$  a Banach space and  $\iota$  the embedding from  $H$  into  $W$  which has to be to be radonifying.

It is usual to identify  $H$  and its dual so that we have the following diagram:

$$W^* \xrightarrow{\iota^*} H^* \simeq H \xrightarrow{\iota} W.$$

Note that  $\iota^*$  is not the identity since we have identified  $H^*$  with  $H$ . As a direct consequence of Lemma [2.6], we have

**Theorem 2.7.** *The triple  $(\iota_{\eta,p}, W_{1,2}, W_{\eta,p})$  is an abstract Wiener space, for any  $\eta \in [0, 1/2)$ .*

The Wiener measure is defined by its characteristic function: For all  $\eta \in W_{\eta,p}^*$ ,

$$\int_{W_{\eta,p}} e^{i \langle \eta, y \rangle_{W_{\eta,p}^*, W_{\eta,p}}} d\mathbf{P}(y) = \exp\left(-\frac{1}{2} \|\iota_{\eta,p}^*(\eta)\|_{W_{1,2}}^2\right).$$

This means that for any  $\eta \in W_{\eta,p}^*$ , the random variable  $\langle \eta, y \rangle_{W_{\eta,p}^*, W_{\eta,p}}$  is a centered Gaussian random variable with variance given by

$$\int_{W_{\eta,p}} \langle \eta, w \rangle_{W_{\eta,p}^*, W_{\eta,p}}^2 d\mathbf{P}(y) = \|\iota_{\eta,p}^*(\eta)\|_{W_{1,2}}^2.$$

By construction,  $\iota_{\eta,p}^*(W_{\eta,p}^*)$  is dense in  $W_{1,2}$  so that we can define the Wiener integral as follows.

**Definition 2.7** (Wiener integral). The Wiener integral, denoted as  $\delta$ , is the isometric extension of the map

$$\begin{aligned} \delta : \iota_{\eta,p}^*(W_{\eta,p}^*) &\subset W_{1,2} \longrightarrow L^2(\mathbf{P}) \\ \iota_{\eta,p}^*(\eta) &\longmapsto \langle \eta, y \rangle_{W_{\eta,p}^*, W_{\eta,p}}. \end{aligned}$$

Remark that for  $\eta \in W_{\eta,p}^*$ , for any  $y \in W_{\eta,p}$ ,

$$(8) \quad \left| \delta(\iota_{\eta,p}^*(\eta))(y) \right| \leq \|\eta\|_{W_{\eta,p}^*} \|y\|_{W_{\eta,p}}.$$

**Definition 2.8.** A function  $F$  from  $W_{\eta,p}$  into  $\mathbf{R}$  is Lipschitz whenever for any  $x$  and  $y$  in  $W_{\eta,p}$ ,

$$|F(x) - F(y)| \leq \|x - y\|_{W_{\eta,p}}.$$

The set of such functions is denoted by  $\text{Lip}(d_{W_{\eta,p}})$ .

**Definition 2.9** (Ornstein-Uhlenbeck semi-group). For any bounded function on  $W_{\eta,p}$ , for any  $\tau \geq 0$ ,

$$P_\tau F(x) = \int_{W_{\eta,p}} F(e^{-\tau}x + \beta_\tau y) \, d\mathbf{P}(y)$$

where  $\beta_\tau = \sqrt{1 - e^{-2\tau}}$ .

The dominated convergence theorem entails that  $P_\tau$  is ergodic: For any  $x \in W_{\eta,p}$ ,

$$P_\tau F(x) \xrightarrow{\tau \rightarrow \infty} \int_{W_{\eta,p}} F \, d\mathbf{P}.$$

Moreover, the invariance by rotation of Gaussian measures implies that

$$\int_{W_{\eta,p}} P_\tau F(x) \, d\mathbf{P}(x) = \int_{W_{\eta,p}} F \, d\mathbf{P}, \text{ for any } \tau \geq 0.$$

Otherwise stated, the Gaussian measure on  $W_{\eta,p}$  is the invariant and stationary measure of the semi-group  $P = (P_\tau, \tau \geq 0)$ . For details on the Malliavin gradient, we refer to [16, 21].

**Definition 2.10.** Let  $X$  be a Banach space. A function  $F : W_{\eta,p} \rightarrow X$  is said to be cylindrical if it is of the form

$$F(y) = \sum_{j=1}^k f_j(\delta h_1(y), \dots, \delta h_k(y)) x_j$$

where for any  $j \in \{1, \dots, k\}$ ,  $f_j$  belongs to the Schwartz space on  $\mathbf{R}^k$ ,  $(h_1, \dots, h_k)$  are elements of  $W_{1,2}$  and  $(x_1, \dots, x_k)$  belong to  $X$ .

For  $h \in W_{1,2}$ ,

$$\langle \nabla F, h \rangle_{W_{1,2}} = \sum_{j=1}^k \partial_j f(\delta h_1(y), \dots, \delta h_k(y)) \langle h_j, h \rangle_{W_{1,2}} x_j,$$

which is equivalent to say

$$\nabla F = \sum_{j=1}^k \partial_j f(\delta h_1(y), \dots, \delta h_k(y)) h_{\otimes} x_j.$$

The space  $\mathbb{D}_{1,2}$  is the closure of the space of cylindrical functions with respect to the norm

$$\|F\|_{1,2}^2 = \|F\|_{L^2(\mathbf{P};X)}^2 + \|\nabla F\|_{L^2(\mathbf{P};W_{1,2}\otimes X)}^2.$$

By induction, higher order gradients are defined similarly. For any  $k \geq 1$ , the norm on the space  $\mathbb{D}_{k,2}$  is given by

$$\|F\|_{k,2}^2 = \|F\|_{L^2(\mathbf{P};X)}^2 + \sum_{j=1}^k \|\nabla^{(j)} F\|_{L^2(\mathbf{P};W_{1,2}^{\otimes j}\otimes X)}^2.$$

According to [20], we have the following properties of Lipschitz continuous functions on  $W_{\eta,p}$ .

**Proposition 1.** *Let  $F \in \text{Lip}(d_{W_{\eta,p}})$  and  $x, y \in W_{\eta,p}$ . For any  $\tau > 0$ ,  $P_\tau F(x)$  belongs to  $\mathbb{D}_{k,2}$  for any  $k \geq 1$ . Furthermore,  $\nabla P_\tau F(x)$  belongs to  $(W_{\eta,p})^*$  and*

$$(9) \quad \langle \nabla P_\tau F(x), h \rangle_{W_{1,2}} = \frac{e^{-t}}{\beta_t} \int_{W_{\eta,p}} F(e^{-t}x + \beta_t y) \delta h(y) \, d\mathbf{P}(y).$$

Moreover, the operator  $\nabla^{(2)} P_\tau F(x)$  is trace-class and

$$(10) \quad \begin{aligned} & \left\langle \nabla^{(2)} P_\tau F(x), h \otimes k \right\rangle_{W_{1,2}^{\otimes 2}} \\ &= \frac{e^{-3t/2}}{\beta_{t/2}^2} \int_{W_{\eta,p}^2} F(e^{-t}x + e^{-t/2}\beta_{t/2} y + \beta_{t/2} z) \delta h(y) \delta k(z) \, d\mathbf{P}(y) \, d\mathbf{P}(z). \end{aligned}$$

Let  $L$  be the formal operator defined by

$$LG(x) = -\langle x, \nabla G(x) \rangle_{W_{\eta,p}, W_{\eta,p}^*} + \text{trace}_{W_{1,2}}(\nabla^{(2)} G(x)).$$

Then, for any  $\tau > 0$ ,  $P_\tau F$  belongs to the domain of  $L$  and

$$\frac{d}{d\tau} P_\tau F(x) = LP_\tau F(x).$$

### 3. RATE OF CONVERGENCE

**3.1. Kolmogorov-Rubinstein distance.** In [14], the proof of Lamperti's Theorem is given for one dimensional processes but it can be straightforwardly adapted to  $\mathbf{R}^d$ -valued random walks and Brownian motion:  $X^{\mathcal{A}}$  becomes the  $\mathbf{R}^d$ -valued process

$$X^{\mathcal{A}}(t) = \sqrt{m} \sum_{a \in \mathcal{A}} X_a r_{a_2}(t) e_{a_1} = \sum_{a \in \mathcal{A}} X_a h_a(t)$$

where  $h_a(t) = \sqrt{m} r_{a_2}(t) e_{a_1}$  and  $(X_a, a \in \mathcal{A})$  is a family of independent identically distributed random variables of mean 0 and variance 1. Furthermore,  $B$  is the  $d$ -dimensional Brownian motion:

$$B(t) = \sum_{i=1}^d B_i(t) e_i.$$

The enriched Brownian motion  $\mathbb{B}$ , is the  $G_2(\mathbf{R}^d)$ -value process defined by

$$\mathbb{B}(t) = 1 \oplus B(t) \oplus \sum_{i,j=1}^d \int_0^t B_i(s) \circ dB_j(s) e_i \otimes e_j, \text{ for any } t \in [0, 1],$$

where the stochastic integrals are to be understood in the Stratonovitch sense. Theorem 13.32 of [12] says that  $S_2(X^{\mathcal{A}})$  converges to  $\mathbb{B}$  in some Hölder type spaces. Our primary goal is to give the rate of this convergence. For, we need to define a distance between probability measures over Hölder spaces. There are several possibilities of such a definition, the best suited for an estimate by the Stein method is the Kolmogorov-Rubinstein<sup>1</sup> distance:

**Definition 3.1** (Kolmogorov-Rubinstein distance). For  $\mu$  and  $\nu$  two probability measures on a metric space  $(W, d_W)$ , their Kolmogorov-Rubinstein distance is given by

$$\text{dist}_{\text{KR}}(\mu, \nu) = \sup_{F \in \text{Lip}(d_W)} \int_W F d\mu - \int_W F d\nu,$$

where

$$\text{Lip}(d_W) = \left\{ F : W \rightarrow \mathbf{R}, \forall w, v \in W, |F(w) - F(v)| \leq d_W(w, v) \right\}.$$

Theorem 11.3.3 of [9] states that the topology induced by this distance on the set of probability measures on  $W$  is the same as the topology of the convergence in law whenever the metric space  $W$  is separable. Unfortunately, as we already mentioned, Hölder spaces are not separable, thus to have a meaningful result, we turn to work on fractional Sobolev spaces. It is of no importance since Sobolev embeddings ensure that convergence in fractional Sobolev spaces induces convergence in Hölder spaces.

Our new goal is then to estimate the Kolmogorov-Rubinstein distance in  $G_2W_{\eta,p}$  between  $\mathbb{B}$  and  $S_2(X^{\mathcal{A}})$ . Remark that

$$\begin{aligned} \pi_2 S_2(X^{\mathcal{A}}) &= \pi_2 \check{S}_2(X^{\mathcal{A}}) \\ &+ 2 \sum_{a \in \mathcal{A}} (X_a^2 - 1) \int_0^t h_a^m(s) \otimes dh_a^m(s) + \sum_{a \in \mathcal{A}} h_a^m(t) \otimes h_a^m(t) \\ &= U_1^m + U_2^m + U_3^m. \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathbb{B}(t) &= \sum_{1 \leq i < j \leq d} \left( \int_0^t B_i(s) \circ dB_j(s) - \int_0^t B_j(s) \circ dB_i(s) \right) [e_i, e_j] \\ &+ 2 \sum_{i=1}^d \int_0^t B_i(s) dB_i(s) e_i \otimes e_i + \sum_{i=1}^d t e_i \\ &= \log \mathbb{B}(t) + U_2 + U_3. \end{aligned}$$

where in  $U_2$ , the stochastic integral is taken in the Itô sense. Direct computations show that  $U_3^m - U_3$  tends to 0 as  $1/m$ . The sequence  $U_2^m$  converges to  $U_2$  as fast as  $X^{\mathcal{A}}$  converges to  $B$ ; a rate which is expected and which will turn

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<sup>1</sup>We stick to the denomination suggested in [22] even if this distance is often called the Wasserstein distance.

out to be much slower than  $1/m$ . In summary, the Kolmogorov-Rubinstein distance between  $\mathbb{B}$  and  $S_2(X^{\mathcal{A}})$  has the same asymptotic behavior as the distance between  $\log \mathbb{B}$  and  $\check{S}_2(X^{\mathcal{A}})$ . Our final objective is then to estimate the Kolmogorov-Rubinstein distance between the distributions of  $\log \mathbb{B}$  and  $\check{S}_2(X^{\mathcal{A}})$  in  $\check{G}_2W_{\eta,p}$ .

**3.2. Reduction to finite dimension.** Should we follow the same procedure as the one we used in [5], we would face the same complications to compute the trace term in some infinite dimensional space. We remark that  $X^{\mathcal{A}}$  belongs to the finite dimensional space

$$V = \text{span}\{h_a^m, a \in \mathcal{A}\} \subset W_{1,2}.$$

Consider  $(g_n, n \geq 1)$  a complete orthonormal basis of  $V^\perp$  in  $W_{1,2}$ . The Itô-Nisio Theorem says that  $B$  can be represented as the  $W_{\eta,p}$ -convergent sum

$$B = \sum_{a \in \mathcal{A}} \delta h_a^m h_a^m + \sum_{n=1}^{\infty} \delta g_n g_n = B_V + B_V^\perp,$$

where  $\delta h$  is the Malliavin divergence (or Wiener integral) associated to  $B$  (see [16]). It turns out that  $B_V$  is also the affine interpolation of  $B$  so that we may use the results of [12] to estimate the distance between  $\check{S}_2(B_V)$  and  $\log \mathbb{B}$ . We can then resort to the Stein's method in finite dimension to estimate only the distance between  $\check{S}_2(X^{\mathcal{A}})$  and  $\check{S}_2(B_V)$ . We can always write

$$\begin{aligned} & \sup_{F \in \text{Lip}(d_{\check{G}_2W_{\eta,p}})} \mathbf{E} [F(\log \mathbb{B})] - \mathbf{E} [F(\check{S}_2(X^{\mathcal{A}}))] \\ & \leq \sup_{F \in \text{Lip}(d_{\check{G}_2W_{\eta,p}})} \mathbf{E} [F(\log \mathbb{B})] - \mathbf{E} [F(\check{S}_2(B_V))] \\ & \quad + \sup_{F \in \text{Lip}(d_{\check{G}_2W_{\eta,p}})} \mathbf{E} [F(\check{S}_2(B_V))] - \mathbf{E} [F(\check{S}_2(X^{\mathcal{A}}))]. \end{aligned}$$

On the one hand, since  $\log \mathbb{B}$  and  $\check{S}_2(B_V)$  live on the same probability space, for  $F \in \text{Lip}(d_{\check{G}_2W_{\eta,p}})$ , according to [12, Proposition 13.20],

$$(11) \quad \mathbf{E} [F(\log \mathbb{B})] - \mathbf{E} [F(\check{S}_2(B_V))] \leq \mathbf{E} \left[ \|\log \mathbb{B} - \check{S}_2(B_V)\|_{\check{G}_2W_{\eta,p}} \right] \leq c m^{-(1/2-\eta)}.$$

It remains to estimate

$$\sup_{F \in \text{Lip}(d_{\check{G}_2W_{\eta,p}})} \mathbf{E} [F(\check{S}_2(B_V))] - \mathbf{E} [F(\check{S}_2(X^{\mathcal{A}}))].$$

Section 4 is devoted to prove

**Theorem 3.1.** *If  $p \geq 3$  and  $X_a$  belongs to  $L^p$ , for any  $\eta \in (1/p, 1/2)$ ,*

$$\sup_{F \in \text{Lip}(\check{G}_2W_{\eta,p})} \mathbf{E} [F(\check{S}_2(B_V))] - \mathbf{E} [F(\check{S}_2(X^{\mathcal{A}}))] \leq c \|X_a\|_{L^p}^3 m^{-(1/2-\eta)}.$$

*Remark 2.* Combine (11) and Theorem 3.1 means that the convergence in the Donsker theorem in  $\check{G}_2W_{\eta,p}$  is of the order  $m^{-(1/2-\eta)}$ . Note that the integrability of the  $X_a$ 's does enlarge the spaces in which the convergence holds, as in the Lamperti Theorem, but it does not modify the rate of convergence.

We now detail the proofs of the main estimates.

#### 4. STEIN METHOD

For  $F \in \text{Lip}(d_{\check{G}_2W_{\eta,p}})$ , we have to estimate

$$\sup_{F \in \text{Lip}(d_{\check{G}_2W_{\eta,p}})} \mathbf{E} [F(\check{S}_2(B_V))] - \mathbf{E} [F(\check{S}_2(X^{\mathcal{A}}))].$$

Recall that

$$X^{\mathcal{A}} = \sum_{a \in \mathcal{A}} X_a h_a^m, \quad B_V = \sum_{a \in \mathcal{A}} \delta h_a^m h_a^m \text{ and } V = \text{span}\{h_a^m, a \in \mathcal{A}\}.$$

Let  $W$  (respectively  $H$ ) be the space  $V$  equipped with the norm of  $W_{\eta,p}(\mathbf{R}^d)$  (respectively of  $W_{1,2}(\mathbf{R}^d)$ ). Since  $V$  is finite dimensional, the difference between  $W$  and  $H$  is tenuous but still of some importance to clarify the situation. We have the following situation

$$\begin{array}{c} W_{\eta,p}^* \subset W^* \xrightarrow{\iota_{\eta,p}^*} H \subset W_{1,2} \\ \downarrow \iota_{\eta,p} \\ W \subset W_{\eta,p} \xrightarrow{\check{S}_2} \check{G}_2W_{\eta,p} \xrightarrow{F} \mathbf{R}. \end{array}$$

For the sake of notations, we set  $\check{F} = F \circ \check{S}_2$ . Note that for  $F \in \text{Lip}(d_{\check{G}_2W_{\eta,p}})$ , for any  $x, y \in W_{\eta,p}$ ,

$$|\check{F}(x) - \check{F}(y)| \leq \|\check{S}_2(x) - \check{S}_2(y)\|_{\check{G}_2W_{\eta,p}}.$$

The Stein-Dirichlet representation formula (see [7]) then stands that

$$\begin{aligned} (12) \quad \mathbf{E} [\check{F}(B_V)] - \mathbf{E} [\check{F}(X^{\mathcal{A}})] &= \mathbf{E} \left[ \int_0^\infty LP_\tau \check{F}(X^{\mathcal{A}}) d\tau \right] \\ &= \mathbf{E} [P_\tau \check{F}(X^{\mathcal{A}}) - \check{F}(X^{\mathcal{A}})] + \mathbf{E} \left[ \int_\tau^\infty LP_\tau \check{F}(X^{\mathcal{A}}) d\tau \right], \end{aligned}$$

for any  $\tau > 0$ .

**4.1. Technical lemmas.** *In what follows,  $c$  is a constant which may vary from line to line.*

The continuity of the semi-group is sufficient to upper-bound the first term of the previous decomposition (12).

**Lemma 4.1.** *There exists  $c > 0$  such that for any  $\tau > 0$ , for any  $F \in \text{Lip}(d_{\check{G}_2W_{\eta,p}})$*

$$|\mathbf{E} [P_\tau \check{F}(X^{\mathcal{A}}) - \check{F}(X^{\mathcal{A}})]| \leq c \beta_\tau.$$



*Proof.* We have

$$\begin{aligned}
& |\mathbf{E} [P_\tau \check{F}(X^\mathcal{A}) - \check{F}(X^\mathcal{A})]| \\
& \leq \mathbf{E} \left[ \int \left| \check{F}(e^{-\tau} X^\mathcal{A} + \beta_\tau y) - \check{F}(X^\mathcal{A}) \right| d\mathbf{P}(y) \right] \\
& \leq \mathbf{E} \left[ \left\| (1 - e^{-\tau}) X^\mathcal{A} + \beta_\tau y \right\|_{W_{\eta,p}} d\mathbf{P}(y) \right] \\
& \quad + \mathbf{E} \left[ \int \left( \iint_{[0,1]^2} U(X^\mathcal{A})_{s,t} d\mu_{\eta,p}(s,t) \right)^{1/p} d\mathbf{P}(y) \right],
\end{aligned}$$

where

$$U(X^\mathcal{A})_{s,t} = \left| \pi_2 \left( \check{S}_2(\beta_\tau^2 X^\mathcal{A} + \beta_\tau y)_{s,t} - \check{S}_2(X^\mathcal{A})_{s,t} \right) \right|^p.$$

Thus,

$$\begin{aligned}
& |\mathbf{E} [P_\tau \check{F}(X^\mathcal{A}) - \check{F}(X^\mathcal{A})]| \\
& \leq c \beta_\tau \left( \|X^\mathcal{A}\|_{W_{\eta,p}} + \int \|y\|_{W_{\eta,p}} d\mathbf{P}^m(y) \right) \leq c \beta_\tau,
\end{aligned}$$

according to Corollary 2.4.  $\square$

As to the control of the second term of (12), we need several technical lemmas. With the notations of Proposition 1, we have:

**Lemma 4.2.** *For any  $F \in \text{Lip}(d_{W_{\eta,p}})$ ,*

$$\begin{aligned}
& \mathbf{E} [LP_t F(X^\mathcal{A})] \\
& = \sum_{a \in \mathcal{A}} \int_0^1 \mathbf{E} \left[ X_a^2 \left\langle \nabla^{(2)} P_t F(X_{-a}^\mathcal{A}) - \nabla^{(2)} P_t F(X_{-a}^\mathcal{A} + r X_a h_a^m), \right. \right. \\
& \quad \left. \left. h_a^m \otimes h_a^m \right\rangle_{W_{1,2}} \right] dr \\
& \quad + \sum_{a \in \mathcal{A}} \mathbf{E} \left[ \left\langle \nabla^{(2)} P_t F(X^\mathcal{A}) - \nabla^{(2)} P_t F(X_{-a}^\mathcal{A}), h_a^m \otimes h_a^m \right\rangle_{W_{1,2}} \right].
\end{aligned}$$

*Proof.* Recall that

$$\begin{aligned}
LP_t F(X^\mathcal{A}) & = - \langle X^\mathcal{A}, \nabla P_t F(X^\mathcal{A}) \rangle_{W_{1,2}} \\
& \quad + \sum_{a \in \mathcal{A}} \left\langle h_a^m \otimes h_a^m, \nabla^{(2)} P_t F(X^\mathcal{A}) \right\rangle_{W_{1,2} \otimes W_{1,2}}.
\end{aligned}$$

By independence,

$$\begin{aligned}
& \mathbf{E} \left[ \langle X^\mathcal{A}, \nabla P_t F(X^\mathcal{A}) \rangle_{W_{1,2}} \right] \\
& = \sum_{a \in \mathcal{A}} \mathbf{E} \left[ X_a \langle h_a^m, \nabla P_t F(X^\mathcal{A}) - \nabla P_t F(X_{-a}^\mathcal{A}) \rangle_{W_{1,2}} \right].
\end{aligned}$$

The fundamental theorem of calculus now states that

$$\begin{aligned} \mathbf{E} \left[ \langle X^{\mathcal{A}}, \nabla P_t F(X^{\mathcal{A}}) \rangle_{W_{1,2}} \right] \\ = \sum_{a \in \mathcal{A}} \int_0^1 \mathbf{E} \left[ X_a^2 \left\langle h_a^m \otimes h_a^m, \nabla^{(2)} P_t F(X_{\leftarrow a}^{\mathcal{A}} + r X_a h_a^m) \right\rangle_{W_{1,2}} \right] dr. \end{aligned}$$

Since  $\mathbf{E} [X_a^2] = 1$ , the result follows by successive cancelations.  $\square$

**Lemma 4.3.** *There exists  $c > 0$  such that for any  $m$ , for any  $0 \leq k \leq m-1$ , we have*

$$(13) \quad \|h_k^m\|_{W_{\eta,p}} \leq \frac{c}{m^{1/2-\eta}},$$

$$(14) \quad \|h_k^m\|_{W_{\eta,p}^*} \leq \frac{c}{\sqrt{m}}.$$

*Proof.* Recall that  $h_k^m$  has been defined in (6). Eqn. (13) easily follows from the definition of the  $W_{\eta,p}$  norm as soon as we remark that for any  $s, t \in [0, 1]$ ,

$$|h_k^m(t) - h_k^m(s)| \leq |t - s| \wedge \frac{1}{m}.$$

As to Eqn. (14), in view of (3), it is sufficient to compute

$$\begin{aligned} \|h_k^m\|_{\mathcal{I}_{\eta',p}^*} &= \|h_k^m\|_{\mathcal{I}_{-\eta',q}} = \sqrt{m} \|I_{0+}^{1+\eta'}(\mathbf{1}_{[k/m, (k+1)/m)})\|_{L^{p^*}} \\ &= \frac{\sqrt{m}}{\Gamma(1+\eta')} \left( \int_0^1 \left( \int_{i/m}^{t \wedge (i+1)/m} (t-s)_+^{\eta'} ds \right)^{p^*} dt \right)^{1/p^*}, \end{aligned}$$

for some  $\eta' > \eta$ , where  $p^*$  is the conjugate of  $p$ . Since  $\eta' > 0$ ,  $|t-s|^{\eta'} \leq 1$  and the result follows.  $\square$

Introduce for any  $a \prec b \in \mathcal{A}$  and any  $0 \leq s < t \leq 1$ ,

$$\begin{aligned} &h_{a,b}^m(s, t) \\ &= \left\{ \int_s^t \left( h_{a_2}^m(r) - h_{a_2}^m(s) \right) dh_{b_2}^m(r) - \int_s^t \left( h_{b_2}^m(r) - h_{b_2}^m(s) \right) dh_{a_2}^m(r) \right\} [e_{a_1}, e_{b_1}]. \end{aligned}$$

**Lemma 4.4.** *Let  $(U_a, a \in \mathcal{A})$  be a family of independent identically distributed random variables which belong to  $L^p$ . Then,*

$$\mathbf{E} \left[ \left| U_a \sum_{b \prec a} U_b h_{a,b}^m \right|_{W_{\eta,p}}^p \right] \leq c m^{-(1/2-\eta)} \mathbf{E} [|U_a|^p]^2.$$

*Proof.* By independence and as in the proof of Theorem 2.3,

$$\mathbf{E} \left[ \left| U_a \sum_{b \prec a} U_b h_{a,b}^m(s, t) \right|^p \right] \leq c \mathbf{E} [|U_a|^p] \mathbf{E} \left[ \left( \sum_{b \prec a} |U_b|^2 |h_{a,b}^m(s, t)|^2 \right)^{p/2} \right].$$

Since  $a$  is fixed and  $b \prec a$ , if  $|t-s| \leq 1/m$ ,  $h_{a,b}^m$  is not zero only for  $b_2 = a_2 - 1$  and then, it is bounded by  $m|t-s|^2$ . If  $|t-s| \geq 1/m$ ,  $h_{a,b}^m$  is not zero for at

most  $\lfloor dm|t-s| \rfloor$  values of  $b$  and then, each term is bounded by  $1/m$ . Hence

$$\mathbf{E} \left[ \left( \sum_{b \prec a} |U_b|^2 |h_{a,b}^m(s, t)|^2 \right)^{p/2} \right] \leq c m^{-p/2} (m|t-s|)^{p/2} \mathbf{E} [|U_b|^p].$$

The result follows by integration with respect to  $\mu_{\eta, p}$ .  $\square$

**4.2. Proof of the main theorem.** We are now in position to prove the main theorem. Actually, the proof in itself, is not very different from the proof of the CLT in finite dimension but for the functional apparatus. The last step is complexified by the computations of some  $\tilde{G}_2 W_{\eta, p}$  norms.

*Proof of Theorem 3.1.* For  $t \geq 0$ ,  $r \in [0, 1]$ ,  $y, z \in W_{\eta, p}$ , let

$$(15) \quad X_{-a}^{\mathcal{A}}(t, r, y, z) = e^{-t} (X_{-a}^{\mathcal{A}} + r X_a h_a^m) + e^{t/2} \beta_{t/2} y + \beta_{t/2}$$

Lemma 4.2 and Proposition 1 imply that

$$\begin{aligned} & \mathbf{E} [LP_t F(X^{\mathcal{A}})] \\ &= \frac{e^{-3t/2}}{\beta_{t/2}^2} \sum_{a \in \mathcal{A}} \int_0^1 \mathbf{E} \left[ \int_{W_{\eta, p}^2} \left( F(X_{-a}^{\mathcal{A}}(t, 0, y, z)) - F(X_{-a}^{\mathcal{A}}(t, r, y, z)) \right) \right. \\ & \quad \left. \times X_a^2 \delta h_a^m(y) \delta h_a^m(z) \, d\mathbf{P}(y) \, d\mathbf{P}(z) \right] dr \\ &+ \frac{e^{-3t/2}}{\beta_{t/2}^2} \sum_{a \in \mathcal{A}} \mathbf{E} \left[ \int_{W_{\eta, p}^2} \left( F(X_{-a}^{\mathcal{A}}(t, 1, y, z)) - F(X_{-a}^{\mathcal{A}}(t, 0, y, z)) \right) \right. \\ & \quad \left. \times \delta h_a^m(y) \delta h_a^m(z) \, d\mathbf{P}(y) \, d\mathbf{P}(z) \right]. \end{aligned}$$

Thus, according to (8), (14) and Cauchy-Schwarz inequality, we get

$$\begin{aligned} (16) \quad & \left| \mathbf{E} [LP_t F(X^{\mathcal{A}})] \right| \leq 2 \|h_a^m\|_{W_{\eta, p}^*}^2 \frac{e^{-3t/2}}{\beta_{t/2}^2} \int_{W_{\eta, p}} \|y\|_{W_{\eta, p}}^2 \, d\mathbf{P}(y) \\ & \times \sum_{a \in \mathcal{A}} \sup_{r \in [0, 1]} \mathbf{E} \left[ (1 + |X_a|^2) \right. \\ & \left. \times \left( \iint_{W_{\eta, p}} \left| F(X_{-a}^{\mathcal{A}}(t, 0, y, z)) - F(X_{-a}^{\mathcal{A}}(t, r, y, z)) \right|^2 \, d\mathbf{P}(y) \, d\mathbf{P}(z) \right)^{1/2} \right]. \end{aligned}$$

Since  $F$  is Lipschitz continuous, it all boils down to estimate

$$\begin{aligned}
& \sup_{\substack{r \in [0,1] \\ y, z \in W_{\eta,p}}} \mathbf{E} \left[ (1 + |X_a|^2) \|X_{\neg a}^{\mathcal{A}}(t, 0, y, z) - X_{\neg a}^{\mathcal{A}}(t, r, y, z)\|_{\check{G}_2 W_{\eta,p}} \right] \\
&= \sup_{\substack{r \in [0,1] \\ y, z \in W_{\eta,p}}} \mathbf{E} \left[ (1 + |X_a|^2) \|X_{\neg a}^{\mathcal{A}}(t, 0, y, z) - X_{\neg a}^{\mathcal{A}}(t, r, y, z)\|_{W_{\eta,p}} \right] \\
&+ \sup_{\substack{r \in [0,1] \\ y, z \in W_{\eta,p}}} \mathbf{E} \left[ (1 + |X_a|^2) \left( \iint \left( \pi_2 \check{S}_2 X_{\neg a}^{\mathcal{A}}(t, r, y, z) \right. \right. \right. \\
&\quad \left. \left. \left. - \pi_2 \check{S}_2 - X_{\neg a}^{\mathcal{A}}(t, 0, y, z) \right)_{u,v}^{p/2} d\mu_{\eta,p}(u, v) \right)^{1/p} \right] \\
&= A_1 + A_2.
\end{aligned}$$

According to (13) and (15),

$$(17) \quad |A_1| \leq e^{-t} \mathbf{E} \left[ (1 + |X_a|^2) |X_a| \right] \|h_a^m\|_{W_{\eta,p}} \leq 2 \mathbf{E} [|X_a|^3] m^{-(1/2-\eta)}.$$

Furthermore,

$$\begin{aligned}
& \left( \pi_2 \check{S}_2 X_{\neg a}^{\mathcal{A}}(t, r, y, z) - \pi_2 \check{S}_2 - X_{\neg a}^{\mathcal{A}}(t, 0, y, z) \right)_{u,v} \\
&= r X_a(t, y, z) \sum_{b \prec a} X_b(t, y, z) h_{a,b}^m(u, v),
\end{aligned}$$

where

$$X_a(t, y, z) = e^{-t} X_a + e^{-t/2} \beta_{t/2} y + \beta_{t/2} z.$$

Hence, according to Hölder inequality,

$$\begin{aligned}
& |A_2| \leq c \mathbf{E} [|X_a|^p]^{2/p} \\
& \times \mathbf{E} \left[ \left( \iint \left| X_a(t, y, z) \sum_{b \prec a} X_b(t, y, z) h_{a,b}^m(u, v) \right|^{p/2} d\mu_{\eta,p}(u, v) \right)^{1/(p-2)} \right]^{(p-2)/p} \\
& \leq c \mathbf{E} [|X_a|^p]^{2/p} \\
& \times \mathbf{E} \left[ \iint \left| X_a(t, y, z) \sum_{b \prec a} X_b(t, y, z) h_{a,b}^m(u, v) \right|^{p/2} d\mu_{\eta,p}(u, v) \right]^{1/p}.
\end{aligned}$$

Lemma 4.4 implies that there exists  $c > 0$  such that for any  $y, z \in W_{\eta,p}$ ,

$$\begin{aligned}
(18) \quad |A_2| & \leq c \mathbf{E} [|X_a|^p]^{2/p} \mathbf{E} [|X_a|^{p/2}]^{2/p} m^{-(1/2-\eta)} \\
& \leq c \mathbf{E} [|X_a|^p]^{3/p} m^{-(1/2-\eta)}.
\end{aligned}$$

Since  $\|h_a^m\|_{W_{\eta,p}^*} \leq c\sqrt{m}$  and  $|\mathcal{A}| = d.m$ , plug (17) and (18) into (16) to obtain the existence of  $c > 0$  such that for any  $t > 0$ ,

$$(19) \quad \left| \mathbf{E} [LP_t F(X^{\mathcal{A}})] \right| \leq c \frac{e^{-3t/2}}{\beta_{t/2}^2} \mathbf{E} [|X_a|^p]^{3/p} m^{-(1/2-\eta)}.$$

From (12), (19) and Lemma 4.1, we infer that

$$\begin{aligned}
 (20) \quad & |\mathbf{E} [\check{F}(B_V)] - \mathbf{E} [\check{F}(X^{\mathcal{A}})]| \\
 & \leq c\beta_{\tau/2} + c \mathbf{E} [|X_a|^p]^{3/p} m^{-(1/2-\eta)} \int_{\tau}^{\infty} \frac{e^{-t}}{\beta_{t/2}^2} dt \\
 & = c \left( \beta_{\tau/2} - 2 \mathbf{E} [|X_a|^p]^{3/p} m^{-(1/2-\eta)} \log \beta_{\tau/2} \right).
 \end{aligned}$$

Optimize (20) with respect to  $\tau$  yields  $\beta_{\tau_{opt}/2} = 2 \mathbf{E} [|X_a|^p]^{3/p} m^{-(1/2-\eta)}$  and

$$|\mathbf{E} [\check{F}(B_V)] - \mathbf{E} [\check{F}(X^{\mathcal{A}})]| \leq c \mathbf{E} [|X_a|^p]^{3/p} m^{-(1/2-\eta)} \log m.$$

The proof is thus complete.  $\square$

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