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# CONVERGENCE RATE IN THE ROUGH DONSKER THEOREM

L. COUTIN AND L. DECREUSEFOND

ABSTRACT. We make precise the rate of convergence in the Lamperti's improvement of the Donsker Theorem on the topology of Hölder continuous functions. Then, we establish that the same rate of convergence holds true for the convergence of the enhanced, in the sense of rough paths theory, random walk towards the enhanced Brownian motion.

## 1. INTRODUCTION

The Donsker theorem says that a random walk

$$X^m(t) = \frac{1}{\sqrt{m}} \sum_{k=1}^{[mt]} X_k$$

where the  $X_k$ 's are independent, identically distributed random variables with mean 0 and variance 1, converges in a functional space to the Brownian motion  $B$ . In the original version (see [8]), the convergence was proved to hold in the space of continuous functions. The first evolution was the paper of Lamperti [13], which proved the convergence in Hölder spaces. Namely, he stated that if the increments of the  $X_k$ 's are  $p$ -integrable then the random walk  $X^m$  converges to  $B$  in  $\text{Hol}(1/2 - 1/p)$ . The higher the integrability, the stronger the topology. There are numerous other extensions which can be made to the Donsker theorem. In the 90s, Barbour [2] estimated the rate of convergence in the space  $\mathcal{C}$  of continuous functions on  $[0, 1]$  equipped with a stronger topology than the usual sup-norm topology. He proved that, provided that  $\mathbf{E}[|X_k|^3]$  is finite, then

$$\sup_{|F|_M \leq 1} \mathbf{E}[F(X^m)] - \mathbf{E}[F(B)] \leq c \frac{\log m}{\sqrt{m}},$$

where  $M$  is roughly speaking, the set of thrice Fréchet differentiable functions on  $\mathcal{C}$  with bounded derivatives and  $|F|_M$  is a function of the supremum of  $D^i F$ ,  $i = 0, \dots, 3$  over  $\mathcal{C}$ . The strategy is to compare  $X^m$  with  $B^m$ , the affine interpolation, of mesh  $1/m$ , of the Brownian motion and then to compare  $B^m$  with  $B$ . The latter comparison is a sample-path comparison since the two processes live in the same probability space. This is the part which yields the  $\log m$  factor. The former comparison is done via the Stein's method in finite dimension. The rate of this convergence is as usual (see [3, 14]) for Gaussian limits, of the order of  $m^{-1/2}$ .

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In [5], we quantified the rate of convergence of  $X^m$  towards  $B$  in Besov-Liouville spaces ([15]), which are one scale of fractional Sobolev spaces. The spaces we considered were not included in Hölder spaces but the method could easily be adapted to obtain convergence rate in Hölder spaces. A slight adaptation of the proofs developed in this paper would show that under the same hypothesis as in [2],

$$\sup_{|F|_L \leq 1} \mathbf{E}[F(X^m)] - \mathbf{E}[F(B)] \leq cm^{2\beta-1}$$

for any  $\beta \in (0, 1/2)$  and  $L$  is the set of thrice Fréchet differentiable functions on  $W_{\beta,3}$  (see below) with bounded derivatives equipped with the norm which is the sum of the sup-norm of  $F$  and of its first three derivatives.

Note that in both [2] and [5], an higher integrability of the  $X_k$ 's would not improve the convergence rates but would give more flexibility on the choice of the topology in which the convergence holds: The higher the integrability, the higher the Hölder exponent may be chosen.

In [11, Theorem 13.3.3], Friz and Victoir essentially showed that a Lamperti's like result holds for the convergence of the enriched random walk in the sense of rough path (see the definition below) to the enriched Brownian motion.

The motivation of this paper was to quantify the rate of this convergence in rough-paths sense. It raises several problems. The limiting process is no longer a Gaussian process: The Lévy area of a Brownian motion is not Gaussian. Hence we cannot expect to have a direct application of the Stein's method. However, there is no more randomness in the Lévy area that there is in the Brownian motion itself: The Lévy area is adapted to the filtration generated by the underlying Brownian motion. Saying that has two consequences. First, that the probability space we have to consider depends only on the Brownian motion. Second, that we have to find functional spaces for which the map which sends a Brownian motion to its Lévy area is not only continuous but also Lipschitz or at least locally Lipschitz. As mentioned in [12], the Besov-Liouville spaces are not well fitted to deal with the iterated integral processes we encounter in rough-paths theory. It is much better to work with the Slobodetsky scale of fractional Sobolev spaces.

In order to avoid some complicated calculations in infinite dimensional spaces, the idea is then to go back to the approach of [2], comparing the random walk with the affine interpolation of the Brownian motion in the Slobodetsky scale of fractional Sobolev spaces. This can be done by an application of the Stein's method in finite dimension. Then, we extend the norm on these spaces to take into account the iterated integrals of the enriched paths and show that the so-called signature map (see below) is Lipschitz continuous. The final result is obtained by considering the known distance between the enriched affine interpolation and the enriched Brownian motion in fractional Sobolev spaces.

Besides the definition of the convenient functional spaces, one of our contribution, interesting in its own, is an improvement of the Stein method. Actually, Stein's method for Gaussian limits in dimension 1 can be applied to  $\mathcal{C}_b^1$  functions but when the limit is in dimension 2 or more, the standard procedure works only for functions thrice differentiable with bounded

derivatives. We show here that we can work with Lipschitz functions in any dimension by splitting the integral of the semi-group into two parts.

Rough paths theory is essentially a deterministic theory, it is therefore tempting to make the estimate we need in a deterministic setting and then to take the expectation of these bounds. This turns out to be a misleading approach. For instance, consider a sequence of centered, independent identically distributed random variables  $(X_n, n \geq 1)$  and let  $S_n = \sum_{i=1}^n X_i$ . If we evaluate the  $p$ -th moment of  $S_n$  with Hölder inequality, we get that this moment is bounded by a constant times  $m^p$ . But if we use, as in the sequel, the Burkholder-Davis-Gundy inequality for discrete time martingale, we get an upper-bound proportional to  $m^{p/2}$ . This martingale argument which is implicitly used in [13] is the key to our work. For the sake of simplicity, this implies to separate the treatment of the symmetric and anti-symmetric parts of the signature. However, this is of no real importance since, as detailed below, the symmetric part of the signature can be handled as a classical  $\mathbf{R}^d$ -valued process.

This paper is organized as follows: In Section 2, we give the necessary notions about fractional Sobolev spaces and rough-paths theory. We also give a detailed proof of Lamperti's result in the fractional Sobolev spaces scale for further use and comparison. In Section 3, we then define the Kolmogorov-Rubinstein distance and show that this distance between the random walk and the affine interpolation of the Brownian motion can be reduced to a problem in finite dimension, should we consider a special set of Lipschitz functions. In Section 4, we then present our development of the Stein-Dirichlet method to estimate this distance.

## 2. PRELIMINARIES

**2.1. Fractional Sobolev spaces.** As in [6, 12], we consider the fractional Sobolev spaces  $W_{\eta,p}$  defined for  $\eta \in (0, 1)$  and  $p \geq 1$  as the the closure of  $\mathcal{C}^1$  functions with respect to the norm

$$|f|_{\eta,p}^p = \int_0^1 |f(t)|^p dt + \iint_{[0,1]^2} \frac{|f(t) - f(s)|^p}{|t - s|^{1+p\eta}} dt ds.$$

For  $\eta = 1$ ,  $W_{1,p}$  is the completion of  $\mathcal{C}^1$  for the norm:

$$|f|_{\eta,p}^p = \int_0^1 |f(t)|^p dt + \int_0^1 |f'(t)|^p dt.$$

They are known to be Banach spaces and to satisfy the Sobolev embeddings [1, 10]:

$$W_{\eta,p} \subset \text{Hol}(\eta - 1/p) \text{ for } \eta - 1/p > 0$$

and

$$W_{\eta,p} \subset W_{\gamma,q} \text{ for } 1 \geq \eta \geq \gamma \text{ and } \eta - 1/p \geq \gamma - 1/q.$$

As a consequence, since  $W_{1,p}$  is separable (see [4]), so does  $W_{\eta,p}$ .

**2.2. Rough paths.** We give a quick introduction to the rough-paths theory. For details, we refer to the monograph [11]. Consider  $T^2(\mathbf{R}^d)$ , the graded algebra of step two:

$$T^2(\mathbf{R}^d) = \mathbf{R} \oplus \mathbf{R}^d \oplus (\mathbf{R}^d \otimes \mathbf{R}^d).$$

We endow  $T^2(\mathbf{R}^d)$  with an algebra structure  $(+, \cdot, \otimes)$  where for all  $(w_0, w_1, w_2), (z_0, z_1, z_2) \in T^2(\mathbf{R}^d), \lambda \in \mathbf{R}$

$$(w_0, w_1, w_2) + (z_0, z_1, z_2) = (w_0 + z_0, w_1 + z_1, w_2 + z_2)$$

$$\lambda \cdot (w_0, w_1, w_2) = (\lambda w_0, \lambda w_1, \lambda w_2)$$

$$(w_0, w_1, w_2) \otimes (z_0, z_1, z_2) = (w_0 z_0, w_0 z_1 + z_0 w_1, w_0 z_2 + z_0 w_2 + w_1 \otimes z_1).$$

Introduce the projection maps: For  $i = 0, 1, 2$

$$\pi_i : T^2(\mathbf{R}^d) \longrightarrow (\mathbf{R}^d)^{\otimes i}$$

$$(w_0, w_1, w_2) \longmapsto w_i,$$

The set

$$\begin{aligned} 1 + t^2(\mathbf{R}^d) &= \{w \in T^2(\mathbf{R}^d), \pi_0(w) = 1\} \\ &= \left\{ g = (1, w_1, w_2), (w_1, w_2) \in \mathbf{R}^d \oplus (\mathbf{R}^d \otimes \mathbf{R}^d) \right\} \end{aligned}$$

is a Lie group with respect to the tensor multiplication  $\otimes$ , [11, Prop. 7.17]. Note that

$$(1, w_1, w_2)^{-1} = (1, -w_1, -w_2 + w_1 \otimes w_1).$$

As usual, a Lie group, like  $1 + t^2(\mathbf{R}^d)$ , leads to a Lie algebra when equipped with notions of product and commutator. Here, the Lie algebra is  $(t^2(\mathbf{R}^d), +, \cdot)$  with product  $\otimes$  and commutator

$$\begin{aligned} [g, w] &= g \otimes w - w \otimes g = 0 \oplus (\pi_1(g) \otimes \pi_1(w) - \pi_1(w) \otimes \pi_1(g)), \\ &\text{for any } w, g \in t^2(\mathbf{R}^d). \end{aligned}$$

The exponential and logarithm maps are useful to go back and forth between  $t^2(\mathbf{R}^d)$  and  $1 + t^2(\mathbf{R}^d)$ :

$$\begin{aligned} \exp : t^2(\mathbf{R}^d) &\longrightarrow 1 + t^2(\mathbf{R}^d) \\ w &\longmapsto 1 + w + \frac{1}{2}(\pi_1 w)^{\otimes 2} \end{aligned}$$

and

$$\begin{aligned} \log : 1 + t^2(\mathbf{R}^d) &\longrightarrow t^2(\mathbf{R}^d) \\ (1 + w) &\longmapsto w - \frac{1}{2}(\pi_1 w)^{\otimes 2}. \end{aligned}$$

We denote by  $\Sigma$  the set of finite partitions  $\sigma = \{t_1, \dots, t_n\}$  of  $[0, 1]$ . A continuous path  $z$  from  $[0, 1]$  into  $\mathbf{R}^d$  is said to have 1-finite variation whenever

$$\sup_{\sigma = \{t_1, \dots, t_n\} \in \Sigma} \sum_{i=1}^{n-1} |z_{t_{i+1}} - z_{t_i}| < \infty.$$

The set of such functions equipped with this quantity as a norm is denoted by  $C^{1\text{-var}}$ .

**Definition 2.1.** The step-2 signature of  $z \in C^{1\text{-var}}$  is given by:

$$\begin{aligned} S_2(z) : [0, 1] &\longrightarrow 1 + t^2(\mathbf{R}^d) \\ t &\longmapsto \left( 1, z_t - z_0, \int_0^t (z_s - z_0) \otimes dz_s \right). \end{aligned}$$

The free nilpotent group of order 2,  $G^2(\mathbf{R}^d)$ , is the closed subgroup of  $1 + t^2(\mathbf{R}^d)$  defined by

$$G^2(\mathbf{R}^d) = \{S_2(z), z \in C^{1-\text{Var}}\}.$$

We also consider  $\check{G}_2(\mathbf{R}^d)$  ( $\check{G}_2$  for short since  $d$  is fixed), the image of  $G_2(\mathbf{R}^d)$  by the logarithm map. For  $z \in C^{1-\text{var}}$ , this corresponds to consider only the anti-symmetric part of  $\pi_2(S_2(z))$ :

$$\begin{aligned} \check{S}_2(z) : [0, 1] &\longrightarrow t^2(\mathbf{R}^d) \\ t &\longmapsto \left( z_t - z_0, \int_0^t [(z_s - z_0), dz_s] \right). \end{aligned}$$

*Remark 1.* Denote by  $(e_i, 1 \leq i \leq d)$  the canonical basis of  $\mathbf{R}^d$ , so that  $(e_i \otimes e_j, 1 \leq i, j \leq d)$  is the canonical basis of  $\mathbf{R}^d \otimes \mathbf{R}^d$ . If

$$z(t) = \sum_{i=1}^d \sum_{k=1}^m z_{ik} h_k(t) e_i$$

where  $(h_1, \dots, h_m)$  are elements of  $C^{1-\text{var}}$ , we have

$$(1) \quad \log S_2(z)(t) = \left( z_t - z_0, \sum_{1 \leq i < j \leq d} \sum_{1 \leq k < l \leq m} z_{ik} z_{jl} \left( \int_0^t h_k(s) dh_l(s) - \int_0^t h_l(s) dh_k(s) \right) [e_i, e_j] \right).$$

For the sake of notations, we set  $\mathcal{A} = \{1, \dots, d\} \times \{1, \dots, m\}$  and define the  $\prec$  relation by:

$$a = (a_1, a_2) \prec b = (b_1, b_2) \iff (a_1 < b_1) \text{ and } (a_2 < b_2).$$

With these notations, Eqn. (1) then becomes

$$\log S_2(z)(t) = \left( z_t - z_0, \sum_{a \prec b} z_a z_b \left( \int_0^t h_{a_2}(s) dh_{b_2}(s) - \int_0^t h_{b_2}(s) dh_{a_2}(s) \right) [e_{a_1}, e_{b_1}] \right).$$

The group  $G^2(\mathbf{R}^d)$  has the structure of a sub-Riemannian manifold. We will not dwell into the meanders of this very rich but intricate structure. It suffices to say that we can proceed equivalently by considering usual norms as follows.

For  $\alpha \in (0, 1)$ , a path  $w = 1 \oplus w_1 \oplus w_2$  is said to be  $\alpha$ -Hölder whenever

$$\rho_\alpha(w) = \max \left( \sup_{s \neq t} \frac{|w_1(t) - w_1(s)|}{|t - s|^\alpha}, \sup_{s \neq t} \frac{|\pi_2(w(s)^{-1} \otimes w(t))|^{1/2}}{|t - s|^\alpha} \right) < \infty.$$

Note that for  $z \in C^{1-\text{var}}$ ,

$$\rho_\alpha(S_2(z)) = \max \left( \sup_{s \neq t} \frac{|z(t) - z(s)|}{|t - s|^\alpha}, \sup_{s \neq t} \frac{\left| \int_s^t (z_r - z_s) \otimes dz_r \right|^{1/2}}{|t - s|^\alpha} \right).$$

**Definition 2.2.** We denote by  $H_\alpha(t^2(\mathbf{R}^d))$ , the vector space of paths  $w$  from  $[0, 1]$  into  $t^2(\mathbf{R}^d)$  such that  $\rho_\alpha(w)$  is finite. It is equipped with the homogeneous norm: For  $w$  and  $v$  in  $H_\alpha(t^2(\mathbf{R}^d))$

$$|w - v|_{H_\alpha(t^2(\mathbf{R}^d))} = \rho_\alpha(w - v) = \rho_\alpha((w_1 - v^{(1)}) \oplus (w_2 - v^{(2)})).$$

Unfortunately, as mentioned in [11, Chapter 8.3], this metric space is complete but not separable, which is unacceptable for our purpose (see Definition 3.1 and the remark below). We thus introduce fractional Sobolev spaces as in [12].

**Definition 2.3.** For any  $\eta \in (0, 1)$ , any  $p \geq 2$ ,  $\check{G}_2W_{\eta,p}$  is the vector space of paths  $w$  from  $[0, 1]$  into  $t^2(\mathbf{R}^d)$  such that

$$|w_1|_{\check{G}_2W_{\eta,p}}^p + \iint_{[0,1]^2} \frac{|\pi_2[w_s^{-1}, w_t]|^{p/2}}{|t-s|^{1+\eta p}} ds dt < \infty.$$

The distance on  $\check{G}_2W_{\eta,p}$  is defined by

$$\begin{aligned} |w - v|_{\check{G}_2W_{\eta,p}} &= |\pi_1(w) - \pi_1(v)|_{W_{\eta,p}} \\ &+ \left( \iint_{[0,1]^2} \frac{|\pi_2[w_s^{-1}, w_t] - \pi_2[v_s^{-1}, v_t]|^{p/2}}{|t-s|^{1+\eta p}} ds dt \right)^{1/p}. \end{aligned}$$

Following [12], we know that  $\check{G}_2W_{\eta,p}$  is a Banach space included into  $H_\alpha(t^2(\mathbf{R}^d))$  for  $\alpha = \eta - 1/p$ , provided  $\alpha > 0$ .

**Lemma 2.1.** For any  $\eta \in (0, 1)$ , any  $p \geq 2$ ,  $\check{G}_2W_{\eta,p}$  is separable.

*Proof.* Consider the map  $\kappa$  defined as

$$\begin{aligned} \kappa : ([0, 1] \rightarrow t_2(\mathbf{R}^d)) &\longrightarrow ([0, 1]^2 \rightarrow t_2(\mathbf{R}^d)) \\ w &\longmapsto \left( (s, t) \mapsto \left( \pi_1(w_t) - \pi_1(w_s), \pi_2[w_s^{-1}, w_t] \right) \right). \end{aligned}$$

Consider the measure  $d\mu_{\eta,p}(s, t) = |t - s|^{-1-\eta p} ds dt$ . Then,

$$(2) \quad |w|_{\check{G}_2W_{\eta,p}} = |\pi_1 \circ \kappa(w)|_{L^p(\mu_{\eta,p})} + |\pi_2 \circ \kappa(w)|_{L^{p/2}(\mu_{2\eta,p/2})}^{1/2}.$$

For any  $p \geq 1$  and  $\eta \in (0, 1)$ ,  $L^p(\mu_{\eta,p})$  is isometrically isomorphic to  $L^p(ds dt)$  hence it is separable. This entails that  $E = L^p(\mu_{\eta,p}) \times L^{p/2}(\mu_{2\eta,p/2})$  is separable. Equation (2) means that the application  $T$  which maps  $w \in \check{G}_2W_{\eta,p}$  to the couple  $(\pi_1 \circ \kappa(w), \pi_2 \circ \kappa(w)) \in E$ , is an isometry. Thus  $\check{G}_2W_{\eta,p}$  is isometrically isomorphic to a closed subspace of the separable space  $E$ , hence it is separable.  $\square$

**2.3. Donsker-Lamperti theorem.** For the sake for completeness and for further comparison, we give the proof of the Donsker-Lamperti theorem in the scale of fractional Sobolev spaces, which induces the convergence in Hölder spaces.

**Definition 2.4.** The random walk associated to the sequence  $(X_k, k \geq 1)$  is defined by

$$X^m(t) = \sqrt{m} \sum_{k=1}^m X_k r_k^m(t) = \sum_{k=1}^m X_k h_k^m(t)$$

where

$$r_k^m(t) = \int_0^t \mathbf{1}_{((k-1)/m, k/m]}(s) ds \text{ and } h_k^m = \sqrt{m} r_k^m.$$

**Theorem 2.2.** *If for any  $k \geq 1$ ,  $X_k$  belongs to  $L^p$  for some  $p \geq 2$ , then there exists  $c > 0$  such that*

$$(3) \quad \sup_{m \geq 1} \frac{\mathbf{E} \left[ \left| \sum_{k=1}^m X_k \sqrt{m} (r_k^m(t) - r_k^m(s)) \right|^p \right]}{|t - s|^{p/2}} < c \mathbf{E} [|X_1|^p]$$

*Proof.* For  $0 \leq s < t \leq 1$  fixed, the discrete time process

$$Y_m^{st} = \sum_{k=1}^m X_k (r_k^m(t) - r_k^m(s))$$

is a martingale with respect to the filtration  $\mathcal{F}_n = \sigma(X_k, 1 \leq k \leq m)$ . The Burkholder-Davis-Gundy [17] entails that

$$m^p \mathbf{E} \left[ \left| \sum_{k=1}^m X_k (r_k^m(t) - r_k^m(s)) \right|^p \right] \leq c m^{p/2} \mathbf{E} \left[ \left| \sum_{k=1}^m X_k^2 (r_k^m(t) - r_k^m(s))^2 \right|^{p/2} \right].$$

If  $|t - s| \leq 1/m$ , there is at most two values of  $k$  such that  $r_k^m(t) - r_k^m(s)$  is not zero. Furthermore,

$$|r_k^m(t) - r_k^m(s)| \leq |t - s|$$

hence

$$|r_k^m(t) - r_k^m(s)|^2 \leq m^{-1} |t - s|.$$

In this situation,

$$\mathbf{E} \left[ \left| \sum_{k=1}^m X_k^2 (r_k^m(t) - r_k^m(s))^2 \right|^{p/2} \right] \leq c m^{-p/2} \mathbf{E} [|X_1|^p] |t - s|^{p/2},$$

so that (3) holds true for  $|t - s| \leq 1/m$ . For  $|t - s| > 1/m$ , we remark that  $r_k^m(t) - r_k^m(s)$  is not null for at most  $[m(t - s)] + 2$  values of  $k$  and since  $r_k^m$  is Lipschitz continuous,  $|r_k^m(t) - r_k^m(s)| \leq 1/m$  for such value of  $k$ . Hence, by convexity inequality,

$$\begin{aligned} \mathbf{E} \left[ \left| \sum_{k=1}^m X_k^2 (r_k^m(t) - r_k^m(s))^2 \right|^{p/2} \right] &\leq c (m|t - s| + 2)^{p/2} m^{-p} \mathbf{E} [|X_1|^p] \\ &\leq c m^{-p/2} \mathbf{E} [|X_1|^p] |t - s|^{p/2}. \end{aligned}$$

Hence, (3) is true for  $|t - s| \geq 1/m$ .  $\square$

It is then straightforward that we have:



**Corollary 2.3.** *Assume that for any  $k \geq 1$ ,  $X_k$  belongs to  $L^p$  for some  $p \geq 2$ . Then, for any  $\eta < 1/2$ ,*

$$\sup_{m \geq 1} \mathbf{E} \left[ |X^m|_{W_{\eta,p}}^p \right] < \infty.$$

*Proof.* Actually,  $(s, t) \mapsto |t - s|^{p/2}$  is  $\mu_{\eta,p}$ -integrable provided that  $p(1/2 - \eta) > 0$ , i.e.  $\eta < 1/2$ .  $\square$

**Corollary 2.4** (Lamperti). *Assume that for any  $k \geq 1$ ,  $X_k$  belongs to  $L^p$  for some  $p \geq 2$ . Then, for any  $1/p < \eta < 1/2$ , the sequence  $(X^m, m \geq 1)$  converges in distribution in  $\text{Hol}(\eta - 1/p)$  to  $B$ .*

*Proof.* It is well-known that the finite dimensional distributions of  $X^m$  converge to that of  $B$ . From the previous lemma, we know that for any  $0 < \eta < \zeta < 1/2$ , there exists  $K$  such that

$$\sup_{m \geq 1} \mathbf{P}(|X^m|_{W_{\zeta,p}} \geq K) \leq \eta.$$

The embedding of  $W_{\zeta,p}$  into  $W_{\eta,p}$  is compact, hence the sequence  $(X^m, m \geq 1)$  is tight in  $W_{\eta,p}$  hence convergent. The result follows by the continuous embedding of  $W_{\eta,p}$  into  $\text{Hol}(\eta - 1/p)$ .  $\square$

### 3. RATE OF CONVERGENCE

**3.1. Kolmogorov-Rubinstein distance.** The proof of Lamperti's Theorem is given for one dimensional processes but it can be straightforwardly adapted to  $\mathbf{R}^d$ -valued random walks and Brownian motion:  $X^m$  becomes the  $\mathbf{R}^d$ -valued process

$$X^m(t) = \sqrt{m} \sum_{a \in \mathcal{A}} X_a r_{a_2}(t) e_{a_1} = \sum_{a \in \mathcal{A}} X_a h_a(t)$$

where  $h_a(t) = \sqrt{m} r_{a_2}(t) e_{a_1}$  and  $(X_a, a \in \mathcal{A})$  is a family of independent identically distributed random variables of mean 0 and variance 1. Furthermore,  $B$  is the  $d$ -dimensional Brownian motion:

$$B(t) = \sum_{i=1}^d B_i(t) e_i.$$

The enriched Brownian motion  $\mathbb{B}$ , is the  $G_2(\mathbf{R}^d)$ -value process defined by

$$\mathbb{B}(t) = 1 \oplus B(t) \oplus \sum_{i,j=1}^d \int_0^t B_i(s) \circ dB_j(s) e_i \otimes e_j, \text{ for any } t \in [0, 1],$$

where the stochastic integrals are to be understood in the Stratonovitch sense. Theorem 13.32 of [11] says that  $S_2(X^m)$  converges to  $\mathbb{B}$  in some Hölder type spaces. Our primary goal is to give the rate of this convergence. For, we need to define a distance between probability measures over Hölder spaces. There are several possibilities of such a definition, the best suited for an estimate by the Stein method is the Kolmogorov-Rubinstein distance:

**Definition 3.1** (Kolmogorov-Rubinstein distance). For  $\mu$  and  $\nu$  two probability measures on a metric space  $(W, d_W)$ , their Kolmogorov-Rubinstein distance is given by

$$\text{dist}_{\text{KR}}(\mu, \nu) = \sup_{F \in \text{Lip}(d_W)} \int_W F \, d\mu - \int_W F \, d\nu,$$

where

$$\text{Lip}(d_W) = \left\{ F : W \rightarrow \mathbf{R}, \forall w, v \in W, |F(w) - F(v)| \leq d_W(w, v) \right\}.$$

Theorem 11.3.3 of [9] states that the topology induced by this distance on the set of probability measures on  $W$  is the same as the topology of the convergence in laws whenever the metric space  $W$  is separable. Unfortunately, as we already mentioned, Hölder spaces are not separable, thus to have a meaningful result, we turn to work on fractional Sobolev spaces. It is of no importance since Sobolev embeddings ensure that convergence in fractional Sobolev spaces induces convergence in Hölder spaces as for the Lamperti Theorem.

Our new goal is then to estimate the Kolmogorov-Rubinstein distance in  $\mathbb{G}_2W_{\eta,p}$  between  $\mathbb{B}$  and  $S_2(X^m)$ . Remark that

$$\begin{aligned} \pi_2 S_2(X^m) &= \pi_2 \check{S}_2(X^m) \\ &+ 2 \sum_{a \in \mathcal{A}} (X_a^2 - 1) \int_0^t h_a^m(s) \otimes dh_a^m(s) + \sum_{a \in \mathcal{A}} h_a^m(t) \otimes h_a^m(t) \\ &= U_1^m + U_2^m + U_3^m. \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathbb{B}(t) &= \sum_{1 \leq i < j \leq d} \left( \int_0^t B_i(s) \circ dB_j(s) - \int_0^t B_j(s) \circ dB_i(s) \right) [e_i, e_j] \\ &+ 2 \sum_{i=1}^d \int_0^t B_i(s) dB_i(s) e_i \otimes e_i + \sum_{i=1}^d t e_i \\ &= \log \mathbb{B}(t) + U_2 + U_3. \end{aligned}$$

where in  $U_2$ , the stochastic integral is taken in the Itô sense. Direct computations show that  $U_3^m - U_3$  tends to 0 as  $1/m$ . The convergence of  $U_2^m$  towards  $U_2$  is of the same kind as the standard Donsker theorem: It follows the same convergence rate as that of the convergence of  $X^m$  to  $B$ ; a rate which is expected and which will turn out to be much slower than  $1/m$ . In summary, the Kolmogorov-Rubinstein distance between  $\mathbb{B}$  and  $S_2(X^m)$  has the same asymptotic behavior as the distance between  $\log \mathbb{B}$  and  $\check{S}_2(X^m)$ . Our final objective is then to estimate the Kolmogorov-Rubinstein distance between the distributions of  $\log \mathbb{B}$  and  $\check{S}_2(X^m)$  in  $\check{\mathbb{G}}_2W_{\eta,p}$ .

**3.2. Reduction to finite dimension.** Should we follow the same procedure as the one we used in [5], we would face the same complications to compute the trace term in some infinite dimensional space. To circumvent this difficulty, we decompose the distance into two parts: Consider  $B^m$  the affine interpolation of  $B$ , we use the results of [11] to estimate the distance

between  $\check{S}_2(B^m)$  and  $\log \mathbb{B}$  and then resort to the Stein's method in finite dimension to estimate the distance between  $\check{S}_2(X^m)$  and  $\check{S}_2(B^m)$ . We can always write

$$\begin{aligned} & \sup_{F \in \text{Lip}(d_{\check{G}_2 W_{\eta,p}})} \mathbf{E} [F(\log \mathbb{B})] - \mathbf{E} [F(\check{S}_2(X^m))] \\ & \leq \sup_{F \in \text{Lip}(d_{\check{G}_2 W_{\eta,p}})} \mathbf{E} [F(\log \mathbb{B})] - \mathbf{E} [F(\check{S}_2(B^m))] \\ & \quad + \sup_{F \in \text{Lip}(d_{\check{G}_2 W_{\eta,p}})} \mathbf{E} [F(\check{S}_2(B^m))] - \mathbf{E} [F(\check{S}_2(X^m))]. \end{aligned}$$

On the one hand, since  $\log \mathbb{B}$  and  $\check{S}_2(B^m)$  live on the same probability space, for  $F \in \text{Lip}(d_{\check{G}_2 W_{\eta,p}})$ , according to [11, Proposition 13.20],

$$(4) \quad \mathbf{E} [F(\log \mathbb{B})] - \mathbf{E} [F(\check{S}_2(B^m))] \leq \mathbf{E} \left[ |\log \mathbb{B} - \check{S}_2(B^m)|_{\check{G}_2 W_{\eta,p}} \right] \leq c m^{-(1/2-\eta)}.$$

It remains to estimate

$$\sup_{F \in \text{Lip}(d_{\check{G}_2 W_{\eta,p}})} \mathbf{E} [F(\check{S}_2(B^m))] - \mathbf{E} [F(\check{S}_2(X^m))].$$

**Theorem 3.1.** *If  $p \geq 3$  and  $X_a$  belongs to  $L^p$ , for any  $\eta \in (1/p, 1/2)$ ,*

$$\sup_{F \in \text{Lip}(\check{G}_2 W_{\eta,p})} \mathbf{E} [F(\check{S}_2(B^m))] - \mathbf{E} [F(\check{S}_2(X^m))] \leq c \|X_a\|_{L^p} m^{-1/2+\eta}.$$

*Proof of Theorem.* According to Theorem 4.5,

$$\mathbf{E} \left[ |P_{\tau_0} F(\check{S}_2(X^m)) - F(\check{S}_2(X^m))| \right] \leq c \|X_a\|_{L^p} m^{-1/2+\eta} \sqrt{1 - e^{-\tau_0}}.$$

Combine this upper-bound with (8) to obtain

$$\begin{aligned} & \sup_{F \in \text{Lip}(\check{G}_2 W_{\eta,p})} \mathbf{E} [F(\check{S}_2(B^m))] - \mathbf{E} [F(\check{S}_2(X^m))] \\ & \leq c m^{-1/2+\eta} \|X_a\|_{L^p} \left( \sqrt{1 - e^{-\tau_0}} + m^{2\eta} \int_{\tau_0}^{\infty} \frac{e^{-5\tau/2}}{1 - e^{-\tau/2}} d\tau \right). \end{aligned}$$

The bracketed term is decreasing with respect to  $\tau_0$  and tends to 1 as  $\tau_0$  goes to infinity, hence the result.  $\square$

We now detail the proofs of the main estimates.

#### 4. IMPROVED STEIN METHOD

**4.1. Finite dimensional Gelfand triplet.** The title of this section may seem pompous at first reading but the construction of the Wiener space we are working on is of crucial importance and not so usual. In view of the previous considerations, our computations are to be held in a finite dimensional space and Gelfand triplet (or abstract Wiener space) are mainly thought to be relevant for infinite dimensional spaces. Actually, our settings should reflect the fact that we are in some sense, projecting an abstract Wiener infinite dimensional space onto a finite dimensional Gelfand triplet.

Recall that  $\mathcal{A} = \{1, \dots, d\} \times \{1, \dots, m\}$ . Let  $\mathcal{V}_m$  be the finite dimensional vector space

$$\mathcal{V}_m = \text{span}\{h_a^m, a \in \mathcal{A}\}$$

We put on  $\mathcal{V}_m$  the norm of  $W_{\eta,p}$ . We denote by  $W$ , the space  $\mathcal{V}_m$  equipped with this norm. To simplify the forthcoming computations, it is useful to introduce

$$g_a^m = \alpha_m^{-1} h_a^m = \alpha_m^{-1} (h_{a_2} \otimes e_{a_1}),$$

where the coefficient  $\alpha_m$  is chosen so that  $|g_a^m|_{W_{\eta,p}} = 1$ .

Furthermore, consider the space  $H$  which is  $\mathcal{V}_m$  equipped with the scalar product:

$$(g_a^m, g_b^m)_H = \delta_{a,b}.$$

Recall that

$$B^m = \sum_{a \in \mathcal{A}} B_a h_a^m = \alpha_m \sum_{a \in \mathcal{A}} B_a g_a^m.$$

Let  $\mathbf{P}^m$  be the distribution of  $B^m$ , it is a centered Gaussian probability measure on  $\mathcal{V}_m$  with covariance matrix  $\alpha_m^2 \text{Id}$ . The triplet  $(H, W, \mathbf{P}^m)$  is a Gelfand triplet:  $\mathbf{P}^m$  is a Gaussian measure on (the Banach space)  $W$ , which contains (as a dense subset) the (Hilbert) space  $H$  and

$$\int_W e^{i\langle \eta, \omega \rangle_{W^*, W}} d\mathbf{P}^m(\omega) = \exp\left(-\frac{1}{2} |\iota^*(\eta)|_H^2\right)$$

where

$$W^* \xrightarrow{\iota^*} H^* \simeq H \xrightarrow{\iota} W.$$

In a finite dimensional setting, all these spaces and their dual are isomorphic and isomorphic to  $\mathbf{R}^{dm}$ , but we cannot identify more than one space with its dual, hence the necessity to define precisely the Gelfand triplet.

**4.2. Malliavin-Dirichlet structure.** In particular, the Cameron-Martin theorem reads as:

$$(5) \quad \mathbf{E} \left[ F\left(B^m + \alpha_m \sum_{a \in \mathcal{A}} y_a g_a^m\right) \right] = \mathbf{E} \left[ F(B^m) \exp\left(\alpha_m \langle y, B^m \rangle_H - \frac{\alpha_m^2}{2} |y|_H^2\right) \right].$$

We denote by  $\nabla F$  the  $H$  valued gradient of  $F$  provided that it exists: For any  $v \in \mathcal{V}_m$ , for any  $g \in$ ,

$$\langle \nabla F(v), g \rangle_{H, H} = \left. \frac{d}{d\varepsilon} F(v + \varepsilon g) \right|_{\varepsilon=0}.$$

The space of twice differentiable functions on  $\mathcal{V}_m$  is denoted by  $\mathcal{C}^2(\mathcal{V}_m)$ . The Fréchet Laplacian is defined by:

$$\Delta F(v) = \text{trace } \nabla^{(2)} F(v) = \sum_{a \in \mathcal{A}} \left\langle \nabla^{(2)} F(v), g_a^m \otimes g_a^m \right\rangle_{H^{\otimes 2}, H^{\otimes 2}}.$$

Let  $L$  be the operator defined on  $\mathcal{C}^2(\mathcal{V}_m)$  by:

$$LF(v) = -\langle \nabla F(v), v \rangle_{H, H} + \alpha_m^2 \Delta F(v).$$

The Cameron-Martin theorem (see Eqn. (5)) yields an integration by parts formula on  $W$  which can be interpreted by saying that the distribution of  $B^m$  on  $\mathcal{V}_m$  is characterized by  $\mathbf{E}[LF] = 0$  for any  $F \in \mathcal{C}^2(\mathcal{V}_m)$ . It induces an Ornstein-Uhlenbeck semi-group  $(P_t, t \geq 0)$  which can be described by the Mehler formula:

$$P_t F(v) = \int_{\mathcal{V}_m} F(e^{-t}v + \beta_t w) d\mathbf{P}^m(w),$$

where  $\beta_t = (1 - e^{-2t})^{1/2}$ .

Introduce for any  $a \prec b \in \mathcal{A}$  and any  $0 \leq s < t \leq 1$ ,

$$\begin{aligned} & \check{g}_{a,b}^m(s, t) \\ &= \left\{ \int_s^t (g_{a_2}^m(r) - g_{a_2}^m(s)) dg_{b_2}^m(r) - \int_s^t (g_{b_2}^m(r) - g_{b_2}^m(s)) dg_{a_2}^m(r) \right\} [e_{a_1}, e_{b_1}]. \end{aligned}$$

**Lemma 4.1.** *For any  $\omega = (\omega_k, 1 \leq k \leq m)$ , we have*

$$\left| \sum_{k=1}^m \omega_k h_k^m \Big|_{\text{var}, [s, t]} \right| \leq \sqrt{m} \sum_{k=1}^m |\omega_k| \left\{ (t-s) \wedge \frac{1}{m} \right\}.$$

Moreover,

$$\left| \sum_{k=1}^m \omega_k h_k^m \Big|_{\eta, p} \right| \leq c m^{-(1/2-\eta)}.$$

*Proof.* For  $k \neq \ell$ ,  $dh_k^m dh_\ell = 0$  hence their contribution to the overall variation are separate:

$$\left\| \sum_{k=1}^m \omega_k h_k^m \Big|_{\text{var}, [s, t]} \right\| = \sum_{k=1}^m |\omega_k| \|h_k^m\|_{\text{var}, [s, t]}.$$

It is easy to see that

$$\|h_k^m\|_{\text{var}, [s, t]} = \sqrt{m} \left( t \wedge \frac{k}{m} - s \vee \frac{k-1}{m} \right) \leq \sqrt{m} \left\{ (t-s) \wedge \frac{1}{m} \right\}.$$

The proof is thus complete.  $\square$

**Lemma 4.2.** *For any  $\tau > 0$ , for any  $v \in \mathcal{V}_m$ , for any  $F \in \text{Lip}(\check{G}_2 W_{\eta, p})$ , we have*

$$\begin{aligned} (6) \quad & \left\langle \nabla^{(2)} P_\tau F(v), g_a^m \otimes g_a^m \right\rangle \\ &= \frac{e^{-3\tau/2}}{\beta_{\tau/2}^2} \iint F(v(\tau, y, z)) (y, g_a^m)_H(z, g_a^m)_H d\mathbf{P}^m(y) d\mathbf{P}^m(z), \end{aligned}$$

where

$$v(\tau, y, z) = e^{-\tau/2}(e^{-\tau/2}v + \beta_{\tau/2}y) + \beta_{\tau/2}z.$$

Moreover, there exists a constant  $c$  such that

$$\begin{aligned} (7) \quad & \left| \left\langle \nabla^{(3)} P_\tau F(v), (g_a^m)^{\otimes(3)} \right\rangle \right| \\ & \leq c \frac{e^{-5\tau/2}}{\beta_{\tau/2}^2} \left( \iint \left| \sum_{b \prec a} (v_b \check{g}_{a,b}^m)_{st} \right|^{p/2} d\mu_{\eta, p}(t, s) \right)^{1/p}. \end{aligned}$$

*Proof.* Following [16, Lemma 4.5], we have

$$\begin{aligned} & \left\langle \nabla^{(2)} P_\tau F(v), g_a^m \otimes g_a^m \right\rangle \\ &= \frac{e^{-\tau}}{\beta_{\tau/2}} \int \left\langle \nabla P_{\tau/2} F(e^{-\tau/2}v + \beta_{\tau/2}y), g_a^m \right\rangle (y, g_a^m)_H \, d\mathbf{P}^m(y), \end{aligned}$$

where we use integration by parts to have the following representation of the first order gradient:

$$\left\langle \nabla P_{\tau/2} F(w), g_a^m \right\rangle = \frac{e^{-\tau/2}}{\beta_{\tau/2}} \int F(e^{-\tau/2}w + \beta_{\tau/2}z) (z, g_a^m)_H \, d\mathbf{P}^m(z).$$

Plugging in this equation in the former yields (6). Moreover,

$$(v + \epsilon g_a^m)(\tau, y, z) = \epsilon e^{-\tau} g_a^m + v(\tau, y, z)$$

and

$$\begin{aligned} & \left| \check{S}_2 \left( \epsilon e^{-\tau} g_a^m + v(\tau, y, z) \right) - \check{S}_2 \left( v(\tau, y, z) \right) \right|_{W_{\eta,p}} \\ & \leq \epsilon e^{-\tau} \left( 1 + \left( \iint \left| \sum_{b \prec a} (v_b \check{g}_{a,b}^m)_{st} \right|^{p/2} \, d\mu_{\eta,p}(t, s) \right)^{1/p} \right). \end{aligned}$$

Hence, (7) holds true.  $\square$

**Lemma 4.3.** *Let  $(U_a, a \in \mathcal{A})$  be a family of independent identically distributed random variables which belong to  $L^p$ . Then,*

$$\mathbf{E} \left[ \left| U_a \sum_{b \prec a} U_b \check{g}_{a,b}^m \right|_{W_{\eta,p}}^p \right] \leq c \mathbf{E} [|U_a|^p]^2.$$

*Proof.* By independence and as in the proof of Theorem 2.2,

$$\mathbf{E} \left[ \left| U_a \sum_{b \prec a} U_b \check{g}_{a,b}^m(s, t) \right|_{W_{\eta,p}}^p \right] \leq c \mathbf{E} [|U_a|^p] \mathbf{E} \left[ \left( \sum_{b \prec a} |U_b|^2 |\check{g}_{a,b}^m(s, t)|^2 \right)^{p/2} \right].$$

Since  $a$  is fixed and  $b \prec a$ , if  $|t-s| \leq 1/m$ ,  $\check{g}_{a,b}^m$  is not zero only for  $b_2 = a_2 - 1$  and then, it is bounded by  $|t-s|/\sqrt{m}$ . If  $|t-s| \geq 1/m$ ,  $\check{g}_{a,b}^m$  is not zero for at most  $[dm|t-s|]$  values of  $b$  and then, each term is bounded by  $1/m$ . Hence

$$\mathbf{E} \left[ \left( \sum_{b \prec a} |U_b|^2 |\check{g}_{a,b}^m(s, t)|^2 \right)^{p/2} \right] \leq c m^{-p/2} (m|t-s|)^{p/2} \mathbf{E} [|U_b|^p].$$

The result follows by integration with respect to  $\mu_{\eta,p}$ .  $\square$

**4.3. Main computations.** Recall that

$$X^m = \alpha_m \sum_{a \in \mathcal{A}} X_a g_a^m \text{ and } B^m = \alpha_m \sum_{a \in \mathcal{A}} B_a g_a^m.$$

and that we expect to estimate the supremum, for  $F \in \text{Lip}(\check{G}_2 W_{\eta,p})$ , of

$$\mathbf{E} [F(\check{S}_2 B^m)] - \mathbf{E} [F(\check{S}_2 X^m)].$$

For the sake of notations, we set  $\check{F} = F \circ \check{S}_2$ . The Stein-Dirichlet representation formula (see [7]) then stands that

$$\begin{aligned} \mathbf{E} [\check{F}(B^m)] - \mathbf{E} [\check{F}(X^m)] &= \mathbf{E} \left[ \int_0^\infty \frac{d}{dt} P_\tau \check{F}(X^m) d\tau \right] \\ &= \mathbf{E} [P_{\tau_0} \check{F}(X^m) - \check{F}(X^m)] + \mathbf{E} \left[ \int_{\tau_0}^\infty LP_\tau \check{F}(X^m) d\tau \right], \end{aligned}$$

for any  $\tau_0 > 0$ .

**Theorem 4.4.** *If  $p \geq 3$  and  $X_a$  belongs to  $L^p$ , for any  $\tau_0 > 0$ , there exists  $c > 0$  such that*

$$(8) \quad \mathbf{E} \left[ \int_{\tau_0}^\infty LP_\tau \check{F}(X^m) d\tau \right] \leq c \|X_a\|_{L^{p/2}} m^{-1/2+3\eta} \int_{\tau_0}^\infty \frac{e^{-5\tau/2}}{1 - e^{-\tau/2}} d\tau.$$

*Proof.* Let  $X_{-a}^m = X^m - X_a g_a^m$ . Since the  $X_a$ 's are independent,

$$\begin{aligned} &\mathbf{E} [\langle \nabla P_\tau \check{F}(X^m), X^m \rangle] \\ &= \alpha_m \mathbf{E} \left[ \sum_{a \in \mathcal{A}} X_a \langle \nabla P_\tau \check{F}(X^m), g_a^m \rangle \right] \\ &= \alpha_m \mathbf{E} \left[ \sum_{a \in \mathcal{A}} X_a \langle \nabla P_\tau \check{F}(X^m) - \nabla P_\tau \check{F}(X_{-a}^m), g_a^m \rangle \right] \\ &= \alpha_m^2 \mathbf{E} \left[ \sum_{a \in \mathcal{A}} X_a^2 \langle \nabla^{(2)} P_\tau \check{F}(X_{-a}^m), g_a^m \otimes g_a^m \rangle \right] \\ &\quad + \alpha_m^3 \mathbf{E} \left[ \sum_{a \in \mathcal{A}} X_a^3 \int_0^1 (1-r) \langle \nabla^{(3)} P_\tau \check{F}(X_{-a}^m + r X_a g_a^m), g_a^{m \otimes 3} \rangle dr \right], \end{aligned}$$

according to the Taylor formula. Since  $\mathbf{E} [X_a^2] = 1$ , we have

$$\begin{aligned} &\mathbf{E} [LP_\tau F(X^m)] \\ &= -\alpha_m^2 \mathbf{E} \left[ \sum_{a \in \mathcal{A}} \langle \nabla^{(2)} P_\tau \check{F}(X_{-a}^m) - \nabla^{(2)} P_\tau \check{F}(X^m), g_a^m \otimes g_a^m \rangle \right] \\ &\quad + \alpha_m^3 \mathbf{E} \left[ \sum_{a \in \mathcal{A}} X_a^3 \int_0^1 (1-r) \langle \nabla^{(3)} P_\tau \check{F}(X_{-a}^m + r \alpha_m X_a g_a^m), g_a^{m \otimes 3} \rangle dr \right] \\ &:= -\alpha_m^2 \frac{e^{-3\tau/2}}{\beta_{\tau/2}^2} A_1 + \alpha_m^3 \frac{e^{-5\tau/2}}{\beta_{\tau/2}^2} A_2. \end{aligned}$$

In view of (6),

$$\begin{aligned} A_1 &= \sum_{a \in \mathcal{A}} \mathbf{E} \left[ \iint [\check{F}(X^m(\tau, y, z)) - \check{F}(X_{-a}^m(\tau, y, z))] \right. \\ &\quad \left. \times (y, g_a^m)_H(z, g_a^m)_H d\mathbf{P}^m(y) d\mathbf{P}^m(z) \right]. \end{aligned}$$

Since  $F$  is Lipschitz continuous, we get

$$|A_1| \leq \alpha_m \sum_{a \in \mathcal{A}} B_1^a + \alpha_m \sum_{a \in \mathcal{A}} B_2^a,$$

where

$$B_1^a = \iint \mathbf{E} \left[ |X^m(\tau, y, z) - X_{-a}^m(\tau, y, z)|_{W_{\eta,p}} \right] \\ \times (y, g_a^m)_H(z, g_a^m)_H \, d\mathbf{P}^m(y) \, d\mathbf{P}^m(z) \\ B_2^a = \iint \mathbf{E} \left[ \left( \iint_{[0,1]^2} \left| \pi_2(S_2(X^m(\tau, y, z)) - S_2(X_{-a}^m(\tau, y, z))) \right|_{s,t} \right)^{p/2} \right. \\ \left. d\mu_{\eta,p}(t, s) \right]^{1/p} \times (y, g_a^m)_H(z, g_a^m)_H \, d\mathbf{P}^m(y) \, d\mathbf{P}^m(z).$$

Since  $g_a^m$  has unit norm in  $W_{\eta,p}$ , we readily have

$$B_1^a \leq e^{-\tau} \mathbf{E} [|X_a|] \left( \int |(y, g_a^m)_H| \, d\mathbf{P}^m(y) \right)^2 \leq c e^{-\tau} \mathbf{E} [|X_a|^p]^{1/p}.$$

Now then,

$$\left( \pi_2 \check{S}_2(X^m(\tau, y, z)) - \pi_2 \check{S}_2(X_{-a}^m(\tau, y, z)) \right)_{s,t} \\ = X_a(\tau, y, z) \sum_{b \prec a} X_b(\tau, y, z) \check{g}_{a,b}^m(s, t).$$

Apply Hölder inequality,

$$|B_2^a| \leq \iint \left( \iint_{[0,1]^2} \mathbf{E} \left[ \left| X_a(\tau, y, z) \sum_{b \prec a} X_b(\tau, y, z) \check{g}_{a,b}^m(s, t) \right|^{p/2} \right] d\mu_{\eta,p}(t, s) \right)^{1/p} \\ \times |(y, g_a^m)_H| |(z, g_a^m)_H| \, d\mathbf{P}^m(y) \, d\mathbf{P}^m(z).$$

For  $y$  and  $z$  fixed, the family  $(X_a(\tau, y, z), a \in \mathcal{A})$  satisfies the hypothesis of Lemma 4.3, hence

$$|B_2^a| \leq c \iint \mathbf{E} \left[ |X_a(\tau, y, z)|^{p/2} \right]^{2/p} |(y, g_a^m)_H| |(z, g_a^m)_H| \, d\mathbf{P}^m(y) \, d\mathbf{P}^m(z) \\ \leq c \|X_a\|_{L^{p/2}}.$$

It follows that for any  $a \in \mathcal{A}$ ,  $B_1^a$  and  $B_2^a$  are bounded with respect to  $m$  by a constant times the  $p$ -th moment of  $X_a$  to the square. Since  $|\mathcal{A}| = dm$ ,  $A_1$  is bounded by  $m$  times this upper-bound. In view of (7), the same kind of computations can be done for  $A_2$ . Combining these upper-bounds, we get the existence of a constant  $c$  such that for any  $\tau \geq \tau_0$ ,

$$|\mathbf{E} [LP_\tau \check{F}(X^m)]| \leq c \|X_a\|_{L^{p/2}} \frac{e^{-5\tau/2}}{\beta_{\tau/2}^2} m \alpha_m^3.$$

Since  $\alpha_m = m^{-1/2+\eta}$ , the result follows.  $\square$

**Theorem 4.5.** For any  $\tau_0 > 0$ ,

$$|\mathbf{E} [P_{\tau_0} \check{F}(X^m) - \check{F}(X^m)]| \leq c \|X_a\|_{L^p} m^{-1/2+\eta} \sqrt{1 - e^{-\tau_0}}.$$



*Proof.*

$$\begin{aligned} & |\mathbf{E} [P_{\tau_0} \check{F}(X^m) - \check{F}(X^m)]| \\ & \leq \mathbf{E} \left[ \int \left| \check{F}(e^{-\tau_0} X^m + \beta_{\tau_0} y) - \check{F}(X^m) \right| d\mathbf{P}^m(y) \right] \\ & \leq \alpha_m \mathbf{E} \left[ |(1 - e^{-\tau_0})X^m + \beta_{\tau_0} y|_{W_{\eta,p}} d\mathbf{P}^m(y) \right] \\ & \quad + \alpha_m \mathbf{E} \left[ \int \left( \iint_{[0,1]^2} U(X^m)_{s,t} d\mu_{\eta,p}(s,t) \right)^{1/p} d\mathbf{P}^m(y) \right], \end{aligned}$$

where

$$U(X^m)_{s,t} = \left| \pi_2(\check{S}_2(\beta_{\tau_0}^2 X^m + \beta_{\tau_0} y)_{s,t} - \check{S}_2(X^m)_{s,t}) \right|^p.$$

Thus,

$$\begin{aligned} & |\mathbf{E} [P_{\tau_0} \check{F}(X^m) - \check{F}(X^m)]| \\ & \leq c \beta_{\tau_0} \alpha_m \left( \|X^m\|_{W_{\eta,p}} + \int |y| d\mathbf{P}^m(y) \right) \\ & \leq c \|X_a\|_{L^p} m^{-1/2+\eta} \sqrt{1 - e^{-\tau_0}}, \end{aligned}$$

according to Corollary 2.3.  $\square$

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