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► **To cite this version:**

Marine Guémas, Antoine Sellier, Franck Pigeonneau. Gravity-driven migration of bubbles and/or solid particles near a free surface. International Conference on Boundary Element and Meshless Techniques XIV, Jul 2013, Palaiseau, France. hal-01519287

**HAL Id: hal-01519287**

<https://minesparis-psl.hal.science/hal-01519287v1>

Submitted on 6 May 2017

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# Gravity-driven migration of bubbles and/or solid particles near a free surface

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**Keywords:** Bubble, free surface, surface tension, Stokes flow, Boundary-integral equation, film drainage.

## Abstract

We investigate the challenging problem of bubble(s) and rigid particle(s) interacting near a free surface. The time-dependent bubble(s) and free surface shapes are determined for a large range of Bond number by solving the creeping flow induced by the bubble(s) and the particle(s) motion. This work extends the boundary-integral formulation handled in a recent work solely dealing with bubble(s) ascending toward a free surface.

## 1. Introduction

The gravity-driven motion of bubble(s) interacting with solid particle(s) in a viscous liquid in presence of a free surface is of high interest in applications such as geophysics, chemistry, glass process, . . . This task is quite involved due to the interactions occurring between the different solid and evolving surfaces. The *axisymmetric* gravity-driven migration of bubble(s) ascending toward a free surface has been numerically investigated either for bubble with equal surface tension in [4] or unequal surface tension in [1]. In contrast, this work considers, still for axisymmetric geometry, the more-involved case of cluster made of both bubble(s) and solid particle(s). This problem is solved adopting regularized and carefully-selected boundary-integral equations enforced on the entire liquid domain.

## 2. Challenging time-dependent problem

### 2.1 Assumptions and relevant axisymmetric quasi-steady Stokes flow

We consider a cluster made of  $M \geq 0$  bubbles  $B_m$  and/or  $N \geq 0$  solid particles  $\mathcal{P}_n$  with  $M + N \geq 1$  immersed in a Newtonian fluid with uniform density  $\rho$  and viscosity  $\mu$ . This liquid is bounded by a free surface and both the cluster and the liquid are subject to the uniform gravity  $\mathbf{g} = -g\mathbf{e}_3$  (with  $g > 0$ ). The bubble  $B_m$ , the solid particle  $\mathcal{P}_n$  and the free surface have smooth and time-dependent surfaces  $S_m(t)$  with uniform surface tension  $\gamma_n$ ,  $\Sigma_n(t)$  and  $S_0(t)$  with uniform surface tension  $\gamma_0$ , respectively.

As illustrated in Figure 1 for  $M = N = 1$ , all surfaces  $S_0(t)$ ,  $S_m(t)$  and /or  $\Sigma_n(t)$  admit unit normal vector  $\mathbf{n}$  directed into the liquid domain  $\mathcal{D}(t)$  and *the same* axis of revolution ( $O, \mathbf{e}_3$ ) (axisymmetric problem).

As the cluster migrates under the gravity, the shapes of the bubble(s) and free surface evolve in time. At initial time, each bubble is spherical with typical radius  $a$  and the free surface is the  $z = 0$  plane. At any time  $t$ , the pressure  $p_0$  above the disturbed free surface  $S_0(t)$  and  $p_m$  inside the disturbed bubble  $B_m$  are assumed to be constant. Each solid particle  $\mathcal{P}_n$  with uniform density  $\rho_n$  has, for symmetry reasons, time-dependent velocity  $U^{(n)}(t)\mathbf{e}_3$ . In addition, the liquid flow has pressure  $p + \rho\mathbf{g}\cdot\mathbf{x}$  (here  $\mathbf{x} = OM$  with  $O$  denoting the origin of our Cartesian coordinates) velocity  $\mathbf{u}$  and stress tensor  $\boldsymbol{\sigma}$ . We denote by  $a$  the bubble(s) and solid particles typical length scale and by  $V$  the typical magnitude of velocities  $\mathbf{u}$  and  $U^{(n)}(t)$ . Assuming that  $\text{Re} = \rho V a / \mu \ll 1$ , inertial effects are negligible

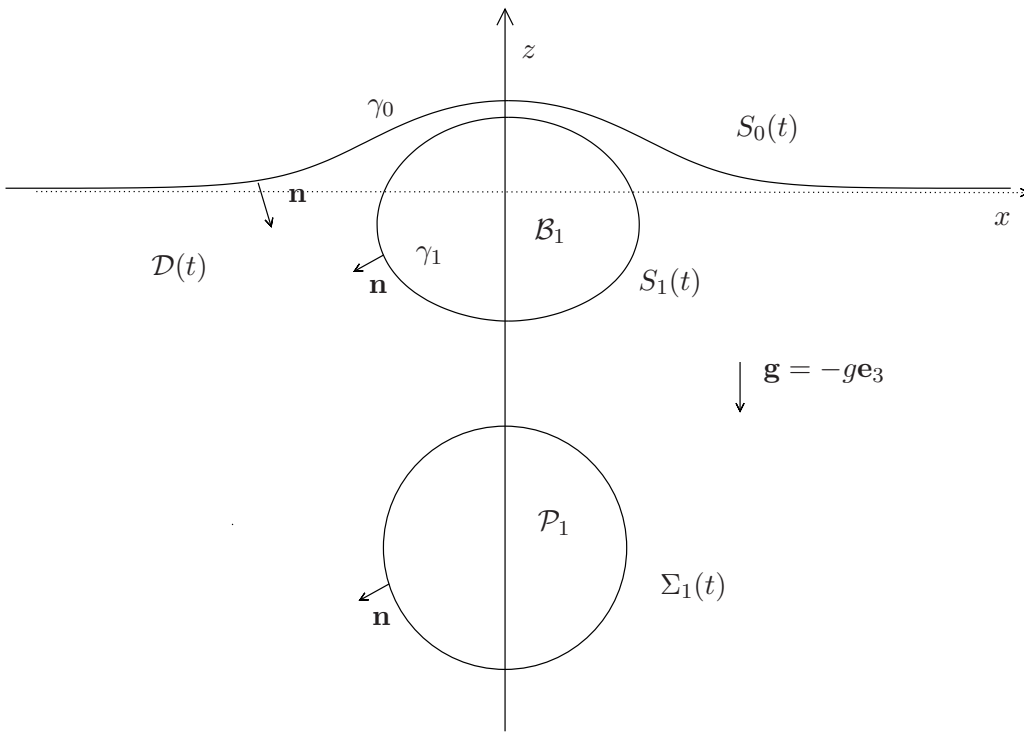


Figure 1: One bubble  $B_1$  and a solid sphere  $\mathcal{P}_1$  moving near a free surface  $S_0(t)$ .

and the flow  $(\mathbf{u}, p)$  obeys the following quasi-steady creeping flow equations and boundary conditions

$$\nabla \cdot \mathbf{u} = 0 \text{ and } \mu \nabla^2 \mathbf{u} = \mathbf{grad} p \text{ in } \mathcal{D}(t), \quad (\mathbf{u}, p) \rightarrow (\mathbf{0}, 0) \text{ as } |\mathbf{x}| \rightarrow \infty, \quad (1)$$

$$\boldsymbol{\sigma} \cdot \mathbf{n} = (\rho \mathbf{g} \cdot \mathbf{x} - p_m + \gamma_m \nabla_S \cdot \mathbf{n}) \mathbf{n} \text{ on } S_m(t) \text{ for } m = 0, \dots, M, \quad (2)$$

$$\mathbf{u} = U^{(n)}(t) \mathbf{e}_3 \text{ on } \Sigma_n(t) \text{ for } n = 1, \dots, N \quad (3)$$

where  $H = [\nabla_S \cdot \mathbf{n}]/2$  is the local average curvature. Assuming bubbles with constant volume, one supplements (1)-(3) with the relations <sup>1</sup>

$$\int_{S_m(t)} \mathbf{u} \cdot \mathbf{n} dS = 0 \text{ on } S_m \text{ for } m=0, \dots, M. \quad (4)$$

For  $N \geq 1$  the velocities  $U^{(n)}(t)$  are unknown. By symmetry, each solid particle  $\mathcal{P}_n$  is torque-free. In addition, each  $\mathcal{P}_n$  with negligible inertia is force-free. This latter property results in the additional conditions

$$\int_{\Sigma_n(t)} \mathbf{e}_3 \cdot \boldsymbol{\sigma} \cdot \mathbf{n} dS = (\rho_n - \rho) \mathcal{V}_n g \text{ for } n = 1, \dots, N \quad (5)$$

where  $\rho_n$  and  $\mathcal{V}_n$  designate the uniform density and volume of the particle  $\mathcal{P}_n$ .

The material surface(s)  $S_m(t)$  have velocity  $\mathbf{V}$ . Since there is no mass transfer across the surfaces  $S_m(t)$ , one has

$$\mathbf{V} \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n} \text{ on } S_m \text{ for } m = 0, \dots, M. \quad (6)$$

## 2.2 Proposed tracking algorithm for the time-dependent entire liquid boundary

We compute the time-dependent shape of the free surface, the bubble(s) and particle(s) surface(s) by running at each time  $t$  the following steps :

<sup>1</sup>Note that (4) indeed also holds for  $m = 0$  because  $\mathbf{u}$  is divergence-free and  $\mathbf{u} \rightarrow \mathbf{0}$ .

Step 1: From the knowledge at time  $t$  of the liquid domain  $\mathcal{D}(t)$ , one first computes the quantity  $\nabla_S \cdot \mathbf{n}$  on each surface  $S_m(t)$ .

Step 2: One then solves at time  $t$  the relations (1)-(5) to get the unknown velocities  $U^{(n)}(t)$  and the fluid velocity  $\mathbf{u}$  on each surface  $S_m(t)$ .

Step 3: The liquid boundary  $\mathcal{D}(t + dt)$  at time  $t + dt$  is obtained by moving between times  $t$  and  $t + dt$  each surfaces  $S_m$  by exploiting the relation (6) and each solid surface  $\Sigma_n$  at the velocity  $U^{(n)}(t)\mathbf{e}_3$ .

One should note that for such a procedure the following issues are of the utmost importance:

- (i) To accurately compute the local average curvature  $(\boldsymbol{\sigma} \cdot \mathbf{n})/2$  on each surface  $S_m$  in Step 1.
- (ii) To efficiently and accurately solve the Stokes problem (1)-(5) in Step 2.
- (iii) To adequately select a time step at in Step 3.

This work introduces a suitable treatment to cope with the previous issue (ii).

### 3. Advocated method

This section presents a new procedure to appropriately solve the problem (1)-(5).

#### 3.1 Auxiliary Stokes flows for cluster involving at least one solid particle.

As soon as  $N \geq 1$ , each velocity  $U^{(n)}$  occurring in (1)-(5) is unknown. Fortunately, it is possible to determine  $U^{(1)}(t), \dots, U^{(N)}(t)$  prior to obtain the liquid flow  $(\mathbf{u}, p)$  ! The trick consists in introducing, for  $n = 1, \dots, N$ , auxiliary Stokes flows  $(\mathbf{u}^{(n)}, p^{(n)})$  obtained without stress on each  $S_m$  and when each solid surface  $\Sigma_q$  is motionless for  $q \neq m$  with the surface  $\Sigma_n$  of  $\mathcal{P}_q$  which translates at the velocity  $\mathbf{e}_3$ . In other words, the flow  $(\mathbf{u}^{(n)}, p^{(n)})$  with stress tensor  $\boldsymbol{\sigma}^{(n)}$  satisfies (1) and the following boundary conditions

$$\mathbf{u}^{(n)} = \delta_{nq}\mathbf{e}_3 \text{ on } \Sigma_q \text{ for } q = 1, \dots, N \quad (7)$$

$$\boldsymbol{\sigma}^{(n)} \cdot \mathbf{n} = \mathbf{0} \text{ on } S_m \text{ for } m = 0, \dots, M. \quad (8)$$

In addition, one supplements (1), (7)-(8) with the additionnal conditions

$$\int_{S_m(t)} \mathbf{u}^{(n)} \cdot \mathbf{n} dS = 0 \text{ on } S_m \text{ for } m = 0, \dots, M. \quad (9)$$

Denoting by  $\partial\mathcal{D}$  the liquid boundary, the reciprocal identity [2] for the flows  $(\mathbf{u}, p)$  and  $(\mathbf{u}^{(n)}, p^{(n)})$  reads

$$\int_{\partial\mathcal{D}} \mathbf{u}^{(n)} \cdot \boldsymbol{\sigma} \cdot \mathbf{n} dS = \int_{\partial\mathcal{D}} \mathbf{u} \cdot \boldsymbol{\sigma}^{(n)} \cdot \mathbf{n} dS. \quad (10)$$

Enforcing the relations (5) by exploiting the aforementioned identity (10) and the boundary conditions (2)-(3) and (7)-(8), one then arrives at the  $N$ -equation linear system

$$\begin{aligned} \sum_{q \geq 1} \left( \int_{\Sigma_q} \mathbf{e}_3 \cdot \boldsymbol{\sigma}^{(n)} \cdot \mathbf{n} dS \right) U^{(q)}(t) &= (\rho - \rho_n) \mathcal{V}_n g \\ &+ \sum_{m \geq 0} \int_{S_m} \mathbf{u}^{(n)} \cdot (\rho \mathbf{g} \cdot \mathbf{x} - p_m + \gamma_m \nabla_S \cdot \mathbf{n}) \mathbf{n} dS \quad \text{for } n = 1, \dots, N. \end{aligned} \quad (11)$$

Furthermore, the pressure  $p_m$  is uniform in the bubble  $B_m$  which leads, in conjunction with (5), to

$$\begin{aligned} \sum_{q \geq 1} \left( \int_{\Sigma_q} \mathbf{e}_3 \cdot \boldsymbol{\sigma}^{(n)} \cdot \mathbf{n} dS \right) U^{(q)}(t) &= (\rho - \rho_n) \mathcal{V}_n g \\ &+ \sum_{m \geq 0} \int_{S_m} \mathbf{u}^{(n)} \cdot (\rho \mathbf{g} \cdot \mathbf{x} + \gamma_m \nabla_S \cdot \mathbf{n}) \mathbf{n} dS \quad \text{for } n = 1, \dots, N. \end{aligned} \quad (12)$$

It is possible (and here admitted) to prove, invoking the energy dissipation in Stokes flow, that (12) is well-posed (i. e. presents a non-singular matrix). Note that, one solely needs to evaluate the surface quantities  $\mathbf{u}^{(n)}$  on each  $S_m$  and  $\boldsymbol{\sigma}^{(n)} \cdot \mathbf{n}$  on each  $\Sigma_q$  to obtain the translational velocity  $U^{(q)}(t) \mathbf{e}_3$  of the particle  $\mathcal{P}_q$ . As shown in the next subsection, those required key surface quantities  $\mathbf{u}^{(n)}$  and  $\boldsymbol{\sigma}^{(n)} \cdot \mathbf{n}$  are calculated by inverting relevant boundary-integral equations on the entire liquid boundary  $\partial \mathcal{D}$ .

## 3.2 Relevant boundary-integral equations

### 3.2.1 Three-dimensionnal formulation

For a Stokes flow  $(\mathbf{u}, p)$  with stress tensor  $\boldsymbol{\sigma}$  obeying (1) with prescribed values of the stress  $\boldsymbol{\sigma} \cdot \mathbf{n}$  on each  $S_m$  and of the velocity  $\mathbf{u}$  on each  $\Sigma_n$ , one has the key coupled *regularized* boundary-integral equations (see for instance [5]),

$$\begin{aligned} -8\mu\pi\mathbf{u}(\mathbf{x}_0) + \sum_{m \geq 0} \int_{S_m} \mu[\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}_0)] \cdot \mathbf{T}(\mathbf{x}, \mathbf{x}_0) \cdot \mathbf{n}(\mathbf{x}) dS - \sum_{n \geq 1} \int_{\Sigma_n} \mathbf{G}(\mathbf{x}, \mathbf{x}_0) \cdot \boldsymbol{\sigma} \cdot \mathbf{n}(\mathbf{x}) dS \\ = \sum_{m \geq 0} \int_{S_m} \mathbf{G}(\mathbf{x}, \mathbf{x}_0) \cdot \boldsymbol{\sigma} \cdot \mathbf{n}(\mathbf{x}) dS \quad \text{for } \mathbf{x}_0 \text{ on } S_m \end{aligned} \quad (13)$$

and

$$\begin{aligned} \sum_{m \geq 0} \int_{S_m} \mu[\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}_0)] \cdot \mathbf{T}(\mathbf{x}, \mathbf{x}_0) \cdot \mathbf{n}(\mathbf{x}) dS - \sum_{n \geq 1} \int_{\Sigma_n} \mathbf{G}(\mathbf{x}, \mathbf{x}_0) \cdot \boldsymbol{\sigma} \cdot \mathbf{n}(\mathbf{x}) dS \\ = +8\mu\pi\mathbf{u}(\mathbf{x}_0) + \sum_{m \geq 0} \int_{S_m} \mathbf{G}(\mathbf{x}, \mathbf{x}_0) \cdot \boldsymbol{\sigma} \cdot \mathbf{n}(\mathbf{x}) dS \quad \text{for } \mathbf{x}_0 \text{ on } \Sigma_n \end{aligned} \quad (14)$$

where the second-rank tensor  $\mathbf{G}$  and third-rank stress tensor  $\mathbf{T}$  are defined as [3]

$$\mathbf{G}(\mathbf{x}, \mathbf{x}_0) = \frac{\mathbf{I}}{|\mathbf{x} - \mathbf{x}_0|} + \frac{(\mathbf{x} - \mathbf{x}_0) \otimes (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|^3}; \quad (15)$$

$$\mathbf{T}(\mathbf{x}, \mathbf{x}_0) = -6 \frac{(\mathbf{x} - \mathbf{x}_0) \otimes (\mathbf{x} - \mathbf{x}_0) \otimes (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|^5}. \quad (16)$$

with  $\mathbf{I}$  the identity tensor. Clearly, solving (13)-(14) permits one to get the unknown vectors  $\mathbf{u}$  on  $S_m$  and  $\boldsymbol{\sigma} \cdot \mathbf{n}$  on  $\Sigma_n$  from the knowledge of  $\mathbf{u}$  on  $\Sigma_n$  and  $\boldsymbol{\sigma} \cdot \mathbf{n}$  on  $S_m$ .

### 3.2.2 Axisymmetric formulation

Since we restrict the analysis to the axisymmetric configuration depicted in Fig.1, we adopt cylindrical coordinates  $(r, \phi, z)$  with  $r = \sqrt{x^2 + y^2}$ ,  $z = x_3$  and  $\phi$  the azimuthal angle in the range  $[0, 2\pi]$ . We set  $\mathbf{u} = u_r \mathbf{e}_r + u_z \mathbf{e}_z = u_\alpha \mathbf{e}_\alpha$  (with  $\alpha = r, z$ ),  $\mathbf{f} = \boldsymbol{\sigma} \cdot \mathbf{n} = f_r \mathbf{e}_r + f_z \mathbf{e}_z = f_\alpha \mathbf{e}_\alpha$  and  $\mathbf{n} = n_r \mathbf{e}_r + n_z \mathbf{e}_z = n_\alpha \mathbf{e}_\alpha$  and introduce the traces  $\mathcal{L}_n$  of  $\Sigma_n$  and  $\mathcal{L}_m$  of  $S_m$  in the  $\phi = 0$  half plane.

Integrating over  $\phi$  the equations (13)-(14), then yields the equivalent coupled boundary equations

$$\begin{aligned} -8\pi u_\alpha(\mathbf{x}_0) + \sum_{m \geq 0} \int_{\mathcal{L}_m} \mu[u_\beta(\mathbf{x}) - u_\beta(\mathbf{x}_0)] C_{\alpha\beta}(\mathbf{x}, \mathbf{x}_0) dl - \sum_{n \geq 1} \int_{\mathcal{L}_n} B_{\alpha\beta}(\mathbf{x}, \mathbf{x}_0) f_\beta n_\beta(\mathbf{x}) dl \\ = \sum_{m \geq 0} \int_{\mathcal{L}_m} B_{\alpha\beta}(\mathbf{x}, \mathbf{x}_0) f_\beta n_\beta(\mathbf{x}) dl \quad \text{for } \mathbf{x}_0 \text{ on } \mathcal{L}_m \end{aligned} \quad (17)$$

and

$$\begin{aligned} \sum_{m \geq 0} \int_{\mathcal{L}_\ddagger} \mu[u_\beta(\mathbf{x}) - u_\beta(\mathbf{x}_0)] C_{\alpha\beta}(\mathbf{x}, \mathbf{x}_0) dl - \sum_{n \geq 1} \int_{\mathcal{L}_n} B_{\alpha\beta}(\mathbf{x}, \mathbf{x}_0) f_\beta n_\beta(\mathbf{x}) dl \\ = 8\pi u_\alpha(\mathbf{x}_0) + \sum_{m \geq 0} \int_{\mathcal{L}_m} B_{\alpha\beta}(\mathbf{x}, \mathbf{x}_0) f_\beta n_\beta(\mathbf{x}) dl \quad \text{for } \mathbf{x}_0 \text{ on } \mathcal{L}_n \end{aligned} \quad (18)$$

for  $\alpha = r, z$ , the differential arc length  $dl$  in the  $\phi = 0$  plane and the so-called single-layer and double-layer  $2 \times 2$  square matrices  $B_{\alpha\beta}(\mathbf{x}, \mathbf{x}_0)$  and  $C_{\alpha\beta}(\mathbf{x}, \mathbf{x}_0)$  given in Pozrikidis [5]. Note that a summation over  $\beta = r, z$  holds in (17)-(18).

### 3.2.3 Resulting boundary-integral equations for the axisymmetric Stokes flow problem

Dealing with our axisymmetric problem (1)-(4), we first evaluate for each axisymmetric flow  $(\mathbf{u}^{(n)}, p^{(n)})$  the needed vectors  $\mathbf{u}^{(n)} = u_\alpha^{(n)} \mathbf{e}_\alpha$  on each  $S_m$  and  $\boldsymbol{\sigma}^{(n)} = f_\beta^{(n)} \mathbf{e}_\beta$  on each  $\Sigma_q$ . We perform this calculation by inverting (17)-(18) for  $u_z^{(n)} = \delta_{nq}$  and  $u_r^{(n)} = 0$  on each  $\Sigma_q$  and  $\boldsymbol{\sigma}^{(n)} \cdot \mathbf{n} = 0$  on each  $S_m$ . Once both the velocity and stress vectors are known on each surfaces  $\Sigma_q$  and  $S_m$ , one then obtains each velocity  $U^{(q)}(t)$  by solving the linear system (12).

Finally, we gain the required velocity  $\mathbf{u} = u_\alpha \mathbf{e}_\alpha$  on each  $S_m$  by inverting one more time (17)-(18) using the boundary conditions (2)-(3), i. e.

$$\begin{aligned} -8\pi u_\alpha(\mathbf{x}_0) + \sum_{m \geq 0} \int_{\mathcal{L}_m} \mu[u_\beta(\mathbf{x}) - u_\beta(\mathbf{x}_0)] C_{\alpha\beta}(\mathbf{x}, \mathbf{x}_0) dl - \sum_{n \geq 1} \int_{\mathcal{L}_n} B_{\alpha\beta}(\mathbf{x}, \mathbf{x}_0) f_\beta n_\beta(\mathbf{x}) dl \\ = \sum_{m \geq 0} \int_{\mathcal{L}_m} B_{\alpha\beta}(\mathbf{x}, \mathbf{x}_0) [\rho \mathbf{g} \cdot \mathbf{x} + \gamma_m \nabla_S \cdot \mathbf{n}] n_\beta(\mathbf{x}) dl \quad \text{for } \mathbf{x}_0 \text{ on } \mathcal{L}_m \end{aligned} \quad (19)$$

and

$$\begin{aligned} \sum_{m \geq 0} \int_{\mathcal{L}_m} \mu[u_\beta(\mathbf{x}) - u_\beta(\mathbf{x}_0)] C_{\alpha\beta}(\mathbf{x}, \mathbf{x}_0) dl - \sum_{n \geq 1} \int_{\mathcal{L}_n} B_{\alpha\beta}(\mathbf{x}, \mathbf{x}_0) f_\beta n_\beta(\mathbf{x}) dl = +8\pi U^{(n)}(t) \mathbf{e}_3(\mathbf{x}_0) \\ + \sum_{m \geq 0} \int_{\mathcal{L}_m} B_{\alpha\beta}(\mathbf{x}, \mathbf{x}_0) [\rho \mathbf{g} \cdot \mathbf{x} + \gamma_m \nabla_S \cdot \mathbf{n}] n_\beta(\mathbf{x}) dl \quad \text{for } \mathbf{x}_0 \text{ on } \mathcal{L}_n \end{aligned} \quad (20)$$

for  $\alpha = r, z$  and  $\gamma_m$  uniform on each surface  $S_m$ . In summary, our approach consists, for  $N \geq 1$  solid particle(s), in inverting  $N + 1$  boundary-integral equations (17)-(18).

## 4. Numerical method

The coupled boundary-integral equation (17)-(18) are numerically inverted by appealing to the following key steps (see for further details [4, 1]):

(i) First, the entire contour  $\mathcal{L} = \mathcal{L}_n \cup \mathcal{L}_n$  is divided into  $N_e$  curved boundary elements with the  $\mathcal{L}_0$  truncated free surface. Each boundary element has  $N_c$  collocation points spread with a uniform distribution. An isoparametric approximation is used for the components  $\mathbf{u}$  and  $\mathbf{f} = \boldsymbol{\sigma} \cdot \mathbf{n}$  on each boundary element.

For the nodes located on  $\mathcal{L}_m$ , the vectors  $\mathbf{U}$  and  $\mathbf{F}_d$  collect the unknown and prescribed components of  $\mathbf{u}$  and  $\mathbf{f}$ . In a similar fashion,  $\mathbf{F}$  and  $\mathbf{U}_d$  are the vectors associated with the unknown and given values of  $\mathbf{f}$  and  $\mathbf{u}$  at the nodes of the solid contours  $\mathcal{L}_n$ . Finally, once the coupled boundary-integral equations (17)-(18) are discretized, these vectors satisfy indeed the  $2N_e N_c$ -equation linear system

$$\mathbf{U} + \mathbf{C} \cdot \mathbf{U} - \mathbf{B}_1 \cdot \mathbf{F} = -\mathbf{B}_2 \cdot \mathbf{F}_d \quad \text{for } \mathbf{x}_0 \text{ on } \cup_{m \geq 0} \mathcal{L}_m, \quad (21)$$

$$\mathbf{C} \cdot \mathbf{U} - \mathbf{B}_1 \cdot \mathbf{F} = -\mathbf{U}_d + \mathbf{B}_2 \cdot \mathbf{F}_d \quad \text{for } \mathbf{x}_0 \text{ on } \cup_{n \geq 1} \mathcal{L}_n. \quad (22)$$

The matrices  $\mathbf{B}_1$ ,  $\mathbf{B}_2$  and  $\mathbf{C}$  involve integrations of the quantities  $B_{\alpha\beta}$  and  $C_{\alpha\beta}$  introduced in §3.2.2 over the entire contours  $\cup_{m \geq 0} \mathcal{L}_m$  and  $\cup_{n \geq 1} \mathcal{L}_n$ .

(ii) One finds the solution  $(\mathbf{U}, \mathbf{F})$  of (21)-(22) by Gaussian elimination.

(iii) The shape of each surface  $S_m$  and the position of each  $\Sigma_n$  if  $N \geq 1$  is tracked in time using the boundary condition (6) and solving the equation  $d\mathbf{x}/dt = \mathbf{u}(\mathbf{x}, t)$  for each nodal point. A Runge-Kutta-Fehlberg method performs this task using a time-step selected by controlling the errors for the second and third-order schemes. Furthermore, as the distance between two surfaces tends to zero, the adjusted time step is then very small and the computations is stopped.

## 5. Conclusions

Preliminary numerical results will be exposed at the oral presentation for a cluster made of one bubble and one spherical solid particle. Furthermore, this particular case will be discussed and compared with the two-bubbles configurations studied in [4, 1]

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