



**HAL**  
open science

# Fluid-Solid Coupling by Conservative Interpolation Methods

Chahrazade Bahbah

► **To cite this version:**

Chahrazade Bahbah. Fluid-Solid Coupling by Conservative Interpolation Methods. [Research Report] Ecole Nationale Supérieure des Mines de Paris. 2017, pp.21. hal-01497711

**HAL Id: hal-01497711**

**<https://minesparis-psl.hal.science/hal-01497711>**

Submitted on 7 Apr 2017

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

**ECOLE NATIONALE SUPERIEURE DES MINES DE  
PARIS**

**CENTRE DE MISE EN FORME DES MATERIAUX**



# **Fluid-Solid Coupling by Conservative Interpolation Methods**

Chahrazade BAHBAH

February 2nd, 2017

## ABSTRACT

---

In numerical simulations, the transfer of fields between the different meshes is a key step. Interpolation is probably the most popular method for transferring data between meshes. Actually, it must ensure the consistency, the continuity and the accuracy of the solutions among the meshes. We present in this research report several conservative interpolation methods from a donor mesh to a target mesh. We start by introducing the basics of the transfer of fields, then we present the common mapping methods used nowadays and finally we give a detailed comparative study of the conservative interpolation methods found in the literature.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Common mapping methods</b>	<b>3</b>
2.1	Linear interpolation . . . . .	4
2.2	Moving least-squares approximation . . . . .	5
2.3	Patch recovery methods . . . . .	5
2.4	Conclusions and remarks . . . . .	6
<b>3</b>	<b>Conservative interpolation methods</b>	<b>7</b>
3.1	Conservative interpolation via intermediate mesh building . . . . .	7
3.1.1	$P_1$ -conservative technique by mesh intersection . . . . .	7
3.1.2	Interpolation via common-refinement or supermesh construction . . . . .	8
3.2	Conservative redistribution via local remapping . . . . .	10
3.3	Matrix based conservative interpolation with restrictions . . . . .	11
<b>4</b>	<b>Conclusion and perspectives</b>	<b>14</b>
<b>5</b>	<b>Annex : Localization algorithms</b>	<b>15</b>

---

# 1 Introduction

Thermal Treatment describes the multifaceted operations in heating furnaces and quenching tanks, performed on a material in the solid state, for the purpose of altering its microstructure and properties. The output of this step is the input of all the following manufacturing steps such as forging, rolling processes and even the prediction of microstructure evolution. Therefore, any lack of control in this upstream operation will affect the global manufacturing chain, and the consequences are then immediate such as prohibiting better quality, higher availability and adaptability of products. In particular, during quenching process, the boiling phenomena taking place is a concentrate of various physical phenomena (see Fig 1) that cannot be modeled as a simple global heat exchange coefficient. The process realism implies an advanced fluid-solid coupling. This subtle step needs to be efficiently reproduced to reach a predictive and exploitable numerical result.

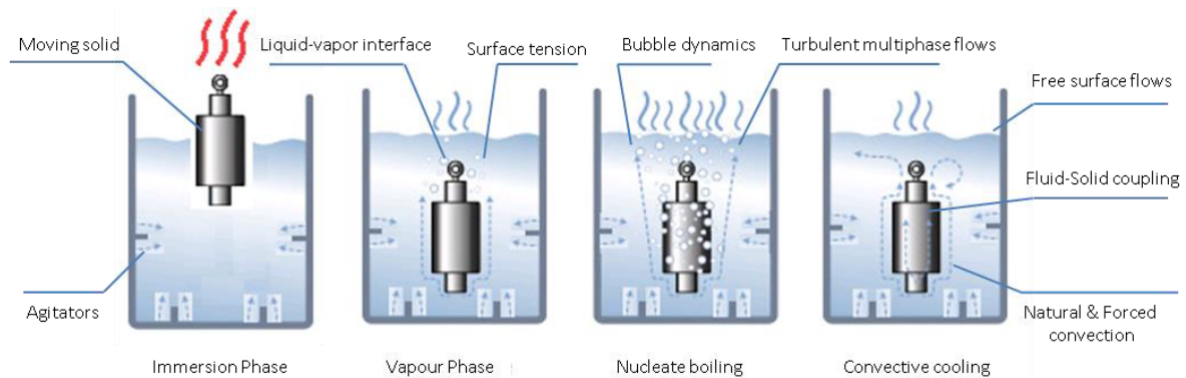


Figure 1: Different steps of the quenching process

Thus, the main purpose of this project is to develop numerical tools in order to simulate the evolution of both the thermal and mechanic properties of the immersed solid during the quenching at an industrial scale. This project will be done in collaboration with Montupet, an industry leader in the manufacture of complex cast aluminium components for the automotive industry worldwide. Recall also that the developed framework will be validated using experimental results given by Montupet.

The initial approach is to consider two domains, see Fig 2. First of all, the fluid-solid domain in which the solid is totally immersed in the fluid. For that, we will refer to the use of our immersed volume framework [1]. Indeed, a full Eulerian framework that simulates the quenching process has been established. It takes into account three ingredients that can be resumed to : (i) geometric : flexibility for multidomain simulation,(ii) fluid mechanics: accounting for turbulent boiling, (iii) physics : phase change and phase transformation, [2]. As for the solid part, we recall that the quenching leads to important deformations such as cracks and defects inside the solid, so we want to study the effect of the residual stresses.



Figure 2: Division into two domains : Fluid-Solid and Solid

Montupet is using Zset, a software dedicated to the solid that is able to simulate the heat treatment of mechanical pieces with phase change and analyze the residual stresses. The fluid mechanical part and the turbulent boiling will be handled during my PhD and implemented in the library CimLib-CFD, that handles different fields of applications : finite element solvers for heat transfer simulations ([3], [4]), anisotropic mesh adaptation ([5], [6], [7], [8], [9]) and high parallel computing ([10], [11]). Thus, a coupling between the two codes and therefore interpolation of fields and fluxes of temperature must be done in an accurate and conservative manner.

The first step of this project consists in transferring the fields of the first fluid-solid domain to the solid domain. Indeed, the two domains are completely different and interpolation is required. Moreover, during adaptive numerical simulations, we use the interpolation of the fields not only when we wish to transfer from a donor mesh to a target mesh, but also when we adapt the mesh following the evolution of the physical properties. However, one serious drawback of many mesh adaptivity algorithms on unstructured meshes is the necessity of interpolating solution fields from the initial mesh to the newly adapted mesh. Such interpolation destroys conservation of important physical quantities and leads to errors in the solution fields.

Thus, this bibliographic report will investigate several conservative interpolation methods from a donor mesh to a target mesh. We will start by introducing the basics of the transfer of fields, we will present the common mapping methods used nowadays and then give a detailed comparative study of the conservative interpolation methods found in the literature.

## 2 Common mapping methods

In numerical simulations, the transfer of fields between the different meshes is a key step. Interpolation (and sometimes extrapolation when a point falls outside the range of the source mesh) is probably the most popular method for transferring data between meshes. Actually, it ensures the consistency, the continuity and the accuracy of the solutions among the meshes. While resolving a mechanical problem, we distinguish for instance the nodal fields of type P1 such as velocity, pressure and temperature, and P0 fields defined at the Gauss points (stress and strain tensor), see Fig 3. While resolving the constitutive equations, we

need to transfer the solutions which are defined on various discretisations using the transfer of fields.

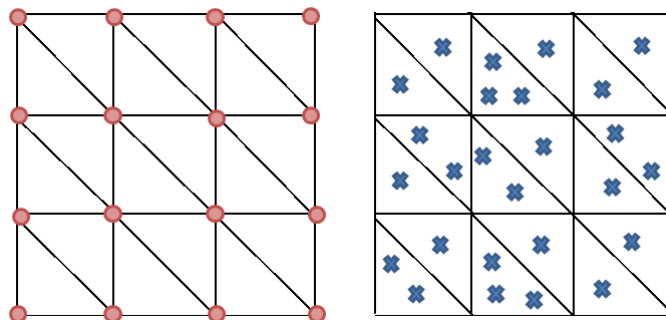
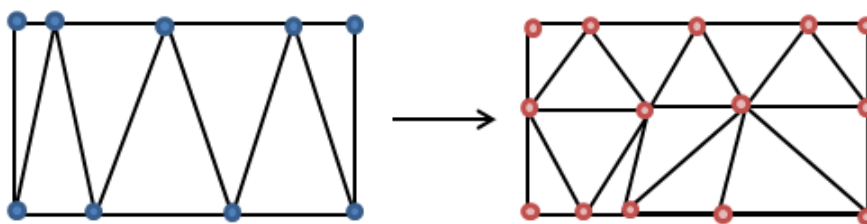
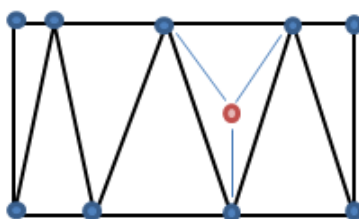


Figure 3: Left: nodal fields - Right: fields defined at Gauss points

It is possible to find in each point of the space, the value of a field discretized in the nodes just by interpolating the nodal values. When applied to the transfer problem, we need to find to which element of the donor mesh  $\mathcal{T}_D$ , the new node of the target mesh  $\mathcal{T}_T$  belongs, this step is called the localization (see Annex in section 5), and then to use interpolation functions in order to compute the value for the new node, see Fig 4.



Representation of the transfer of the fields  
 Left : Donor mesh  $\mathcal{T}_D$  - Right : Target mesh  $\mathcal{T}_T$



Position of the new node in the donor mesh  
 Computation of the solution in this new node with interpolation

Figure 4: Interpolation method applied to a transfer of field

## 2.1 Linear interpolation

The most common interpolation functions are polynomial because they are easier to integrate and to derive, unlike the trigonometric functions that lead to additional computational

time [12]. The choice of the interpolation polynomial is based on the type and the degree of the finite element defined in the initial mesh. Most of the time, the interpolation polynomials are defined throughout the shape functions of the elements but are not necessary of the same degree of the element at issue. Nowadays, the standard interpolation method is the linear interpolation. It is a two-step procedure defined as follows : first of all, one needs to determine the position of the new node in the old mesh; for that, several localization algorithms can be used and will be detailed in the annex of this report (section 5). For each node  $p_T \in \mathcal{T}_T$ , a containing element  $K_D$  is identified in the donor mesh  $\mathcal{T}_D$  using the localization algorithm, and then the solution  $u_D$  is evaluated at the physical location of the target node  $p_T$ . The linear interpolation uses the source functions, it assigns the value at the node  $p_T$  of the target mesh to be :

$$\Pi_1 u(p_T) = \sum_i \Phi_i(p_T) u(p_i) \quad (1)$$

with  $\Pi_1$  the P1-interpolation operator,  $\Phi_i$  the shape functions associated with the source mesh and  $u(p_i)$  the solution defined in the nodes of the element that contains the new vertex.

## 2.2 Moving least-squares approximation

The moving least square method developed by [13] is a method for reconstructing continuous functions from a set of unorganized point samples via the calculation of a weighted least squares measure around the point at which the reconstructed value is requested. The approximation of the exact solution at a point is defined based on its nodal values at a limited number of neighbor points : only the closest points to the current optimum are taken into account. It is computed as follows:

$$u_{app}(x) = p^T(x) a(x) \quad (2)$$

with  $p$  the basis functions and  $a$  the adjusting coefficients that are computed by minimizing a norm of the weighted difference between the estimated values at nodes and the nodal values  $u_i$  (see [14] for details):

$$J(a) = \sum_i w_i(\|x_i - x\|) (p^T(x_i - x) a - u_i)^2 \quad (3)$$

The basic principle of this method is that the influence of a node is governed by a decreasing weighting function  $w_i$ , that is equal to zero outside the domain of influence of the node. The weight function plays an important role, it gives a local character to the approximation by influencing the way that the coefficients  $a_i$  depend on the location of the designed point  $x$ .

## 2.3 Patch recovery methods

In finite element method, the use of numerical integration, approximation and interpolation leads to an accumulation of the errors. We recall the concept of superconvergent patch recovery methods (SPR) that was first introduced by Zienkiewicz and Zhu [15],[16] in order to estimate the errors made when using a finite element method. The use of a unique interpolation polynomial coupled with a patch recovering procedure was suggested. The basic principle of this approach consists in recovering the value of the nodal fields by least square



fit method and then interpolating the nodal values using standard shape functions. For that, an element patch is defined at each node (see Fig 5), it contains all the elements to which the node belongs to. For each node, an improved solution is computed by determining a polynomial expansion over the patch.

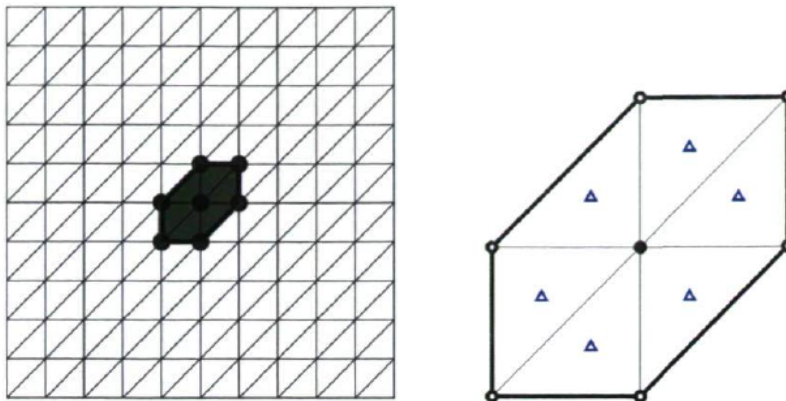


Figure 5: Top, Example of an element patch in a regular mesh. Bottom : Sampling points (triangles) on a patch, Right P1, Left P2

However, for a given patch, only the values on the integration points are conserved; the derivatives values over the boundaries are not taken into account. For that, when an interpolation point is located in the intersection between two patches, a weighted average procedure of the different values must be done. Moreover, an error estimate is defined in order to minimize the error between the finite element solution gradient and the new improved solution obtained using a least square projection of the gradient. A study developed by Zhang [17] shows that the patch recovery technique of Zienkiewicz and Zhu gives ultra convergent results when finite element spaces have the same order and local uniform meshes are used. Over the years, several techniques inspired from the SPR method were proposed. For instance, the Recovery by Equilibrium in patches (REP) proposed by Boroomand and Zienkiewicz [18], which avoids the use of superconvergent points and uses a weighted form of equilibrium equation to produce recovered solution. Gu and al. [19] increased the robustness and the accuracy of the SPR approach for non linear problems using integration points as sampling points and introducing additional nodes. The stress recovery method was later used to recover nodal fields from integration points in order to transfer data between a donor and a target mesh [20]. Kumar and Forment [21] proposed a consistent technique easier to implement in a parallel environment that deals with the boundary points with the same order of accuracy as the interior points.

## 2.4 Conclusions and remarks

In fluid-solid interaction applications, when we wish to interpolate a field, the data transfer must be numerically accurate and physically conservative. However, the mapping methods listed above suffer from many drawbacks. First of all, conservation : the integral of the interpolant on the target mesh is not the same as the integral of the field on the donor mesh :

$$\int_{\mathcal{T}_D} u \neq \int_{\mathcal{T}_T} \pi_1 u \quad (4)$$

Due to the accumulation of numerical errors during interpolation, the conservation of mass and energy is not necessarily respected which is crucial for industrial applications.

Secondly, the maximum and the minimum of the solution will be lost during interpolation :

$$\min_{q \in \mathcal{T}_D} u(q) \leq \Pi_1 u(p_T) \leq \max_{q \in \mathcal{T}_D} u(q) \quad (5)$$

Therefore, we will focus in the following section on the various methods found in the literature that answer the problem of conservation and maximum principle.

### 3 Conservative interpolation methods

Several conservative interpolation operators that preserve the global integrals of the solution fields can be found in the literature. Some of them require building an auxiliary mesh, either as the intersection between the elements of the different meshes or as the union of the donor and target meshes in order to facilitate the use of projection operators. Some techniques depend upon the use of a weighted averaging procedure and others on the resolution of an optimization problem.

#### 3.1 Conservative interpolation via intermediate mesh building

Many conservative interpolation methods based on the use of an auxiliary mesh can be found in the literature, in the following section we decide to detail the most recent methods : the P1-conservative algorithm and the one based on the construction of a supermesh.

##### 3.1.1 $P_1$ -conservative technique by mesh intersection

In [22], the author proposes a P1-conservative interpolation operator that satisfies the maximum principle. This operator is based on local mesh intersections. The approach begins with the localization procedure based on the barycentric coordinates presented in the previous section, then uses the mesh intersection algorithm. It is a local procedure that consists in computing the intersection between an element of the target mesh and all the elements of the donor mesh that it overlaps. This algorithm gives us a precise intersection list that contains for each element of the new mesh, all the triangles of the background mesh that it overlays. For each couple of triangles  $K_T$  and  $K_D^j$ , a new mesh of the intersection with the following definition is introduced :

$$\mathcal{T}_j = K_T \cap K_D^j \quad (6)$$

The solution is piecewise linear by element. For each triangle of the donor mesh, we know the values of:

- The mass  $m_{K_D} = \int_{K_D} u$
- The constant gradient of the solution  $\nabla u_{K_D}$

Hence, we use a quadrature formula in order to compute the exact quantity of mass and the value of the gradient of the element  $K_T$  on the target mesh :

$$m_{K_T} = \int_{K_T} \pi_1^c u = \sum_j \int_{\mathcal{T}_j} u \quad (7)$$

$$(\nabla \pi_1^c u)|_{K_T} = \frac{\sum_j \int_{\mathcal{T}_j} \nabla u}{|K_T|} \quad (8)$$

This scheme is  $P_1$ -exact, respects the mass conservation and fulfills the maximum principle by reconstructing the mass field and its gradient with the elemental intersections between both the donor and target mesh, see [23] for more details.

The obtained numerical results show the influence of the conservative interpolation method, see Fig 6.

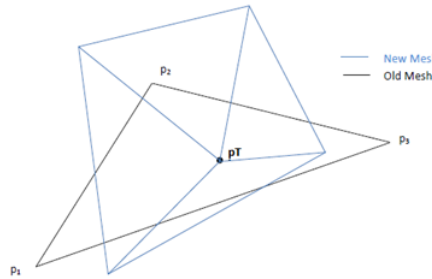


Figure 6: Gaussian analytical function. Top left, 3D representation of the function. Top right, the mass variation for the transfer  $\mathcal{T}_D \rightarrow \mathcal{T}_T$ . Bottom left, error for the transfer  $\mathcal{T}_D \rightarrow \mathcal{T}_T$ . Bottom right, error for the transfer  $\mathcal{T}_D \rightarrow \mathcal{T}_T \rightarrow \mathcal{T}_D$ , [22]

The properties of this algorithm have been verified numerically on analytical examples and adaptive simulations. While preserving the mass, the results with the  $P_1$ -conservative interpolation scheme are more accurate and ensure better conservation of the mass than the ones obtained with the classical linear interpolation. However, the difficulty of this algorithm lies in the construction of the list of intersections : many degenerated topological cases can be encountered.

### 3.1.2 Interpolation via common-refinement or supermesh construction

Another recent technique is the construction of a common-refinement or supermesh. Grandy [24] proceeds by mapping from donor to target mesh by calculating the intersection volume of overlapping polyhedra between the donor and the target mesh, as opposed to Bailey [25] that approximates the solution over the area of intersection using a Galerkin projection.

One of the main fields of application of conservative interpolation is the transmission of loads between interfaces, for instance in coupled problems such as fluid structure. Cebra and Lohner [26], proposed the conservation of the load along the interface using a node-projection scheme, whereas Jiao and Heath [27] proposed a method also based on the use of an auxiliary mesh that is highly recommended by Jamain et al [28]. The idea is to define a common-refinement which is composed of the intersection of the fluid and solid meshes

along the interface, the algorithms used for surface meshes are described in [29]. It is an accurate, conservative data transfer algorithm based on the use of a common-refinement of two non matching surface meshes in order to allow an efficient Galerkin projection of the results, which minimizes the  $L^2$ -norm of the interpolation error, see [27]. Several methods are compared for transferring data once the common-refinement has been constructed, and it has been shown that the common-refinement based scheme with  $L_2$  minimization on linear basis functions leads to errors which grow more slowly with iteration number compared to other interpolation methods.

On the other hand, Farrell [30] was the first to present a minimally diffusive bounded interpolation algorithm between unrelated unstructured meshes that ensures the maintenance of the maximum principle and mass conservation. It is based on building a supermesh as the union of the original and the target mesh. The purpose of this union is to facilitate the use of projection operators as mesh to mesh interpolators. An example of the construction of a supermesh is shown in Fig 7. The supermesh must respect the following conditions :

$$N_S \supseteq N_D \cup N_T \tag{9}$$

with  $N_S$  the nodes of the supermesh  $\mathcal{T}_S$ , and  $N_D, N_T$  the nodes belonging to the parent meshes, respectively  $\mathcal{T}_D$  and  $\mathcal{T}_T$ . By definition, if a node belongs to a parent mesh it has to be in the supermesh; and for each element  $K_S \in \mathcal{T}_S$ , the intersection of  $K_S$  with any element of the parent meshes, has to be either empty or the element itself. The utility of this supermesh is that it provides a decomposition of elements in  $\mathcal{T}_D$  and  $\mathcal{T}_T$  as elements in  $\mathcal{T}_S$ . This is done using mapping functions  $\chi_{SD}$  and  $\chi_{TS}$  that allow to determine the parent element of  $K_S \in \mathcal{T}_S$ , in other words, whether it belongs to the donor or the target mesh, and respectively the children, see [31] (section 2) for more details.

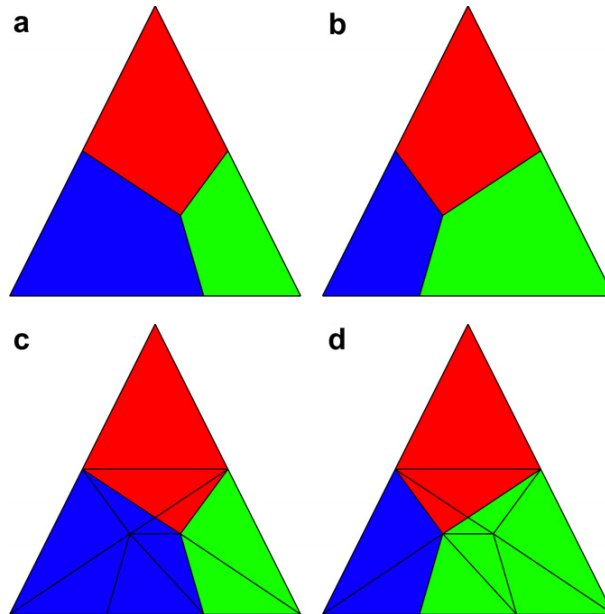


Figure 7: (a) and (b) Quadrilateral meshes, (c) A triangular supermesh of (a) and (b) coloured to show the elements of (a) - (d) is the same supermesh coloured to show the elements of (b), [31]

Let us recall that  $u$  is the function whose integral need to be conserved :

$$\int_{\mathcal{T}_D} u = \int_{\mathcal{T}_T} \pi^c u \quad (10)$$

Let us use a discrete form of eq. (11) over the meshes  $\mathcal{T}_D$  and  $\mathcal{T}_T$  :

$$\sum_{K_D \in \mathcal{T}_D} \int_{K_D} u = \sum_{K_T \in \mathcal{T}_T} \int_{K_T} \pi^c u \quad (11)$$

We express  $K_T$  using the mapping functions, so the problem reduces to interpolating from  $\mathcal{T}_D$  to  $\mathcal{T}_S$  in a conservative manner such that :

$$\sum_{K_D \in \mathcal{T}_D} \int_{K_D} u = \sum_{K_S \in \mathcal{T}_S} \int_{K_S} \pi^c u \quad (12)$$

We have the elemental integrals of the solution, the last step is to compute its nodal values by means of a local Galerkin projection, [31]. We obtain the coming equation :

$$M_T u_T = M_{TD} u_D \quad (13)$$

with  $u_T$ ,  $u_D$  and  $M_T$  respectively the discrete solutions on the target and donor meshes, and the mass matrix composed of the basis functions of the target mesh. The difficulty arises in computing the right hand-side of the equation (14). Indeed, the mixed matrix  $M_{TD}$  is composed of multiplications of basis functions on the donor and target meshes. The issue that arises in the evaluation of this integral is that the donor mesh basis functions are only guaranteed to be continuous over each element of the donor mesh. If the integral of the basis functions is evaluated at Gauss quadrature points on the target mesh, it will lead to a loss of conservation and accuracy. The use of the supermesh ensures the continuity of the basis functions of the donor mesh. The mixed matrix is assembled by decomposing the elements of  $\mathcal{T}_T$  into its children (elements belonging to the supermesh  $\mathcal{T}_S$ ) using the mapping functions presented earlier, hence by construction this scheme is conservative.

An example of the use of the supermesh construction and the Galerkin projection for conservative interpolation with control-volume meshes is given by Adam and al. [32]. It is achieved by first mapping the control-volume field into a finite element representation on the donor mesh by Galerkin projection, then interpolating onto the target mesh by constructing a finite element supermesh from the intersection of the donor and target meshes and finally projecting back to a control-volume representation on the target mesh.

The algorithm based on the construction of a supermesh makes possible the use of Galerkin projection between unrelated unstructured meshes. It was proved to be efficient and accurate in three dimensions and also for adaptive meshing simulations.

### 3.2 Conservative redistribution via local remapping

The techniques for achieving the conservation of certain quantities during the interpolation consist in either interpolating the results to a new random grid after computing the intersections between the target and donor mesh, as presented in the previous section, or by locally modifying the original one, it is called local remapping. These methods grew out of the

development of arbitrary Lagrangian-Eulerian (ALE) methods, [33]. In ALE settings, there is a Lagrangian phase in which the nodes and the elements are advected with the flow, and then a rezoning phase in which the nodes of the mesh are moved to a more optimal configuration, then a remapping phase in which the Lagrangian solution is interpolated onto the rezoned grid.

Indeed, conservative remapping is a simple form of data transfer that consists in computing the value of an element of the target mesh by calculating the weighted average of the values of the source elements in contact with it. Dukowicz [34] first introduced this remapping method that computes the intersection between meshes with a surface integral and then builds the interpolated field for quadrilateral meshes : the problem of computing the volume intersection of old and new cells into a surface integral is simplified by using the divergence theorem. The approach proposed by Grandy [24] differs from that of Dukowicz [34] in the manner in which the volume of intersection is computed. J. D. Ramshaw [35] extended this approach for all kind of 2D domains. In 1985, this method was modified with not only considering the case with constant density cell, but also a more general case with a linear density distribution, which improved the accuracy of the scheme (less diffusion), [36]. A conservative local interpolation step in ALE algorithms is given by Margolin and Shashkov [37], it is based on estimating the mass exchanged between the neighboring cells at their common interfaces. It is an accurate scheme that divides the new cells by taking into account the intersection with the old ones and guarantees the mass conservation along the interfaces; the mass is exchanged between the neighboring cells. It was extended to 3D domains by Garimella et al. [38]. The authors defined an underlying function that represents the density throughout the domain; the only given information about this density function is its mean value in each cell of the old mesh :

$$\bar{g}(K_D) = \frac{\int_{K_D} g(r) dV}{V(K_D)} \quad (14)$$

with  $r = (x, y, z)$ ,  $K_D$  an element of the donor mesh  $\mathcal{T}_D$ ,  $g$  the underlying function,  $\bar{g}$  the mean value of the function  $g$ , and  $V(K_D)$  the volume of the cell  $K_D$ .

This function will be used to define an efficient linearity and bound preserving remapping method. Indeed, the first step of the algorithm is the reconstruction of the underlying function. The gradient of the function in all the cells of the old mesh is computed in order to obtain a reconstructed piecewise linear function. Then, a quadrature formula is used to compute the approximate integration of the reconstructed function onto the new mesh, in order to obtain the mean values in the new cells (see section 4.2.1 of [38] for details). The last step is the conservative redistribution of the mass in each cell in order to conserve the local bounds. Kucharik and Shashkov [39] recommended the use of local remappers that are simpler and computationally more efficient than global remappers. Indeed, all the possible coincidences between donor and target nodes, edges, and facets have been handled without altering the position of nodes, thus allowing to conserve mass to nearly machine precision. Moreover, the remapping methods can also be applied to unstructured meshes [40], however, they are restricted to the use of an ALE framework.

### 3.3 Matrix based conservative interpolation with restrictions

A conservative formulation based on the projection of the physical properties has been proposed by Chippada and al, [41]. It is not based on grid operations or algorithms but on the

physics of the problem. In this article, the authors present the example of the transport of the velocity field. The idea is to compute an approximation of the solution field defined in the initial mesh; this approximated solution has to satisfy the mass conservation law. It will not only depend on the initial solution but also on a corrector term that ensures mass conservation without modifying the vorticity of the velocity field. Brancherie and al. [42] presented an adaptive remeshing technique based on the diffusive approximation and preserves local equilibrium and ensures conservation of dissipated and strain energy, whereas, Chesshire and Henshaw [43] discussed conservative interpolation of fluxes between overlapping structured meshes; the interpolation coefficients are assumed to be free parameters, then constraints are applied to the integration weights and the interpolation coefficients. Pont, Codina and Baiges, [44] proposed an approach so that one will not be restrained to mass and momentum equations. Thus, the following formulation is proposed. First of all, the discretised solution field on the target mesh is computed using a classic interpolation. For that, two choices are possible : either the use of a projection in the  $L^2$  norm, or of a standard Lagrangian interpolation. Let us assume we computed the interpolated solution  $u_{h,T}$  on the target mesh. This solution has the drawback of being non-conservative. So, the idea proposed by the authors consists in obtaining  $\tilde{u}_{h,T}$  by resolving an optimization problem with Lagrange multipliers. A series of constraints, such as the conservation of the mass or energy are applied to the interpolated field through the Lagrange multipliers. The interpolated field must satisfy the two following restrictions :

- $\tilde{u}_{h,T}$  must be the nearest solution to  $u_{h,T}$  in the  $L^2$  norm
- $\tilde{u}_{h,T}$  must respect some physical properties of  $u_{h,D}$ , solution on the donor mesh

In order to verify the second condition, restriction operators must be defined :

$$R_{n,i} : V_{h,n} \rightarrow \mathbb{R} \quad (15)$$

where  $n$  refers either to donor mesh or target mesh,  $i = 1, \dots, m$  represents the restriction counter.  $V_{h,n}$  is the subspace of discretised functions,  $V_{h,n} \subset V$ , the space of functions where the continuous solution lives. It is proposed to minimize this functional :

$$L : V_{h,T} \times \mathbb{R}^m \rightarrow \mathbb{R}$$

$$L(v_{h,T}, \mu) = \frac{1}{2} \left\| \sum_{n_T} N_T^{n_T} (V_T^{n_T} - U_T^{n_T}) \right\|_{L^2(\Omega)}^2 - \sum_{i=1}^m \lambda_i \left( \sum_{n_T} R_{T,i}^{n_T} V_T^{n_T} - \sum_{n_D} R_{D,i}^{n_D} U_D^{n_D} \right) \quad (16)$$

where  $n_D$  and  $n_T$  are the nodes on the donor and target mesh and  $\lambda_i$  the Lagrange multipliers.  $N_T^{n_T}$  represents the shape functions on the target mesh;  $U_D^{n_D}, U_T^{n_T}$  and  $V_T^{n_T}$  are respectively the nodal values of  $u_{h,D}, u_{h,T}$  and  $v_{h,T}$ . After differentiating the functional  $L$  with respect to the unknowns, we obtain an algebraic system that is different whether we consider a linear or non linear problem, more details in [44] (section 2.2 & 2.3). Solving the obtained system of equations will allow determining the new solution  $\tilde{U}_T$  that satisfies the conservation of mass and energy.

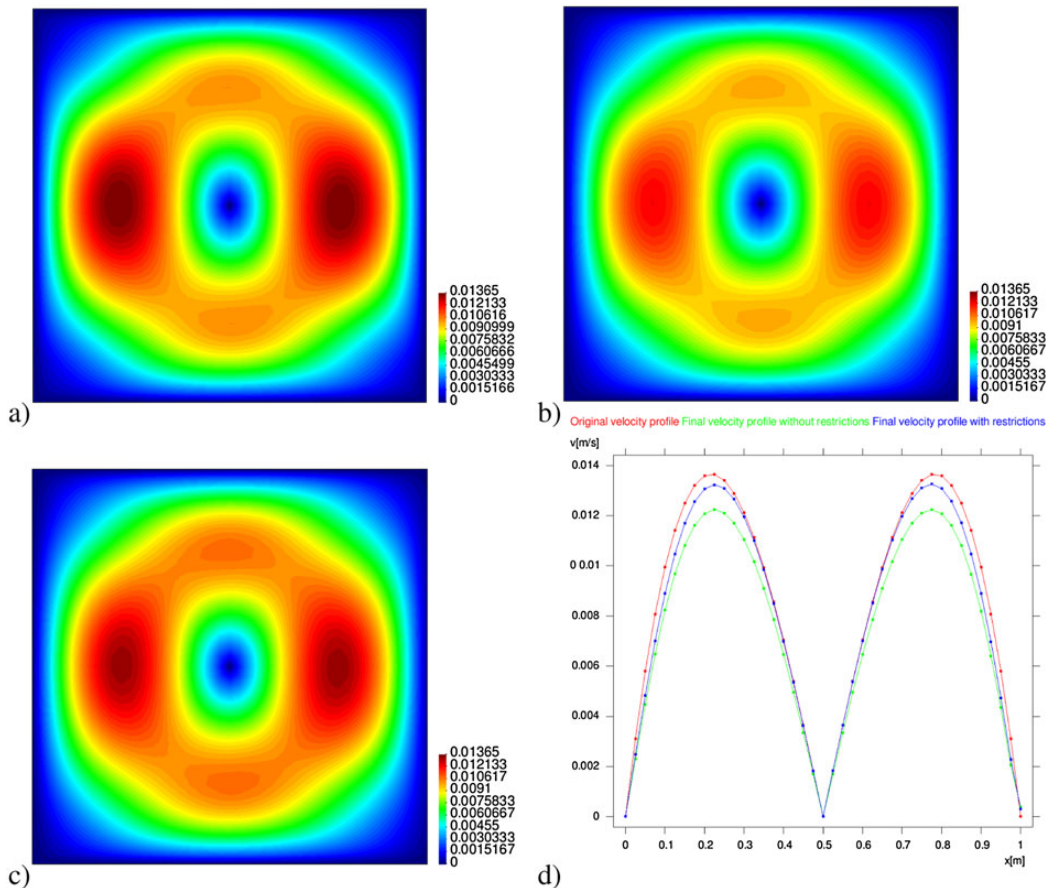


Figure 8: Top left, original function. Top right, final profile of the function after interpolating without restrictions. Bottom left, final profile of the function after interpolating with the restrictions. Bottom right, superposition of all the profiles at  $y = 0.5$ , [44]

Figure 8, shows how the imposition of momentum and  $L^2$  norm conservation reduces most of the dissipation. These results are very encouraging. Indeed, the method is easy to implement and general for different resolutions. However, its extension to mesh adaptation as well as to anisotropic meshing remains a challenge.



## 4 Conclusion and perspectives

The interpolation of numerical solutions between computational meshes is a well known procedure used in many applications : mesh adaptivity to reduce the computational cost, coupling between different physical domains or moving meshes which are used in order to follow rigid bodies in their movement [Phd Thesis Wafa DALDOUL]. The accuracy of the data transfer is the main concern since the accumulation of generalized diffusion can lead to numerical errors and create convergence problems. Thus, this bibliographic reports gives a concise review of the data transfer methods in the literature, both conservative and non conservative.

Indeed, many conservative interpolation methods were introduced over the years. Among them, the remapping methods that are appealing, but we are using an Eulerian framework and these methods where only tested in ALE settings. Some approaches depend upon the use of an auxiliary mesh, either as the intersection of the elements of the initial and final mesh, or as a union of the elements. These methods have the advantage of being local, and therefore easily parallelized. Moreover, they provide satisfying results when tested in a context of adaptive meshing. Finally, another type of methods that are very attractive : the matrix based with restrictions. Indeed, they are based on applying restrictions to interpolated solutions, which can easily be implemented. Moreover, the interpolation with restriction was only tested with simple cases that do not require mesh adaptation. Therefore, it could be interesting to extend it to adaptive anisotropic meshing, [45].

In the context of this research project, the method that we will be implementing needs to be operational in a parallel environment, robust and optimal in computational time. On the other side, the conservative property is mandatory for industrial applications. Today, the linear interpolation scheme is used in CEMEF, it works in parallel and deals with triangular and tetrahedral meshes. First of all, we will focus on modifying this method to increase the accuracy and ensure a conservative transfer in terms of mass and energy. Different approaches can be investigated. Indeed, the supermesh construction proposed by Farell ( section 3.1.2) and the matrix based (section 3.3) will be analyzed, implemented and tested for different applications. And finally, we will apply conservative interpolation to the industrial problem, in order to transfer accurately the fields from a fluid-solid mesh to a solid mesh.

## 5 Annex : Localization algorithms

The localization algorithm consists in identifying the element of a mesh that contains a given point. The localization of a point in a mesh is necessary when we shall transfer data from one mesh to another. Hence, the algorithm consists in finding the elements of the old mesh that contains the new nodes of the target mesh. For that, different approaches can be used, a detailed comparison of the efficiency of the different methods can be found in [46]. We recall here two major contributions.

### Topological search method

Alauzet, [22] proposes a topological method based on the barycentric coordinates. Here, for the sake of simplicity, we will consider a two dimensional space. Let us consider the following situation: we have a node  $p_T$  that belongs to the new mesh  $\mathcal{T}_T$  and we want to find the element  $K_D \in \mathcal{T}_D$  in which it is located. The element  $K_D$  is defined by its three vertices  $[P_0, P_1, P_2]$ , and the barycentric coordinates are computed as follows :

$$\beta_i = \frac{A_{K_i}}{A_K} \quad (17)$$

with  $i = [0, 1, 2]$ ,  $A_K$  the surface of the triangle  $K_D$  and  $A_{K_i}$ , the surface of the triangle whose vertex  $p_i$  is replaced with the vertex of the new mesh  $p_T$ .

The idea is to study the sign of  $\beta_i$  (Fig 9), three cases are possible :

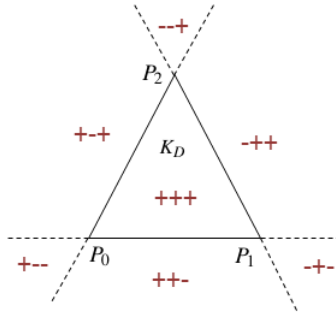


Figure 9: Sign of the barycentric coordinates

- if all the barycentric coordinates are positive, then the new node  $p_T$  belongs to  $K_D$
- if only one  $\beta_i$  is negative, then we have to move to the neighbor  $K_i$  that shares the edge  $\vec{e}_i$  with  $K_D$
- if two  $\beta_i$  are negative, then we have to try the two neighbors.

We reiterate this process till all the coordinates are positive which means we found the triangle in which  $p_T$  is located, an example of the localization algorithm in a three dimensional space is given in Fig 10.

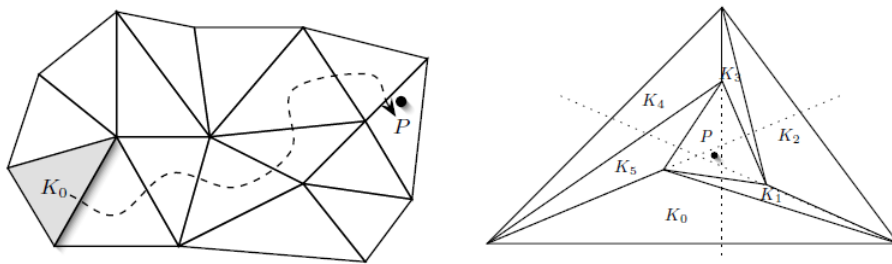


Figure 10: Left, a possible path to locate the vertex  $P$  of the new mesh starting from the triangle  $K_0$  of the donor mesh. Right, cyclic path leading to an already checked element. Starting from  $K_0$ , triangles  $K_1$ ,  $K_2$ ,  $K_3$ ,  $K_4$  and  $K_5$  are visited, bringing us back to  $K_0$ , [22]

### Tree search algorithms

An alternative to the topological approach is the use of tree search algorithms. For instance, the quadtree algorithm for a two dimensional case that consists in partitioning the two dimensional space into four quad (Fig 11) and the octree algorithm by subdividing the 3D space into 8 regions..

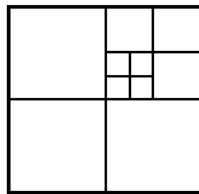


Figure 11: Quadtree algorithm

We start with one rectangle, and then for each node of the new mesh we check :

- if that new node fits completely inside the rectangle
- if yes, we subdivide the rectangle into four children, then recursively do the first step
- if not, then continue to the next rectangle until there are no nodes left

The vertex of the new mesh is located in the smallest rectangle that can contain it. Here in CEMEF, we use the octree algorithm but also the R-tree [47]. Indeed, the structure is designed so that a spatial search requires visiting only a small number of nodes. In this case, the rectangles are not regular, we group nearby elements and represent them with their minimum bounding rectangle.

## References

- [1] E. Hachem, G. Jannoun, J. Veysset, M. Henri, R. Pierrot, I. Poitroult, E. Massoni, T. Coupez, 2013. Modeling of heat transfer and turbulent flows inside industrial furnaces. *Simulation Modelling Practice and Theory*, Vol. 30, pp. 35-53.
- [2] E. Hachem, T. Kloczko, H. Digonnet, and T. Coupez, 2012. Stabilized finite element solution to handle complex heat and fluid flows in industrial furnaces using the immersed volume method. *International Journal for Numerical Methods in Fluids*, 68(1) 99-121.
- [3] S. Brogniez, C. Farhat, E. Hachem, A high-order discontinuous Galerkin method with Lagrange multipliers for advection diffusion problems, 2013. *Computer Methods in Applied Mechanics and Engineering*, Vo. 264, pp. 49-66.
- [4] E. Hachem, S. Feghali, T. Coupez, R. Codina, 2015. A three-field stabilized finite element method for fluid-structure interaction: elastic solid and rigid body limit. *International Journal for Numerical Methods in Engineering*, Vol. 104, pp. 566 - 584
- [5] Y. Mesri, H. Guillard, T. Coupez, 2012. Automatic coarsening of three dimensional anisotropic unstructured meshes for multigrid applications. *Journal of Applied Mathematics and Computation*. 218 (21), 10500-10519.
- [6] E. Hachem, G. Jannoun, J. Veysset, T. Coupez, 2014. On the stabilized finite element method for steady convection-dominated problems with anisotropic mesh adaptation. *Applied Mathematics and Computation*, Vol. 232, pp. 581-594.
- [7] Y. Mesri. Gestion et contrôle des maillages anisotropes non structurés : Applications a l'aérodynamique. Thèse de doctorat de l'Université de Nice-Sophia Antipolis, 2007
- [8] Y. Mesri, M. Khalloufi, E. Hachem, 2016. On optimal simplicial meshes for minimizing the Hessian-based errors. *Applied Numerical Mathematics*, Vol. 109, pp. 235-249.
- [9] G. Jannoun, E. Hachem, J. Veysset, T. Coupez, 2015. Anisotropic meshing with time-stepping control for unsteady convection-dominated problems, *Applied Mathematical Modelling*, Vol. 39, pp. 1899-1916
- [10] Y. Mesri, H. Digonnet, and T. Coupez. Hierarchical adaptive multi-mesh partitioning algorithm on heterogeneous systems. *Computational science and Engineering*. V. 74, pp. 299-306, 2011
- [11] Y. Mesri, JM. Gratien, O. Ricois, R. Gayno, 2013. Parallel Adaptive Mesh Refinement for Capturing Front Displacements : Application to Thermal EOR Processes, SPE-166058-MS, SPE paper
- [12] A. Ern and J. L. Guermond. *Elements finis : théorie, applications, mise en oeuvre*. *Mathématiques & Applications* 36. Springer.
- [13] **P. Breitkopf, G. Touzot, P. Villon, 1998. Consistency approach and diffusive derivation in element free methods based on moving least-squares approximation. *Computer Assisted Mechanics and Engineering Sciences* 5 479-501.**

- [14] **P. Breitkopf, H. Naceur, A. Rassineux, P. Villon, 2005. Moving least squares response surface approximation : Formulation and metal forming applications. Computers and Structures 83 1411-1428.**
- [15] **O. C. Zienkiewicz and J. Z. Zhu, 1992. The superconvergent patch recovery (SPR) and adaptive finite element refinement. Computer Methods in Applied Mechanics and Engineering 101 207-224.**
- [16] **O. C. Zienkiewicz and J. Z. Zhu, 1995. Superconvergence and the superconvergent patch recovery. Finite Elements in Analysis and Design 19 11-23.**
- [17] Z. Zhang, J. Z. Zhu, 1995. Analysis of the superconvergent patch recovery technique and a posteriori error estimator in the finite element method (I). Computer Methods in Applied Mechanics and Engineering 123 173-187.
- [18] B. Boroomand and O. C. Zienkiewicz, 1997. Recovery by equilibrium in patches (REP). International Journal for Numerical Methods in Engineering 40 137-164.
- [19] H. Gu, Z. Zong, K. C. Hung, 2004. A modified superconvergent patch recovery method and its application to large deformation problem. Finite Elements in Analysis and Design 40 665-687.
- [20] A. R. Khoei, S. A. Gharehbaghi, 2007. The superconvergence patch recovery technique and data transfer operators in 3D plasticity problems. Finite Elements in Analysis and Design 43 630-648.
- [21] S. Kumar, L. Fourment, 2012. Remapping method for transferring data between two meshes using a modified iterative SPR approach for parallel resolution. Key Engineering materials Vols. 504-506 pp 455-460.
- [22] **F. Alauzet and M. Mehrenberger, 2010. P1-conservative solution interpolation on unstructured triangular meshes. International Journal for Numerical Methods in Engineering 84 (13) 1552-1588.**
- [23] **F. Alauzet, 2015. A parallel matrix-free conservative solution interpolation on unstructured tetrahedral meshes. [Research Report] RR-8785, INRIA Paris-Rocquencourt.**
- [24] J. Grandy, 1999. Conservative remapping and region overlays by intersecting arbitrary polyhedra. Journal of Computational Physics 148 (2) 433-466.
- [25] R. H. Bailey, 1987. An algorithm for the conservative interpolation of data between two-dimensional structured or unstructured triangular meshes. Master's thesis, University College Swansea.
- [26] J. R. Cebal, R. Lohner, 1997. Conservative load projection and tracking for fluid-structure problems. AIAA journal 35(4) 687-692.
- [27] **X. Jiao and M. T. Heath, 2004. Common-refinement-based data transfer between non-matching meshes in multiphysics simulations. International Journal for Numerical Methods in Engineering 61 (14) 2402-2427.**

- [28] R. K. Jaiman, X. Jiao, P. H. Geubelle and E. Loth, 2005. Assessment of conservative load transfer for fluid-solid interface with non matching meshes. *International Journal for Numerical Methods in Engineering* 64 (15) 2014-2038.
- [29] **X. Jiao and M. T. Heath, 2004. Overlaying surface meshes (part 1): algorithms. *International Journal of Computational Geometry and Applications* 14 379-402.**
- [30] **P. E. Farrell, M. D. Pigott, C. C. Pain, G. J. Gorman and C. R. Wilson, 2009. Conservative interpolation between unstructured meshes via supermesh construction. *Computer Methods in Applied Mechanics and Engineering* 198 (33) 2632-2642.**
- [31] **P. E. Farrell, J. R. Maddison, Conservative interpolation between volume meshes by local galerkin projection, 2011. *Computer Methods in Applied Mechanics and Engineering* 200 (1) 89-100.**
- [32] A. Adam, D. Pavlidis, J. R. Percival, P. Salinas, Z. Xie, F. Fang, C. C. Pain, A. H. Muggeridge and M. D. Jackson, 2016. Higher-order conservative interpolation between control-volume meshes: Application to advection and multiphase flow problems with dynamic mesh adaptivity. *Journal of Computational Physics* 321 512-531.
- [33] J. Donea, A. Huerta, J. Ph. Ponthot and A. Rodriguez-Ferran, 2004. Arbitrary Lagrangian-Eulerian Methods. *Encyclopedia of Computational Mechanics*. Chapter 14.
- [34] J. K. Dukowicz, 1984. Conservative rezoning (remapping) for general quadrilateral meshes. *Journal of Computational Physics* 54 411-424.
- [35] J. D. Ramshaw, 1985. Conservative rezoning algorithm for generalized two-dimensional meshes. *Journal of Computational Physics* 59 (2) 193-199.
- [36] **J. K. Dukowicz and J. W. Kodis, 1987. Accurate conservative remapping (rezoning) for arbitrary lagrangien-eulerian computations. *SIAM Journal on Scientific and Statistical Computing* 8 (3) 305-321.**
- [37] **L. G. Margolin, M. Shashkov, 2003. Second-order sign-preserving conservative interpolation (remapping) on general grids. *Journal of Computational Physics* 184 (1) 266-298.**
- [38] **R. Garimella, M. Kucharik, M. Shashkov, 2007. An efficient linearity and bound preserving conservative interpolation (remapping) on polyhedral meshes. *Computers & fluids* 36 (2) 224-237.**
- [39] M. Kucharik and M. Shashkov, 2007. Extension of efficient, swept-integration-based conservative remapping method for meshes with changing connectivity. *International Journal for Numerical Methods in Fluids*, 56(8) 1359-1365.
- [40] Z. Lin, S. Jiang, S. Wu, L. Kuang, 2011. A local rezoning and remapping method for unstructured mesh. *Computer Physics Communications*, 182 1361-1376.
- [41] **S. Chipada, C .N. Dawson, M. L. Martinez and M. F. Wheeler, 1997. A projection method for constructing a mass conservative velocity field. *Computer Methods in Applied Mechanics and Engineering* 157 (1) 1-10.**

## REFERENCES

---

- [42] D. Brancherie, P. Villon, A. Ibrahimbegovic, A. Rassineux and P. Breitkopf, 2005. Field transfer in nonlinear structural mechanics based on diffuse approximation. International Conference on Computational Plasticity COMPLAS VIII.
- [43] G. Cheshire and W. D. Henshaw, 1994. A scheme for conservative interpolation on overlapping grids. SIAM Journal on Scientific and Statistical Computing 15 (4) 819-845.
- [44] **A. Pont, R. Codina, J. Baiges, 2016. Interpolation with restrictions between finite element meshes for flow problems in ALE setting. International Journal for Numerical Methods in Engineering. DOI: 10.1002/nme.5444**
- [45] A. Dervieux, Y. Mesri, F. Alauzet, A. Loseille, L. Hascoet, and B. Koobus, 2008. Continuous Mesh Adaptation Models for CFD. Computational Fluid Dynamics Journal 16 (4), 346-355.
- [46] R. Lohner, 1994. Robust, vectorized search algorithms for interpolation on unstructured grids. Journal of Computational Physics 118 380-387.
- [47] A. Guttman, 1984. R-Trees: A dynamic index structure for spatial searching. ACM SIGMOD Record 14 (2) 4757.