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# Stabilization of photon-number states via single-photon corrections: a first convergence analysis under an ideal set-up

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**Abstract**—This paper presents a first mathematical convergence analysis of a Fock states feedback stabilization scheme via single-photon corrections. This measurement-based feedback has been developed and experimentally tested in 2012 by the cavity quantum electrodynamics group of Serge Haroche and Jean-Michel Raimond. Here, we consider the infinite-dimensional Markov model corresponding to the ideal set-up where detection errors and feedback delays have been disregarded. In this ideal context, we show that any goal Fock state can be stabilized by a Lyapunov-based feedback for any initial quantum state belonging to the dense subset of finite rank density operators with support in a finite photon-number sub-space. Closed-loop simulations illustrate the performance of the feedback law.

## I. INTRODUCTION

In [8], a photon-number states (Fock state) feedback stabilization scheme via single-photon corrections was described and experimentally tested. Such control problem is relevant for quantum information applications [6], [4]. The quantum state  $\rho$  corresponds to the density operator of a microwave field stored inside a super-conducting cavity and described as a quantum harmonic oscillator. At each sample step  $k \in \mathbb{N}$ , a probe atom is launched inside the cavity. The measurement outcome  $y_k$  detected by a sensor is the energy-state of this probe atom after its interaction with the microwave field. Each probe atom is considered as a two-level system: either it is detected in the lowest energy state  $|g\rangle$ , or the highest energy state  $|e\rangle$ . Consequently, the measurement outcomes corresponds to a discrete-valued output  $y_k$  with only two distinct possibilities:  $g$  or  $e$ . Similarly, the control inputs  $u_k$  are also discrete-valued with 3 distinct possibilities:  $-1, 0, +1$ . The open-loop value  $u_k = 0$  corresponds to a dispersive atom/field interaction: it achieves in fact a Quantum Non-Demolition measurement of Fock states [2]. The two other values  $u_k = \pm 1$  correspond to resonant atom/field interactions where the probe atom and the field exchange energy quanta: these values achieve single-photon corrections.

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Although the feedback law proposed and implemented in [8] considered imperfect detections on  $y_k$  and delays in the control, here we focus on an ideal-set up, that is, detection errors and control delays have been disregarded. Theorem 2 shows that, by adding an arbitrarily small term to the Lyapunov function used in [8], one ensures almost sure global stabilization of any goal Fock state for the closed-loop ideal set-up. This is achieved by relying on an infinite-dimensional Markov model of the ideal set-up that takes into account the back-action of the measurement outcome  $y_k$  on the quantum state  $\rho_{k+1}$ .

Loosely speaking, in [8], the control value  $u_k$  at each sampling step  $k$  was chosen so as to minimize the conditional expectation of the Lyapunov function  $V(\rho_k) = \text{Tr}(d(N)\rho_k)$ , where  $N$  is the photon-number operator,  $d(n) = (n - \bar{n})^2$  and  $\bar{\rho} = |\bar{n}\rangle\langle\bar{n}|$  is the goal Fock state. However, in closed-loop, the difference between such  $V$  and its conditional expectation is not strictly positive: such  $V$  does not become a strict Lyapunov function in closed-loop and additional arguments have to be considered to prove convergence. These additional arguments are related to Lasalle invariance. They are well established in a smooth context where the control  $u$  is a smooth function of the state  $\rho$ . This cannot be the case here since  $u$  is a discrete-valued control. In order to overcome such technical difficulties, we propose, similarly to [1], to add the arbitrarily small term  $-\epsilon \sum_{n=0}^{\infty} (\langle n|\rho_k|n\rangle)^2$  to  $V(\rho_k)$ , where  $\epsilon > 0$ . This slightly modified control-Lyapunov function becomes then a strict-Lyapunov function in closed-loop that simplifies notably the convergence analysis. Moreover, the developed convergence analysis is done in the infinite-dimensional setting in the following sense: we show that, for any initial density operator  $\rho_0$  with a finite photon-number support ( $\rho_0|n\rangle = 0$  for  $n$  large enough), the closed-loop trajectory  $k \mapsto \rho_k$  remains also with a finite photon-number support with a uniform bound on the maximum photon-number. This almost finite-dimensional behavior simplifies the convergence analysis despite the fact that such condition on  $\rho_0$  is met on a dense subset of density operators (Hilbert-Schmidt topology on the Banach space of Hilbert-Schmidt self-adjoint operators).

The paper is organized as follows. Section II presents the ideal Markov model of the experimental set-up of the controlled microwave super-conducting cavity reported in [8] and precisely formulates the Fock state stabilization problem here treated (see Definition 1). Section III establishes the proposed solution to the control problem in two distinct parts. Firstly, Section III-A considers the case where the initial condition  $\rho_0$  is a diagonal density operator (see Theorem 1).

Only the main ideas of the convergence proof are outlined. The technical details are given in Section V. Afterwards, in Section III-B, the main result of the paper is presented: the general solution is obtained from Theorem 1 for  $\rho_0$  belonging to a dense subset (see Theorem 2). The simulation results are exhibited in Section IV. The proof of some intermediate results and computations required in Sections III and V are presented in Appendices B–G. Finally, the concluding remarks are given in Section VI.

## II. IDEAL MARKOV MODEL

Denote by  $\mathcal{H}$  the separable complex Hilbert space  $L_2(\mathbb{C})$  with orthonormal basis  $\{|n\rangle, n \in \mathbb{N}\}$  of Fock states (photon-number). Hence,  $\mathcal{H} = \{\sum_{n \in \mathbb{N}} \psi_n |n\rangle, (\psi_0, \psi_1, \dots) \in l_2(\mathbb{C})\}$ . Let  $\mathbb{D}$  be the set of all density operators on  $\mathcal{H}$ , that is, the set of trace-class, self-adjoint, non-negative operators on  $\mathcal{H}$  with unit trace. The sample step, corresponding to a sampling period around  $100\mu\text{s}$ , is indexed by  $k \in \mathbb{N} = \{0, 1, 2, \dots\}$ ,  $u_k \in \{-1, 0, 1\}$  is the control,  $\rho_k \in \mathbb{D}$  the quantum state and  $y_k \in \{g, e\}$  the measurement outcome. The ideal Markov model of the controlled microwave superconducting cavity used in [8] is given by:

$$\rho_{k+1} = \begin{cases} \rho_{k+1}^g = \frac{M_g(u_k)\rho_k M_g^\dagger(u_k)}{\text{Tr}(M_g(u_k)\rho_k M_g^\dagger(u_k))} & \text{when } y_k = g, \\ \rho_{k+1}^e = \frac{M_e(u_k)\rho_k M_e^\dagger(u_k)}{\text{Tr}(M_e(u_k)\rho_k M_e^\dagger(u_k))} & \text{when } y_k = e, \end{cases} \quad (1)$$

where the measurements outcomes  $y_k = g$  and  $y_k = e$  occur with probabilities<sup>1</sup>  $p_{g,k} = \text{Tr}(M_g(u_k)\rho_k M_g^\dagger(u_k))$  and  $p_{e,k} = \text{Tr}(M_e(u_k)\rho_k M_e^\dagger(u_k)) = 1 - p_{g,k}$ , respectively,  $u_k = 0$  corresponds to a dispersive interaction of the launched atom with the cavity field (Quantum Non-Demolition measurement of photons)

$$M_g(0) = \cos\left(\frac{\phi_0 N + \phi_R}{2}\right), \quad M_e(0) = \sin\left(\frac{\phi_0 N + \phi_R}{2}\right), \quad (2)$$

when  $u_k = +1$  the atom enters the cavity in the state  $|e\rangle$  with a resonant interaction with the cavity field

$$M_g(+1) = \frac{\sin\left(\frac{\theta_0}{2}\sqrt{N}\right)}{\sqrt{N}} \mathbf{a}^\dagger, \quad M_e(+1) = \cos\left(\frac{\theta_0}{2}\sqrt{N+1}\right), \quad (3)$$

when  $u_k = -1$  it enters in  $|g\rangle$  with a resonant interaction

$$M_g(-1) = \cos\left(\frac{\theta_0}{2}\sqrt{N}\right), \quad M_e(-1) = \mathbf{a} \frac{\sin\left(\frac{\theta_0}{2}\sqrt{N}\right)}{\sqrt{N}}, \quad (4)$$

and  $\phi_0, \phi_R, \theta_0 \in \mathbb{R}$  are adjustable control parameters. For each  $u \in \{-1, 0, 1\}$ ,  $M_g(u)$  and  $M_e(u)$  are (linear) operators on  $\mathcal{H}$  defined in the obvious way<sup>2</sup> according to the definitions in Appendix A. They are indeed well-defined operators on  $\mathcal{H}$ , despite the fact that  $\mathbf{a}$  and  $\mathbf{a}^\dagger$  are unbounded

<sup>1</sup>As usual in quantum physics, it is here assumed that the measurement outcome  $y_k = y$  cannot occur when  $\text{Tr}(M_y(u_k)\rho_k M_y^\dagger(u_k)) = 0$ , for  $y = g, e$ .

<sup>2</sup>For instance,  $M_g(+1)|n\rangle = \left(\sin\left(\frac{\theta_0}{2}\sqrt{N}\right)/\sqrt{N}\right) \sqrt{n+1}|n+1\rangle = \sin\left(\frac{\theta_0}{2}\sqrt{n+1}\right)|n+1\rangle$ . In order for the definition of  $M_e(-1)$  to be consistent, it is assumed  $\sin(0)/0 = 1$ .

operators. It is clear that  $M_g(u), M_e(u)$  are bounded operators on  $\mathcal{H}$  with  $M_g^\dagger(u)M_g(u) + M_e^\dagger(u)M_e(u) = \mathbf{I}$  (identity operator),  $M_e(-1) = M_g^\dagger(+1) = \mathbf{a} \sin\left(\frac{\theta_0}{2}\sqrt{N}\right)/\sqrt{N}$ , and  $M_g(-1), M_g(0), M_e(0), M_e(+1)$  are self-adjoint. It is easy to see that if the initial condition  $\rho_0$  is a density operator then, for all realizations of the ideal Markov process (1)–(4),  $\rho_k$  is a density operator for  $k \in \mathbb{N}$ .

Notice that  $\bar{\rho} = |\bar{n}\rangle\langle\bar{n}|$  is a steady state of the Markov process (1)–(4) with  $u_k = 0$ , where  $\bar{n} \in \mathbb{N}$  is arbitrary. The control problem here treated is given as follows:

*Definition 1:* For the ideal Markov process (1)–(4), the control problem is to find a feedback law  $u_k = f(\rho_k)$  such that, given an initial condition  $\rho_0$  and  $\bar{n} \in \mathbb{N}$ , the closed-loop trajectory  $\rho_k$  converges almost surely towards the goal Fock state  $\bar{\rho} = |\bar{n}\rangle\langle\bar{n}|$  as  $k \rightarrow \infty$ .

The almost sure convergence above is with respect to the probabilities amplitudes  $P_n(\rho) = \text{Tr}(|n\rangle\langle n|\rho) = \langle n|\rho|n\rangle$  of  $\rho$ , that is,  $\lim_{k \rightarrow \infty} P_n(\rho_k) = P_n(\bar{\rho})$  for each  $n \in \mathbb{N}$ . In other words,  $\lim_{k \rightarrow \infty} P_{\bar{n}}(\rho_k) = 1$  and  $\lim_{k \rightarrow \infty} P_n(\rho_k) = 0$  when  $n \neq \bar{n}$ . The solution proposed in this paper for the control problem above is developed in the next section.

## III. STABILIZATION OF FOCK STATES

Given any operator  $A: \mathcal{H} \rightarrow \mathcal{H}$ , let  $A_{mn} = \langle m|A|n\rangle$  for  $m, n \in \mathbb{N}$ . Hence,  $A_{nn}$  is the  $n$ -th diagonal element of  $A$ , while  $A_{mn}$  with  $m \neq n$  correspond to its “off-diagonal” elements. One says that the operator  $A$  is *diagonal* when  $A_{mn} = 0$  for all  $m, n \in \mathbb{N}$  with  $m \neq n$ . One shall begin by solving the control problem given in Definition 1 in the particular case where the initial condition  $\rho_0$  is diagonal (see Theorem 1 in Section III-A). Afterwards, in Section III-B, the solution to the general non-commutative case is presented (see Theorem 2): its solution relies essentially on the diagonal case.

### A. Diagonal case

For each  $n^* \in \mathbb{N}$ , define<sup>3</sup>

$$D_{n^*} = \{\rho \in \mathbb{D} \mid \rho \text{ is diagonal and } \rho|n\rangle = 0, \forall n > n^*\}.$$

Consider the set  $D_* = \bigcup_{n^* \in \mathbb{N}} D_{n^*} \subset \mathbb{D}$ . Note that  $D_{n^*} \subset D_{n^*+1}$ , and that each element  $\rho$  of  $D_*$  is “finite dimensional” in the following sense:  $\rho \in \mathbb{D}$  is in  $D_{n^*}$  if and only if  $\rho = \sum_{n=0}^{n^*} \rho_{nn} |n\rangle\langle n|$ , and  $\rho \in D_{n^*}$  may be considered as an operator from  $\mathcal{H}$  to the finite-dimensional space  $\mathcal{H}_{n^*} = \text{span}\{|0\rangle, \dots, |n^*\rangle\}$ , or as a density matrix on  $\mathcal{H}_{n^*}$ . One defines the functions  $n_{\min}: D_* \rightarrow \mathbb{N}$ ,  $n_{\max}: D_* \rightarrow \mathbb{N}$  and  $n_{\text{length}}: D_* \rightarrow \mathbb{N}$  respectively by:

- $n_{\min}(\rho)$  is the smallest  $n \in \mathbb{N}$  such that  $\rho|n\rangle \neq 0$ ;
- $n_{\max}(\rho)$  is the greatest  $n \in \mathbb{N}$  such that  $\rho|n\rangle \neq 0$ ;
- $n_{\text{length}}(\rho) = n_{\max}(\rho) - n_{\min}(\rho)$ .

It is clear that, given  $\rho \in D_*$ , one has  $\rho \in D_{n^*}$  if and only if  $n_{\max}(\rho) \leq n^*$ . The next result exhibits the properties of the state  $\rho_k$  of (1)–(4) with respect to these functions.

<sup>3</sup>Note that if  $\rho = |n\rangle\langle n|$  for some  $n \in \mathbb{N}$ , then  $\rho \in D_n$ .

**Proposition 1:** For every realization of the ideal Markov process (1)–(4) with initial condition  $\rho_0 \in D_*$ , one has that  $\rho_k \in D_*$  for all  $k \in \mathbb{N}$  with:

- If  $u_k = 0$  or  $u_k = -1$ , then  $n_{\max}(\rho_{k+1}) \leq n_{\max}(\rho_k)$  and  $n_{\text{length}}(\rho_{k+1}) \leq n_{\text{length}}(\rho_k)$ ;
- If  $u_k = +1$ , then  $n_{\max}(\rho_{k+1}) \leq n_{\max}(\rho_k) + 1$  and  $n_{\text{length}}(\rho_{k+1}) \leq n_{\text{length}}(\rho_k)$ .

*Proof:* See Appendix B. ■

Take a goal photon-number  $\bar{n} \in \mathbb{N}$ . As in [1], consider the following Lyapunov function  $V_\epsilon: D_* \rightarrow \mathbb{R}$  defined as

$$V_\epsilon(\rho) = \text{Tr}(d(\mathbf{N})\rho) - \epsilon \sum_{n \in \mathbb{N}} \rho_{nn}^2, \quad \text{for } \rho \in D_*, \quad (5)$$

where  $\epsilon > 0$  is a real number and  $d(n) = (n - \bar{n})^2$  as defined in [8]. The feedback law  $u: D_* \rightarrow \{-1, 0, 1\}$  is given by

$$u = f(\rho) \triangleq \underset{v \in \{-1, 0, 1\}}{\text{Argmin}} \mathbb{E}[V_\epsilon(\rho_{k+1}) \mid \rho_k = \rho, u_k = v]. \quad (6)$$

Note that for each  $\rho \in D_*$  and  $n^* \geq n_{\max}(\rho)$ ,  $d(\mathbf{N})\rho$  in (5) is a well-defined self-adjoint, non-negative, trace-class operator on  $\mathcal{H}$ , by considering  $d(\mathbf{N})$  as an operator on  $\mathcal{H}_{n^*}$  and  $\rho$  as an operator from  $\mathcal{H}$  to  $\mathcal{H}_{n^*}$ . Indeed,  $d(\mathbf{N})\rho = \sum_{n=0}^{n^*} \rho_{nn}(n - \bar{n})^2 |n\rangle\langle n|$ . Thus, (5) is well-defined. Moreover, since  $\mathcal{H}_{n^*}$  is invariant under  $\rho \in D_*$  for  $n^* \geq n_{\max}(\rho)$ , it is clear that  $\text{Tr}(d(\mathbf{N})\rho) = \text{Tr}_{\mathcal{H}_{n^*}}(d(\mathbf{N})\rho)$ , where on the right-hand side one considers  $\rho$  as an operator on the finite-dimensional space  $\mathcal{H}_{n^*}$  and the trace is taken over  $\mathcal{H}_{n^*}$ .

We have the following convergence result when  $\rho_0 \in D_*$ :

**Theorem 1:** Let  $\bar{n} \in \mathbb{N}$  and  $\epsilon > 0$ . In (2)–(4), assume that  $\phi_0/\pi$  and  $(\theta_0/\pi)^2$  are irrational numbers, and take  $\phi_R = \pi/2 - \bar{n}\phi_0$ . Consider the closed-loop Markov process (1)–(4) with  $u_k = f(\rho_k)$ , where the feedback law  $f$  is as in (6). Then, given any initial condition  $\rho_0 \in D_*$ , one has that  $\rho_k$  converges almost surely towards  $\bar{\rho} = |\bar{n}\rangle\langle\bar{n}|$  as  $k \rightarrow \infty$ .

Its proof is decomposed into two steps:

**First Step.** Choose  $\bar{n} \in \mathbb{N}$  and  $\epsilon > 0$ . Let  $n_0 = n_{\text{length}}(\rho_0)$ ,  $r_0 = n_{\min}(\rho_0)$ . Then, there exists an integer  $m_0 > n_0 + r_0 + \bar{n} + 1$  (depending on  $n_0, r_0, \bar{n}$  and  $\epsilon$ ) such that, for all closed-loop realizations  $\rho_k$ , one has  $\rho_k \in D_{m_0}$  for  $k \in \mathbb{N}$ .

**Second Step.** Choose irrational numbers  $\phi_0/\pi$  and  $(\theta_0/\pi)^2$  in (2)–(4), and take  $\phi_R = \pi/2 - \bar{n}\phi_0$ . In  $D_{m_0}$ ,  $V_\epsilon$  is a strict super-martingale: for all density operators  $\rho$  in  $D_{m_0}$ , one has

$$\mathbb{E}[V_\epsilon(\rho_{k+1}) \mid \rho_k = \rho, u_k = f(\rho)] - V_\epsilon(\rho) = -Q_{V_\epsilon}(\rho, f(\rho)),$$

where  $Q_{V_\epsilon}(\rho, f(\rho)) \geq 0$ , and  $Q_{V_\epsilon}(\rho, f(\rho)) = 0$  if and only if  $\rho = \bar{\rho}$ . The almost sure convergence follows then from usual results on strict super-martingales for Markov processes with compact state spaces.

The complete proof of the two steps above is presented in Section V. The general case where the initial condition  $\rho_0$  is not necessarily diagonal is treated in the next subsection.

## B. General case

Consider, for each  $n^* \in \mathbb{N}$ ,

$$\mathbb{D}_{n^*} = \{\rho \in \mathbb{D} \mid \rho|n\rangle = 0, \forall n > n^*\} \subset \mathbb{D}_{n^*+1},$$

and let  $\mathbb{D}_* = \bigcup_{n^* \in \mathbb{N}} \mathbb{D}_{n^*} \supset D_*$ . It is clear that  $\rho \in \mathbb{D}$  is in  $\mathbb{D}_{n^*}$  if and only if  $\rho = \sum_{m,n=0}^{n^*} \rho_{mn} |m\rangle\langle n|$ . Consequently,  $\mathbb{D}_*$  is a dense subset of  $\mathbb{D}$  when  $\mathbb{D}$  is endowed with the subspace topology induced from the Hilbert-Schmidt norm. Indeed, let  $\mathcal{J}_2$  be the complex Banach space of all Hilbert-Schmidt operators on  $\mathcal{H}$  with the Hilbert-Schmidt norm  $\|B\|_2 = (\sum_{m,n \in \mathbb{N}} |B_{mn}|^2)^{1/2}$ , for  $B \in \mathcal{J}_2$  [7], [3]. Since  $\mathbb{D} \subset \mathcal{J}_2$  and  $\rho \in \mathbb{D}_{n^*}$  has the form  $\rho = \sum_{m,n=0}^{n^*} \rho_{mn} |m\rangle\langle n|$ , the density property of  $\mathbb{D}_*$  in  $\mathbb{D}$  is clear.

One has that  $\rho \in \mathbb{D}_{n^*}$  may be considered as an operator from  $\mathcal{H}$  to the finite-dimensional space  $\mathcal{H}_{n^*}$ , or as a density matrix on  $\mathcal{H}_{n^*}$ . Hence,  $d(\mathbf{N})\rho$  is a well-defined trace-class operator on  $\mathcal{H}$ , by considering  $d(\mathbf{N})$  as an operator on  $\mathcal{H}_{n^*}$  and  $\rho \in \mathbb{D}_{n^*}$  as an operator from  $\mathcal{H}$  to  $\mathcal{H}_{n^*}$ . Indeed,  $d(\mathbf{N})\rho = \sum_{m,n=0}^{n^*} \rho_{mn}(m - \bar{n})^2 |m\rangle\langle n|$ , and it is trace-class because its range is finite-dimensional [7], [3]. Consequently, the Lyapunov function  $V_\epsilon$  in (5), the feedback in (6) and  $n_{\max}$  can be extended to  $\mathbb{D}_*$ .

Define the map  $\Delta: \mathbb{D}_* \rightarrow D_* \subset \mathbb{D}_*$  as  $\Delta\rho = \sum_{n=0}^{n_{\max}(\rho)} \rho_{nn} |n\rangle\langle n|$ . Note that  $\Delta$  extracts the diagonal of  $\rho \in \mathbb{D}_*$ . It is easy to see that  $n_{\max}(\Delta\rho) = n_{\max}(\rho)$  and  $(\Delta\rho)_{nn} = \rho_{nn}$ ,  $\rho \in \mathbb{D}_*$ . Moreover,  $\Delta\rho = \rho$  when  $\rho \in D_*$ . Other properties of the map  $\Delta$  are given in the next result:

**Proposition 2:** Let  $\rho \in \mathbb{D}_*$ ,  $u \in \{-1, 0, 1\}$ ,  $y = g, e$ . Take  $\alpha = \text{Tr}(\mathbf{M}_y(u)\rho\mathbf{M}_y^\dagger(u))$ . Then:

- $\text{Tr}(A\rho) = \text{Tr}(A\Delta\rho)$ , for every diagonal bounded operator  $A: \mathcal{H} \rightarrow \mathcal{H}$ ;
- $V_\epsilon(\rho) = V_\epsilon(\Delta\rho)$ , for  $\epsilon > 0$ ;
- $\alpha^{-1} \mathbf{M}_y(u)\rho\mathbf{M}_y^\dagger(u)$  belongs to  $\mathbb{D}_*$  with  $\Delta(\alpha^{-1} \mathbf{M}_y(u)\rho\mathbf{M}_y^\dagger(u)) = \alpha^{-1} \mathbf{M}_y(u)(\Delta\rho)\mathbf{M}_y^\dagger(u)$ ;
- $[\mathbf{M}_y(u)(\Delta\rho)\mathbf{M}_y^\dagger(u)]_{nn} = [\mathbf{M}_y(u)\rho\mathbf{M}_y^\dagger(u)]_{nn}$ , for all  $n \in \mathbb{N}$ . In particular,  $\alpha = \text{Tr}(\mathbf{M}_y(u)(\Delta\rho)\mathbf{M}_y^\dagger(u))$ .

*Proof:* See Appendix G. ■

Now, let  $\epsilon > 0$  and  $\bar{\rho} = |\bar{n}\rangle\langle\bar{n}|$ , where  $\bar{n} \in \mathbb{N}$ . Assume that  $\rho_0 \in \mathbb{D}_*$ . Let  $\rho_k$ ,  $k \in \mathbb{N}$ , be the corresponding closed-loop trajectory for a fixed realization of (1)–(4) with feedback  $u_k = f(\rho_k)$ , where  $f$  is as in (6). It is immediate from the proposition above that:

- $\rho_k \in \mathbb{D}_*$ , for  $k \in \mathbb{N}$ ;
- $\Delta\rho_k \in D_*$ ,  $k \in \mathbb{N}$ , is the corresponding closed-loop trajectory of (1)–(4) for the initial condition  $\Delta\rho_0$ , the same realization (and with the same transition probabilities  $p_{e,k}$  and  $p_{g,k}$ ), as well as the same feedback  $u_k = f(\rho_k) = f(\Delta\rho_k)$ ;
- $\text{Tr}(|n\rangle\langle n|\rho_k) = \text{Tr}(|n\rangle\langle n|\Delta\rho_k)$ , for any  $n \in \mathbb{N}$ .

From these arguments, Theorem 1 and the fact that  $\Delta\bar{\rho} = \bar{\rho}$ , one immediately obtains the following *generic* solution to the control problem, that is, when the initial condition  $\rho_0$  belongs to the dense subset  $\mathbb{D}_*$  of  $\mathbb{D}$ :

**Theorem 2:** Let  $\bar{n} \in \mathbb{N}$  and  $\epsilon > 0$ . In (2)–(4), assume that  $\phi_0/\pi$  and  $(\theta_0/\pi)^2$  are irrational numbers, and take  $\phi_R = \pi/2 - \bar{n}\phi_0$ . Consider the closed-loop Markov process (1)–(4) with  $u_k = f(\rho_k)$ , where the feedback law  $f$  is as in (6). Then, given any initial condition  $\rho_0 \in \mathbb{D}_*$ , one has that  $\rho_k$  converges almost surely towards  $\bar{\rho} = |\bar{n}\rangle\langle\bar{n}|$  as  $k \rightarrow \infty$ .

#### IV. SIMULATION RESULTS

This section presents the closed-loop simulation results concerning the application of Theorem 2 above to the ideal Markov process (1)–(4). The quantum experimental results exhibited in [8] used the following control parameter values in (2)–(4):  $\phi_0/\pi = 0.252$  and  $\theta_0/\pi \approx 2/\sqrt{n+1}$ . However, according to the assumptions in Theorem 2,  $\phi_0/\pi$  and  $(\theta_0/\pi)^2$  should be irrational numbers. Hence, here one chooses  $\phi_0/3.14 = 0.252$  and  $\theta_0/3.14 = 2/\sqrt{n+1}$ . One takes  $\rho_0 = \sum_{n=0}^{15} |n\rangle\langle n|/16 \in \mathbb{D}_*$  as the initial condition,  $\bar{n} = 10$  for the goal Fock state  $\bar{\rho} = |\bar{n}\rangle\langle\bar{n}|$ , and  $\epsilon = 10^3$  as the gain for the feedback  $u_k = f(\rho_k)$  in (5)–(6). Figure 1 exhibits the simulation results for one closed-loop realization with such choices and a final sample step of 120. It shows: the dynamics of the populations of  $\rho_k$  (top), the controls  $u_k$  (middle) and the simulated outcomes  $y_k$  (bottom). The populations of  $\rho_k$  correspond to the following observables:  $A_1 = \sum_{n=0}^{\bar{n}-1} |n\rangle\langle n|$  ( $n < \bar{n}$ ),  $A_2 = |\bar{n}\rangle\langle\bar{n}|$  ( $n = \bar{n}$ ),  $A_3 = \sum_{n>\bar{n}} |n\rangle\langle n|$  ( $n > \bar{n}$ ). Therefore, one sees from the dynamics of the populations that  $\rho_k$  converges to  $\bar{\rho}$  as  $k \rightarrow \infty$ , which is in accordance with Theorem 2. Note that  $\langle\bar{n}|\rho_k|\bar{n}\rangle \approx 1$  and  $u_k = 0$  for all  $k > 45$ .

Recall that Theorem 2 assumes that  $\epsilon > 0$ . In order to further analyze the performance of the Lyapunov-based feedback law here proposed, we now make a comparison with the one used experimentally in [8], which corresponds to take  $\epsilon = 0$  in (5), i.e. to disregard the term  $-\epsilon \sum_{n \in \mathbb{N}} \rho_{nn}^2$ . Figure 2 presents the simulation results for one closed-loop realization of such case. The control parameters,  $\rho_0$  and  $\bar{n} = 10$  are the same as above. Note that  $\langle\bar{n}|\rho_k|\bar{n}\rangle \approx 1$  and  $u_k = 0$  for all  $k > 78$ . In order to make a comparison in terms of the speed of convergence, define the settling time  $k_s$  to be the smallest  $\tilde{k} \in \mathbb{N}$  such that  $\langle\bar{n}|\rho_k|\bar{n}\rangle > 0.9$  for all  $k \geq \tilde{k}$ . One has  $k_s = 45$  for the case  $\epsilon = 10^3$  above, and  $k_s = 78$  for  $\epsilon = 0$ . Therefore, in the two realizations here simulated, the choice of  $\epsilon = 10^3$  reduced the settling time  $k_s$  by nearly 42% with respect to  $\epsilon = 0$ . This behavior is typical on an average basis, thereby justifying the term  $-\epsilon \sum_{n \in \mathbb{N}} \rho_{nn}^2$  in (5). Table I shows the average value  $\bar{k}_s$  and the standard deviation  $\sigma$  of  $k_s$  for  $\epsilon \in \{0, 0.1, 1, 10, 10^2, 10^3, 10^4, 10^5\}$ , where a total of 5000 realizations were simulated for each  $\epsilon$ . Notice that when  $\epsilon$  is relatively large or relatively small in comparison to  $\epsilon = 10^3$ , the average settling time  $\bar{k}_s$  deteriorated. Furthermore, although for  $\epsilon = 10^5$  one has that  $\bar{k}_s$  increased by nearly 22% in comparison to  $\epsilon = 10^3$ , the standard deviation  $\sigma$  decreased by nearly 62%. Computer simulations have suggested that a choice of  $\epsilon > 0$  which may perhaps significantly improve  $\bar{k}_s$  generally depends on the initial condition  $\rho_0$  and on the goal Fock state  $\bar{\rho} = |\bar{n}\rangle\langle\bar{n}|$ , and it has to be determined heuristically.

#### V. PROOF OF THEOREM 1 (DIAGONAL CASE)

##### Proof of the First Step:

Let  $\epsilon > 0$ . Define  $V: \mathbb{D}_* \rightarrow \mathbb{R}$  and  $W: \mathbb{D}_* \rightarrow \mathbb{R}$  as

$$V(\rho) = \text{Tr}(d(\mathbf{N})\rho), \quad W(\rho) = -\sum_{n \in \mathbb{N}} \rho_{nn}^2, \quad (7)$$

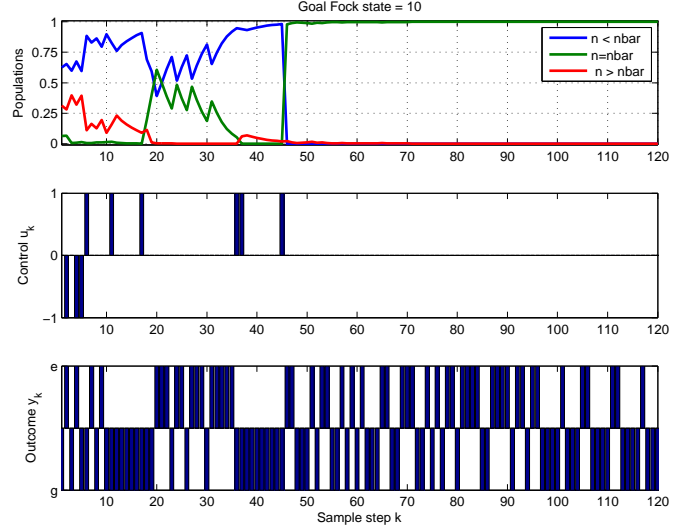


Fig. 1. Simulation of one closed-loop realization with gain  $\epsilon = 10^3$ : convergence of  $\rho_k$  towards  $\bar{\rho}$  (top), controls  $u_k$  (middle), and outcomes  $y_k$  (bottom). Notice that  $\langle\bar{n}|\rho_k|\bar{n}\rangle \approx 1$  and  $u_k = 0$  for all  $k > 45$ .

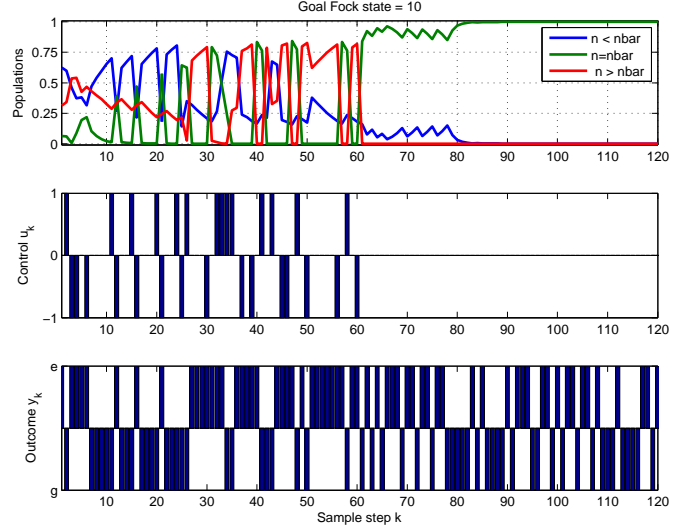


Fig. 2. Simulation of one closed-loop realization with gain  $\epsilon = 0$ : convergence of  $\rho_k$  towards  $\bar{\rho}$  (top), controls  $u_k$  (middle), and outcomes  $y_k$  (bottom). Notice that  $\langle\bar{n}|\rho_k|\bar{n}\rangle \approx 1$  and  $u_k = 0$  for all  $k > 78$ .

TABLE I  
AVERAGE SETTLING TIME  $\bar{k}_s$  AND STANDARD DEVIATION  $\sigma$  AS A FUNCTION OF THE GAINS  $\epsilon$ , CONSIDERING 5000 REALIZATIONS

$\epsilon = 0$ $\bar{k}_s = 79.94$ $\sigma = 164.97$	$\epsilon = 0.1$ $\bar{k}_s = 79.95$ $\sigma = 166.61$	$\epsilon = 1$ $\bar{k}_s = 81.24$ $\sigma = 174.29$	$\epsilon = 10$ $\bar{k}_s = 71.33$ $\sigma = 150.95$
$\epsilon = 10^2$ $\bar{k}_s = 60.41$ $\sigma = 119.39$	$\epsilon = 10^3$ $\bar{k}_s = 44.18$ $\sigma = 44.12$	$\epsilon = 10^4$ $\bar{k}_s = 47.05$ $\sigma = 37.37$	$\epsilon = 10^5$ $\bar{k}_s = 53.77$ $\sigma = 16.84$

respectively. Note that  $V_\epsilon = V + \epsilon W$ . Define:

- $Q_W(\rho, u) = W(\rho) - \mathbb{E}[W(\rho_{k+1}) \mid \rho_k = \rho, u_k = u]$ ,
- $Q_V(\rho, u) = V(\rho) - \mathbb{E}[V(\rho_{k+1}) \mid \rho_k = \rho, u_k = u]$ ,
- $Q_{V_\epsilon}(\rho, u) = V_\epsilon(\rho) - \mathbb{E}[V_\epsilon(\rho_{k+1}) \mid \rho_k = \rho, u_k = u]$ ,

for  $\rho \in D_*$  and  $u \in \{-1, 0, 1\}$ . The proof of Theorem 1 is a straightforward consequence of the next proposition:

**Proposition 3:** Let  $\epsilon > 0$  and  $n_0, r_0, \bar{n} \in \mathbb{N}$ . There exists an integer  $m_0 > n_0 + r_0 + \bar{n} + 1$  (depending on  $\epsilon, n_0, r_0, \bar{n}$ ) such that, for each  $\rho \in D_*$  with  $n_{\text{length}}(\rho) \leq n_0$ , if  $n_{\text{max}}(\rho) = m_0$ , then

$$Q_{V_\epsilon}(\rho, -1) > \max\{Q_{V_\epsilon}(\rho, 0), Q_{V_\epsilon}(\rho, +1)\}.$$

In fact, given  $\rho_0 \in D_*$ , let  $n_0 = n_{\text{length}}(\rho_0)$  and  $r_0 = n_{\text{min}}(\rho_0)$ . Note that  $n_{\text{max}}(\rho_0) = n_0 + r_0 < m_0$ . By Proposition 1,  $\rho_k \in D_*$  with  $n_{\text{length}}(\rho_k) \leq n_0$ , for all  $k \in \mathbb{N}$ . Since  $u = f(\rho)$  maximizes  $Q_{V_\epsilon}(\rho, f(\rho))$ , Proposition 3 implies that when  $n_{\text{max}}(\rho_k) = m_0$  for some  $k \in \mathbb{N}$ , then the input  $u_k$  will be always be equal to  $-1$ , and hence Proposition 1 ensures that  $n_{\text{max}}(\rho_{k+1}) \leq n_{\text{max}}(\rho_k) = m_0$ . Therefore,  $n_{\text{max}}(\rho_k) \leq m_0$ ,  $k \in \mathbb{N}$ , showing the First Step.

The following two lemmas are instrumental for showing Proposition 3. Their proofs are given in Appendix D and Appendix E, respectively.

**Lemma 1:** Given an arbitrary nonzero  $\theta_0 \in \mathbb{R}$ , fix any  $a \in \mathbb{R}$  such that  $0 < a < 1/2$ . For all nonzero  $N_0, N \in \mathbb{N}$ , there exists an integer  $\bar{N} > N$  big enough such that,

$$0 < 1/2 - a \leq \sin^2\left(\frac{\theta_0}{2}\sqrt{n}\right) \leq 1/2 + a,$$

for  $n = \bar{N}, \bar{N} + 1, \dots, \bar{N} + N_0 - 1$ .

**Lemma 2:** Let  $\rho \in D_*$ . Then:

- $|Q_W(\rho, u)| \leq 1$ , for each  $u \in \{-1, 0, 1\}$ ;
- $Q_V(\rho, 0) = 0$ ;
- $Q_V(\rho, +1) = -\sum_{n \in \mathbb{N}} \rho_{nn} [2(n - \bar{n}) + 1] \sin^2\left(\frac{\theta_0}{2}\sqrt{n+1}\right)$ ;
- $Q_V(\rho, -1) = \sum_{n \in \mathbb{N}} \rho_{nn} [2(n - \bar{n}) - 1] \sin^2\left(\frac{\theta_0}{2}\sqrt{n}\right)$ .

The proof of Proposition 3 is shown in the sequel.

*Proof:* Let  $\epsilon > 0$  and  $n_0, r_0, \bar{n} \in \mathbb{N}$ . One has to show that there exists  $m_0 > n_0 + r_0 + \bar{n} + 1$  such that, if  $\rho \in D_*$  with  $n_{\text{length}}(\rho) \leq n_0$ , then  $u = -1$  always maximizes  $Q_{V_\epsilon}(\rho, u)$  whenever  $n_{\text{max}}(\rho) = m_0$ . From Lemma 2 and the fact that  $Q_{V_\epsilon} = Q_V + \epsilon Q_W$ , to complete the proof it suffices to show that:

- If  $\rho \in D_*$  is such that  $n_{\text{length}}(\rho) \leq n_0$  and  $n_{\text{max}}(\rho) \geq n_0 + \bar{n}$ , then  $Q_V(\rho, +1) \leq 0$ ;
- There exists  $m_0 > n_0 + r_0 + \bar{n} + 1$  such that  $Q_V(\rho, -1) > 2\epsilon$ , whenever  $\rho \in D_*$  is such that  $n_{\text{length}}(\rho) \leq n_0$  and  $n_{\text{max}}(\rho) = m_0$ .

Note that

$$Q_V(\rho, +1) = -\sum_{n=n_{\text{min}}(\rho)}^{n_{\text{max}}(\rho)} \rho_{nn} [2(n - \bar{n}) + 1] \sin^2\left(\frac{\theta_0}{2}\sqrt{n+1}\right),$$

for any  $\rho \in D_*$ . Thus, if  $n_{\text{length}}(\rho) \leq n_0$  and  $n_{\text{max}}(\rho) \geq \bar{n} + n_0$ , then  $n_{\text{min}}(\rho) \geq \bar{n}$ , and hence the first claim is shown.

Now, fix  $0 < a < 1/2$  and let<sup>4</sup>  $N \geq \frac{1}{2} \left[ \frac{2\epsilon}{1/2-a} + 2\bar{n} + 1 \right]$ . Applying Lemma 1 for  $N_0 = n_0 + r_0 + 1$  and such choice of  $N$ , one gets  $\bar{N} > N$  in which  $0 < 1/2 - a \leq \sin^2\left(\frac{\theta_0}{2}\sqrt{n}\right)$ ,

for  $n = \bar{N}, \bar{N} + 1, \dots, \bar{N} + n_0 + r_0$ . Take  $m_0 = \bar{N} + n_0 + r_0$ . Let  $\rho \in D_*$  with  $n_{\text{length}}(\rho) \leq n_0$  and  $n_{\text{max}}(\rho) = m_0$ . Note that  $m_0 > n_0 + r_0 + \bar{n} + 1$  and  $n_{\text{min}}(\rho) \geq \bar{N} + r_0$ . From Lemma 2 and the inequality above for  $1/2 - a$ , one obtains

$$\begin{aligned} Q_V(\rho, -1) &= \sum_{n=n_{\text{min}}(\rho)}^{m_0} \rho_{nn} [2(n - \bar{n}) - 1] \sin^2\left(\frac{\theta_0}{2}\sqrt{n}\right) \\ &\geq \sum_{n=n_{\text{min}}(\rho)}^{m_0} \rho_{nn} [2(n - \bar{n}) - 1] (1/2 - a) \\ &\geq \sum_{n=n_{\text{min}}(\rho)}^{m_0} \rho_{nn} [2(\bar{N} - \bar{n}) - 1] (1/2 - a) \\ &= [2(\bar{N} - \bar{n}) - 1] (1/2 - a) \sum_{n=n_{\text{min}}(\rho)}^{m_0} \rho_{nn}. \end{aligned}$$

Using the fact that  $\sum_{n=n_{\text{min}}(\rho)}^{m_0} \rho_{nn} = 1$  and  $\bar{N} > \frac{1}{2} \left[ \frac{2\epsilon}{1/2-a} + 2\bar{n} + 1 \right]$ , one shows the second claim, thereby completing the proof of Proposition 3. ■

### Proof of the Second Step:

Let  $\epsilon > 0$ . Recall that, by definition,  $Q_{V_\epsilon} = Q_V + \epsilon Q_W$ . Using the same notation of the First Step, the central idea of the proof is to show that, given  $\rho \in D_{m_0}$ , one has that  $Q_{V_\epsilon}(\rho, f(\rho)) \geq 0$ , and that  $Q_{V_\epsilon}(\rho, f(\rho)) = 0$  if and only if  $\rho = \bar{\rho}$ . The following lemma is instrumental for the proof of such property. Its proof is presented in Appendix F.

**Lemma 3:** Assume that  $\phi_0/\pi$  is an irrational number in (2), and take  $\phi_R = \pi/2 - \bar{n}\phi_0$ , where  $\bar{n} \in \mathbb{N}$ . Let  $\rho \in D_*$ . Then:

- $Q_W(\rho, 0) \geq 0$ , and  $Q_W(\rho, 0) = 0$  if and only if  $\rho = |n\rangle\langle n|$  for some  $n \in \mathbb{N}$ ;
- $Q_W(\rho, +1) = Q_W(\rho, -1) = 0$  whenever  $\rho = |n\rangle\langle n|$  for some  $n \in \mathbb{N}$ .

One has that  $m_0 > \bar{n}$ , and  $(\theta_0/\pi)^2$  is an irrational number by assumption. Recall that  $\sin^2(x) = 0$  if and only if  $x = \ell\pi$ , where  $\ell$  is an integer. First we show that  $Q_{V_\epsilon}(\bar{\rho}, f(\bar{\rho})) = 0$ . By Lemma 2:  $Q_V(\bar{\rho}, +1) = -\sin^2\left(\frac{\theta_0}{2}\sqrt{\bar{n}+1}\right) < 0$ ;  $Q_V(\bar{\rho}, -1) = -\sin^2\left(\frac{\theta_0}{2}\sqrt{\bar{n}}\right) < 0$  when  $\bar{n} > 0$ , and  $Q_V(\bar{\rho}, -1) = 0$  when  $\bar{n} = 0$ ; and  $Q_V(\bar{\rho}, 0) = 0$ . As  $Q_W(\bar{\rho}, u) = 0$ , for  $u \in \{-1, 0, 1\}$ , and  $u = f(\bar{\rho})$  maximizes  $Q_{V_\epsilon}(\bar{\rho}, u)$ , one has that  $Q_{V_\epsilon}(\bar{\rho}, f(\bar{\rho})) = 0$ .

Now, let  $\rho \in D_{m_0} \subset D_*$ . Since  $u = f(\rho)$  maximizes  $Q_{V_\epsilon}(\rho, u)$ , it follows that

$$Q_{V_\epsilon}(\rho, f(\rho)) \geq Q_V(\rho, 0) + \epsilon Q_W(\rho, 0) = \epsilon Q_W(\rho, 0) \geq 0.$$

Suppose  $Q_{V_\epsilon}(\rho, f(\rho)) = 0$ . Hence,  $Q_W(\rho, 0) = 0$ , and so  $\rho = |n\rangle\langle n|$  for some  $n \in \{0, 1, \dots, m_0\}$ . It suffices to show that  $Q_{V_\epsilon}(\rho, f(\rho))|_{|n\rangle\langle n|} > 0$  for  $n \in \{0, 1, \dots, m_0\}$  with  $n \neq \bar{n}$ . Assume that  $n > \bar{n}$ . It is clear that  $u = f(|n\rangle\langle n|) = -1$  and  $Q_{V_\epsilon}(\rho, f(\rho))|_{|n\rangle\langle n|} = [2(n - \bar{n}) - 1] \sin^2\left(\frac{\theta_0}{2}\sqrt{n}\right) > 0$ . Assume now that  $n < \bar{n}$ . Then,  $u = f(|n\rangle\langle n|) = +1$  and  $Q_{V_\epsilon}(\rho, f(\rho))|_{|n\rangle\langle n|} = -[2(n - \bar{n}) + 1] \sin^2\left(\frac{\theta_0}{2}\sqrt{n+1}\right) > 0$ . This completes the proof of the referred property.

The remaining part of the proof of the Second Step is a straightforward consequence of the standard stochastic convergence result below:

<sup>4</sup>As  $N$  is an integer, it follows that  $N \geq \bar{n} + 1$ .

**Theorem 3:** [5, Theorem 1, p. 195] Let  $\Omega$  be a probability space and let  $W$  be a measurable space. Consider that  $X_k: \Omega \rightarrow W$ ,  $k \in \mathbb{N}$ , is a Markov chain with respect to the natural filtration. Let  $Q: W \rightarrow \mathbb{R}$  and  $V: W \rightarrow \mathbb{R}$  be measurable non-negative functions with  $V(X_k)$  integrable for all  $k \in \mathbb{N}$ . If  $\mathbb{E}[V(X_{k+1}) | X_k] - V(X_k) = -Q(X_k)$ , for  $k \in \mathbb{N}$ , then  $\lim_{k \rightarrow \infty} Q(X_k) = 0$  almost surely.

Indeed, let  $\mathcal{J}_1$  be the complex Banach space of all trace-class operators on  $\mathcal{H}$  with the trace norm  $\|\cdot\|_1$ , that is,  $\|B\|_1 = \text{Tr}(|B|)$ , where  $|B| \triangleq \sqrt{B^\dagger B}$ , for  $B \in \mathcal{J}_1$ . Recall that  $\|B\| \leq \|B\|_1$  and<sup>5</sup>  $|\text{Tr}(AB)| \leq \|A\| \|B\|_1$ , for every  $B \in \mathcal{J}_1$  and each bounded operator  $A: \mathcal{H} \rightarrow \mathcal{H}$ , where  $\|\cdot\|$  is the usual operator norm (*sup* norm of bounded operators) [7], [3]. Consider the subspace topology on  $D_{m_0}$  with respect to  $\mathcal{J}_1$ . One has that the closed-loop trajectory  $\rho_k$ ,  $k \in \mathbb{N}$ , is a Markov chain with phase space  $D_{m_0}$  (with respect to the natural filtration and the Borel algebra on  $D_{m_0}$ ). It is clear that  $D_{m_0}$  is compact, and that  $Q_\epsilon$  and  $V_\epsilon - \alpha_\epsilon$  are non-negative and continuous on  $D_{m_0}$ , for all  $\epsilon > 0$ , where  $\alpha_\epsilon \triangleq \min_{\rho \in D_{m_0}} V_\epsilon(\rho)$ . The theorem above implies that  $\rho_k$  converges almost surely towards  $\bar{\rho}$  as  $k \rightarrow \infty$  (with respect to the trace norm). This completes the proof of Theorem 1.

## VI. CONCLUDING REMARKS

This paper provided a convergence analysis of Fock states stabilization via single-photon corrections under an ideal set-up, that is, assuming perfect measurement detection and no control delays. In terms of convergence speed, the simulation results here presented have justified the inclusion of the term  $-\epsilon \sum_{n \in \mathbb{N}} \rho_{nn}^2$  in the Lyapunov-based feedback law (5)–(6). It is straightforward to verify that the convergence analysis developed in this paper remains valid for: (i) any other function  $d(n)$  in (5) satisfying  $d(\bar{n}) = 0$ ,  $d(n)$  is increasing for  $n > \bar{n}$  and  $d(n)$  is decreasing for  $n < \bar{n}$ ; and (ii)  $\epsilon > 0$  dependent on  $n$ , that is, to take the term  $-\sum_{n \in \mathbb{N}} \epsilon_n \rho_{nn}^2$ . However, it is an open problem how to choose the function  $d(n)$  and the gains  $\epsilon_n > 0$  so as to achieve the best convergence speed.

Finally, the feedback law used in [8], which corresponds to  $\epsilon = 0$ , was tailored for an experimental set-up with measurement imperfections and control delays. The convergence analysis of such realistic situation will be investigated in the future.

## VII. ACKNOWLEDGMENTS

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## APPENDIX

### A. Basic properties of the operators $\mathbf{N}$ , $\mathbf{a}$ and $\mathbf{a}^\dagger$

Fix  $n^* \in \mathbb{N}$  and let  $\mathcal{H}_{n^*} = \text{span}\{|0\rangle, \dots, |n^*\rangle\}$ . Consider the (linear) operators  $\mathbf{N}: \mathcal{H}_{n^*} \rightarrow \mathcal{H}_{n^*}$ ,  $\mathbf{a}: \mathcal{H}_{n^*} \rightarrow \mathcal{H}_{n^*-1} \subset \mathcal{H}_{n^*}$ ,  $\mathbf{a}^\dagger: \mathcal{H}_{n^*} \rightarrow \mathcal{H}_{n^*+1}$  defined respectively as  $\mathbf{N}|n\rangle = n|n\rangle$ ,  $\mathbf{a}|0\rangle = 0$ ,  $\mathbf{a}|n\rangle = \sqrt{n}|n-1\rangle$  for  $n \geq 1$ ,

<sup>5</sup>One also recalls that if  $A$  is a bounded operator on  $\mathcal{H}$  and  $B \in \mathcal{J}_1$ , then  $AB, BA \in \mathcal{J}_1$  with  $\text{Tr}(AB) = \text{Tr}(BA)$ .

$\mathbf{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ . Note that these operators cannot be extended to  $\mathcal{H}$ . Let  $f: \mathbb{N} \rightarrow \mathbb{R}$  be a function. Define the operator  $f(\mathbf{N}): \mathcal{H}_{n^*} \rightarrow \mathcal{H}_{n^*}$  by  $f(\mathbf{N})|n\rangle = f(n)|n\rangle$ , for each  $n = 0, \dots, n^*$ . It is clear that  $f(\mathbf{N})$  can be extended to  $\mathcal{H}$  whenever  $f$  is a bounded function. Given  $f: \mathbb{N} \rightarrow \mathbb{R}$  and an integer  $m$ , one defines  $g: \mathbb{N} \rightarrow \mathbb{R}$  as:  $g(n) = f(n+m)$ , when  $n+m \geq 0$ ; and  $g(n) = 0$ , when  $n+m < 0$ . One abuses notation letting  $f(\mathbf{N}+m)$  stand for  $g(\mathbf{N})$ . Given two functions  $f, g: \mathbb{N} \rightarrow \mathbb{R}$ , it is clear that  $f(\mathbf{N})g(\mathbf{N}) = g(\mathbf{N})f(\mathbf{N}) = (fg)(\mathbf{N})$  and  $(f+g)(\mathbf{N}) = f(\mathbf{N}) + g(\mathbf{N})$ . Furthermore:  $\mathbf{a}\mathbf{a}^\dagger = \mathbf{N} + \mathbf{I}$ ,  $\mathbf{a}^\dagger\mathbf{a} = \mathbf{N}$ ,  $\mathbf{a}f(\mathbf{N}) = f(\mathbf{N}+1)\mathbf{a}$ ,  $\mathbf{a}^\dagger f(\mathbf{N}) = f(\mathbf{N}-1)\mathbf{a}^\dagger$ .

### B. Proof of Proposition 1

Fix any  $\rho \in D_*$  and let  $n \in \mathbb{N}$ . In particular,  $\rho|n\rangle = \rho_{nn}|n\rangle$ . It then follows from (2)–(4) that:

$$\mathbf{M}_g(0)\rho\mathbf{M}_g^\dagger(0)|n\rangle = \rho_{nn} \cos^2\left(\frac{\phi_0 n + \phi_R}{2}\right)|n\rangle,$$

$$\mathbf{M}_e(0)\rho\mathbf{M}_e^\dagger(0)|n\rangle = \rho_{nn} \sin^2\left(\frac{\phi_0 n + \phi_R}{2}\right)|n\rangle,$$

$$\mathbf{M}_g(+1)\rho\mathbf{M}_g^\dagger(+1)|n\rangle = \begin{cases} 0, & \text{for } n = 0, \\ \rho_{n-1, n-1} \sin^2\left(\frac{\theta_0}{2}\sqrt{n}\right)|n\rangle, & n \geq 1, \end{cases}$$

$$\mathbf{M}_e(+1)\rho\mathbf{M}_e^\dagger(+1)|n\rangle = \rho_{nn} \cos^2\left(\frac{\theta_0}{2}\sqrt{n+1}\right)|n\rangle,$$

$$\mathbf{M}_g(-1)\rho\mathbf{M}_g^\dagger(-1)|n\rangle = \rho_{nn} \cos^2\left(\frac{\theta_0}{2}\sqrt{n}\right)|n\rangle,$$

$$\mathbf{M}_e(-1)\rho\mathbf{M}_e^\dagger(-1)|n\rangle = \rho_{n+1, n+1} \sin^2\left(\frac{\theta_0}{2}\sqrt{n+1}\right)|n\rangle.$$

Therefore:

$$\mathbf{M}_g(0)\rho\mathbf{M}_g^\dagger(0) = \sum_{n=n_{\min}(\rho)}^{n_{\max}(\rho)} \rho_{nn} \cos^2\left(\frac{\phi_0 n + \phi_R}{2}\right)|n\rangle\langle n|, \quad (8)$$

$$\mathbf{M}_e(0)\rho\mathbf{M}_e^\dagger(0) = \sum_{n=n_{\min}(\rho)}^{n_{\max}(\rho)} \rho_{nn} \sin^2\left(\frac{\phi_0 n + \phi_R}{2}\right)|n\rangle\langle n|, \quad (9)$$

$$\mathbf{M}_g(+1)\rho\mathbf{M}_g^\dagger(+1) = \sum_{n=n_{\min}(\rho)+1}^{n_{\max}(\rho)+1} \rho_{n-1, n-1} \sin^2\left(\frac{\theta_0}{2}\sqrt{n}\right)|n\rangle\langle n|, \quad (10)$$

$$\mathbf{M}_e(+1)\rho\mathbf{M}_e^\dagger(+1) = \sum_{n=n_{\min}(\rho)}^{n_{\max}(\rho)} \rho_{nn} \cos^2\left(\frac{\theta_0}{2}\sqrt{n+1}\right)|n\rangle\langle n|, \quad (11)$$

$$\mathbf{M}_g(-1)\rho\mathbf{M}_g^\dagger(-1) = \sum_{n=n_{\min}(\rho)}^{n_{\max}(\rho)} \rho_{nn} \cos^2\left(\frac{\theta_0}{2}\sqrt{n}\right)|n\rangle\langle n|, \quad (12)$$

$$\mathbf{M}_e(-1)\rho\mathbf{M}_e^\dagger(-1) = \sum_{n=\max\{0, n_{\min}(\rho)-1\}}^{n_{\max}(\rho)-1} \rho_{n+1, n+1} \sin^2\left(\frac{\theta_0}{2}\sqrt{n+1}\right)|n\rangle\langle n|. \quad (13)$$

By assumption,  $\rho_0 \in D_*$ . Then, (1), (8)–(13) above and induction on  $k$  show the assertions in Proposition 1.

### C. Computation of $Q_V(\rho, u)$

Fix any  $\rho \in D_*$  and  $\bar{n} \in \mathbb{N}$ . Recall that  $V(\rho) = \text{Tr}(d(\mathbf{N})\rho)$ , where  $d: \mathbb{N} \rightarrow \mathbb{R}$  be given by  $d(n) = (n - \bar{n})^2$ . Note that (1) implies that, for each  $u \in \{-1, 0, 1\}$ ,

$$\begin{aligned} &\mathbb{E}[V(\rho_{k+1}) | \rho_k = \rho, u_k = u] \\ &= \text{Tr}\left(d(\mathbf{N})\mathbf{M}_g(u)\rho\mathbf{M}_g^\dagger(u)\right) + \text{Tr}\left(d(\mathbf{N})\mathbf{M}_e(u)\rho\mathbf{M}_e^\dagger(u)\right). \end{aligned} \quad (14)$$

Take  $u = 0$ . From (8)–(9) in Appendix B, one has

$$\begin{aligned} \mathbb{E}[V(\rho_{k+1}) \mid \rho_k = \rho, u_k = 0] \\ &= \text{Tr} \left( d(\mathbf{N}) \mathbf{M}_g(0) \rho \mathbf{M}_g^\dagger(0) \right) + \text{Tr} \left( d(\mathbf{N}) \mathbf{M}_e(0) \rho \mathbf{M}_e^\dagger(0) \right) \\ &= \text{Tr} \left( d(\mathbf{N}) \left[ \mathbf{M}_g(0) \rho \mathbf{M}_g^\dagger(0) + \mathbf{M}_e(0) \rho \mathbf{M}_e^\dagger(0) \right] \right) \\ &= \text{Tr} (d(\mathbf{N}) \rho) = V(\rho). \end{aligned}$$

In particular,

$$Q_V(\rho, 0) = 0. \quad (15)$$

Now, take  $u = +1$ . Then, (14) above and (10)–(11) in Appendix B provide that

$$\begin{aligned} \mathbb{E}[V(\rho_{k+1}) \mid \rho_k = \rho, u_k = +1] \\ &= \text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{\mathbf{N} + 1} \right) d(\mathbf{N} + 1) \rho \right) \\ &\quad + \text{Tr} \left( \cos^2 \left( \frac{\theta_0}{2} \sqrt{\mathbf{N} + 1} \right) d(\mathbf{N}) \rho \right). \end{aligned}$$

By summing and subtracting  $\text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{\mathbf{N} + 1} \right) d(\mathbf{N}) \rho \right)$ ,

$$\begin{aligned} \mathbb{E}[V(\rho_{k+1}) \mid \rho_k = \rho, u_k = +1] \\ &= \text{Tr} (d(\mathbf{N}) \rho) \\ &\quad + \text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{\mathbf{N} + 1} \right) [d(\mathbf{N} + 1) - d(\mathbf{N})] \rho \right) \\ &= V(\rho) + \text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{\mathbf{N} + 1} \right) [d(\mathbf{N} + 1) - d(\mathbf{N})] \rho \right). \end{aligned}$$

In particular,

$$\begin{aligned} Q_V(\rho, +1) &= -\text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{\mathbf{N} + 1} \right) [d(\mathbf{N} + 1) - d(\mathbf{N})] \rho \right), \\ &= -\sum_{n \in \mathbb{N}} \rho_{nn} [2(n - \bar{n}) + 1] \sin^2 \left( \frac{\theta_0}{2} \sqrt{n + 1} \right). \quad (16) \end{aligned}$$

Finally, take  $u = -1$ . Using (14) above and (12)–(13) in Appendix B,  $\mathbb{E}[V(\rho_{k+1}) \mid \rho_k = \rho, u_k = -1] = \text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{\mathbf{N}} \right) d(\mathbf{N} - 1) \rho \right) + \text{Tr} \left( \cos^2 \left( \frac{\theta_0}{2} \sqrt{\mathbf{N}} \right) d(\mathbf{N}) \rho \right)$ .

By summing and subtracting  $\text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{\mathbf{N}} \right) d(\mathbf{N}) \rho \right)$ ,

$$\begin{aligned} \mathbb{E}[V(\rho_{k+1}) \mid \rho_k = \rho, u_k = -1] \\ &= \text{Tr} (d(\mathbf{N}) \rho) + \text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{\mathbf{N}} \right) [d(\mathbf{N} - 1) - d(\mathbf{N})] \rho \right) \\ &= V(\rho) + \text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{\mathbf{N}} \right) [d(\mathbf{N} - 1) - d(\mathbf{N})] \rho \right). \end{aligned}$$

In particular,

$$\begin{aligned} Q_V(\rho, -1) &= -\text{Tr} \left( \sin^2 \left( \frac{\theta_0}{2} \sqrt{\mathbf{N}} \right) [d(\mathbf{N} - 1) - d(\mathbf{N})] \rho \right), \\ &= \sum_{n \in \mathbb{N}} \rho_{nn} [2(n - \bar{n}) - 1] \sin^2 \left( \frac{\theta_0}{2} \sqrt{n} \right). \quad (17) \end{aligned}$$

#### D. Proof of Lemma 1

Assume that  $N_0$  is even (otherwise one may take  $N_0 + 1$  instead of  $N_0$  in this proof). Define the function  $\eta: \mathbb{N} \rightarrow \mathbb{R}$  by

$$\eta(\ell) = \left[ \frac{2}{\theta_0} \left( \ell \frac{\pi}{2} + \frac{\pi}{4} \right) \right]^2. \quad (18)$$

By definition, one has  $\frac{\theta_0}{2} \sqrt{\eta(\ell)} = \ell \frac{\pi}{2} + \frac{\pi}{4}$  for all  $\ell \in \mathbb{N}$ . Let  $h = \pi/4 - \arcsin \left( \sqrt{1/2 - a} \right)$ . Using the definition of

$h$  and the symmetries<sup>6</sup> of the function  $\sin^2(\cdot)$ , it is easy to show that

$$1/2 - a \leq \sin^2(x + \pi/4) \leq a + 1/2, \quad \forall x \in [-h, h]. \quad (19)$$

Let  $\bar{\ell} \in \mathbb{N}$  be even and big enough such that the following two conditions are simultaneously met:

$$\eta(\bar{\ell}) > N_0/2 + N, \quad \frac{1}{8} \theta_0 N_0 / \sqrt{\eta(\bar{\ell}) - N_0/2} \leq h. \quad (20)$$

Now, take  $\bar{N} = \lfloor \eta(\bar{\ell}) \rfloor - \frac{N_0}{2} + 1 > N$ , where  $\lfloor \eta \rfloor$  denotes the greatest integer which is less or equal to  $\eta$ . By construction,  $\eta(\bar{\ell})$  is in-between the points  $\bar{N} + N_0/2 - 1$  and  $\bar{N} + N_0/2$ , and hence it is in the interval  $[\bar{N}, \bar{N} + N_0 - 1]$ . Then, for  $n = \bar{N}, \dots, \bar{N} + N_0 - 1$ , one has that  $|n - \eta(\bar{\ell})| < N_0/2$ . Consider the function  $\phi(x) = \frac{\theta_0}{2} \sqrt{x}$ . From the fact that  $\phi'(x) = \frac{\theta_0}{4\sqrt{x}}$ , by the mean value theorem applied to the function  $\phi$  and the second inequality in (20), one obtains

$$\left| \frac{\theta_0}{2} \sqrt{n} - \frac{\theta_0}{2} \sqrt{\eta(\bar{\ell})} \right| < h, \quad \text{for } n = \bar{N}, \dots, \bar{N} + N_0 - 1.$$

Then, the proof follows easily from (18), (19) and the fact that  $\sin^2(x - \ell\pi/2) = \sin^2(x)$ , for every even  $\ell \in \mathbb{N}$ .

#### E. Proof of Lemma 2

**Proof of the first claim:** Let  $u \in \{-1, 0, 1\}$ ,  $\rho \in D_*$ . Recall that  $W(\rho) = -\sum_{n \in \mathbb{N}} \rho_{nn}^2$ . Since  $\text{Tr}(\rho) = \sum_{n \in \mathbb{N}} \rho_{nn} = 1$ , then  $-1 = -\sum_{n \in \mathbb{N}} \rho_{nn} \leq W(\rho) \leq 0$ . Now, by (1),  $\mathbb{E}[W(\rho_{k+1}) \mid \rho_k = \rho, u_k = u] = p_{g,k} W(\rho_{k+1}^g) + p_{e,k} W(\rho_{k+1}^e)$ , where  $p_{g,k}, p_{e,k} \geq 0$  with  $p_{g,k} + p_{e,k} = 1$ . Thus  $-1 \leq \mathbb{E}[W(\rho_{k+1}) \mid \rho_k = \rho, u_k = u] \leq 0$ . Since  $Q_W(\rho, u)$  is the difference of two numbers that are in-between  $-1$  and  $0$ , one concludes that  $|Q_W(\rho, u)| \leq 1$ .

The second, third and fourth claims, are immediate from (15), (16) and (17) in Appendix C, respectively.

#### F. Proof of Lemma 3

**Proof of the first claim:** Let  $\rho \in D_{m_0}$ . By (8)–(9) in Appendix B,  $\mathbf{M}_g(0) \rho \mathbf{M}_g^\dagger(0) + \mathbf{M}_e(0) \rho \mathbf{M}_e^\dagger(0) = \rho$ . Taking  $\rho_k = \rho$  in  $u_k = 0$  in (1), define

$$\rho^y \triangleq \rho_{k+1}^y = \frac{\mathbf{M}_y(0) \rho \mathbf{M}_y^\dagger(0)}{\text{Tr}(\mathbf{M}_y(0) \rho \mathbf{M}_y^\dagger(0))}, \quad \text{for } y = g, e.$$

Hence,  $\alpha \rho^g + (1 - \alpha) \rho^e = \rho$ , where  $\alpha \triangleq p_{g,k} = \text{Tr}(\mathbf{M}_g(0) \rho \mathbf{M}_g^\dagger(0))$ . In particular,  $\alpha \rho_{nn}^g + (1 - \alpha) \rho_{nn}^e = \rho_{nn}$ , for  $n \in \mathbb{N}$ . Note that, if  $\alpha = 0$ , then  $\mathbf{M}_g(0) \rho \mathbf{M}_g^\dagger(0) = 0$ , and so  $\rho^e = \rho$ . Similarly,  $\alpha = 1$  implies  $\rho^g = \rho$ . Thus, the identity  $\alpha \rho_{nn}^g + (1 - \alpha) \rho_{nn}^e = \rho_{nn}$ , for  $n \in \mathbb{N}$ , still holds when  $\alpha = 0$  or  $\alpha = 1$ . From (1), (7) and  $\alpha = p_{g,k}$ , one has

$$\begin{aligned} Q_W(\rho, 0) &= W(\rho) - [p_{g,k} W(\rho_{k+1}^g) + p_{e,k} W(\rho_{k+1}^e)] \\ &= \sum_{n \in \mathbb{N}} \alpha (\rho_{nn}^g)^2 + (1 - \alpha) (\rho_{nn}^e)^2 - [\alpha \rho_{nn}^g + (1 - \alpha) \rho_{nn}^e]^2 \\ &= \alpha(1 - \alpha) \sum_{n \in \mathbb{N}} [\rho_{nn}^g - \rho_{nn}^e]^2 \geq 0, \quad (21) \end{aligned}$$

<sup>6</sup>More precisely,  $\sin^2(\pi/2 - x) = 1 - \cos^2(\pi/2 - x) = 1 - \sin^2(x)$ .



thereby showing the first part of the first claim.

If  $\rho = |m\rangle\langle m|$  for some  $m \in \mathbb{N}$  with  $0 < \alpha < 1$ , then (8)–(9) in Appendix B imply that  $\rho^g = \rho^e = \rho$ , and so  $Q_W(\rho, 0) = 0$ . Now, one shows that  $\rho = |m\rangle\langle m|$  for some  $m \in \mathbb{N}$  whenever  $Q_W(\rho, 0) = 0$ . Suppose  $Q_W(\rho, 0) = 0$ . Then, (21) implies that  $\alpha = 0$ , or  $\alpha = 1$ , or  $\rho_{nn}^g = \rho_{nn}^e$  for all  $n \in \mathbb{N}$  with  $0 < \alpha < 1$ . Assume that  $\alpha = 0$ . Hence,  $M_g(0)\rho M_g^\dagger(0) = \sum_{n \in \mathbb{N}} \rho_{nn} \cos^2(\frac{\phi_0 n + \phi_R}{2}) |n\rangle\langle n| = 0$  by (8) in Appendix B. Suppose that  $\rho \neq |m\rangle\langle m|$  for every  $m \in \mathbb{N}$ . Thus, there exists  $n_1, n_2 \in \mathbb{N}$  with  $n_1 \neq n_2$ ,  $\rho_{n_1, n_1} > 0$ ,  $\rho_{n_2, n_2} > 0$ . Recall that  $\sin(x_1) = \pm \sin(x_2)$  if and only if  $x_1 + x_2 = \ell\pi$  or  $x_2 - x_1 = \ell\pi$ , where  $\ell$  is an integer. Therefore,  $\sin(\frac{\phi_0 n_1 + \phi_R}{2}) = \pm \sin(\frac{\phi_0 n_2 + \phi_R}{2})$ , which contradicts the assumptions that  $\phi_0/\pi$  is an irrational number and  $\phi_R = \pi/2 - \bar{n}\phi_0$ . One has shown that  $\rho = |m\rangle\langle m|$  for some  $m \in \mathbb{N}$  whenever  $\alpha = 0$ . If  $\alpha = 1$ , or  $\rho_{nn}^g = \rho_{nn}^e$  for all  $n \in \mathbb{N}$  with  $0 < \alpha < 1$ , then from similar arguments and computations one also concludes that  $\rho = |m\rangle\langle m|$  for some  $m \in \mathbb{N}$ .

**Proof of the second claim:** Let  $m \in \mathbb{N}$  and take  $\rho = |m\rangle\langle m| \in D_*$ . It is clear that  $W(\rho) = -\sum_{n \in \mathbb{N}} \rho_{nn}^2 = -1$ . From (10)–(13) in Appendix B, one has that:

$$\begin{aligned} (M_g(+1)\rho M_g^\dagger(+1))_{nn} &= \delta(n, m+1) \sin^2\left(\frac{\theta_0}{2}\sqrt{m+1}\right), \\ (M_e(+1)\rho M_e^\dagger(+1))_{nn} &= \delta(n, m) \cos^2\left(\frac{\theta_0}{2}\sqrt{m+1}\right), \\ (M_g(-1)\rho M_g^\dagger(-1))_{nn} &= \delta(n, m) \cos^2\left(\frac{\theta_0}{2}\sqrt{m}\right), \\ (M_e(-1)\rho M_e^\dagger(-1))_{nn} &= \delta(n+1, m) \sin^2\left(\frac{\theta_0}{2}\sqrt{m}\right), \end{aligned} \quad (22)$$

where  $\delta(n, m)$  is the usual Kronecker delta:  $\delta(n, m) = 0$  if  $n \neq m$ , and  $\delta(n, m) = 1$  if  $n = m$ . In particular:

$$\begin{aligned} \text{Tr}\left(M_g(+1)\rho M_g^\dagger(+1)\right) &= \sin^2\left(\frac{\theta_0}{2}\sqrt{m+1}\right), \\ \text{Tr}\left(M_e(+1)\rho M_e^\dagger(+1)\right) &= \cos^2\left(\frac{\theta_0}{2}\sqrt{m+1}\right), \\ \text{Tr}\left(M_g(-1)\rho M_g^\dagger(-1)\right) &= \cos^2\left(\frac{\theta_0}{2}\sqrt{m}\right), \\ \text{Tr}\left(M_e(-1)\rho M_e^\dagger(-1)\right) &= \sin^2\left(\frac{\theta_0}{2}\sqrt{m}\right), \end{aligned}$$

$$\sum_{n \in \mathbb{N}} \left( \frac{M_y(u)\rho M_y^\dagger(u)}{\text{Tr}\left(M_y(u)\rho M_y^\dagger(u)\right)} \right)_{nn}^2 = 1, \text{ for } u = \pm 1, y = g, e$$

(assuming no division by 0). Now, using (1) and the above computations, one gets

$$\begin{aligned} \mathbb{E}[W(\rho_{k+1}) \mid \rho_k = \rho, u_k = \pm 1] &= p_{g,k} W(\rho_{k+1}^g) + p_{e,k} W(\rho_{k+1}^e) \\ &= - \sum_{y=g,e} \left[ \text{Tr}\left(M_y(\pm 1)\rho M_y^\dagger(\pm 1)\right) \times \right. \\ &\quad \left. \times \sum_{n \in \mathbb{N}} \left( \frac{M_y(\pm 1)\rho M_y^\dagger(\pm 1)}{\text{Tr}\left(M_y(\pm 1)\rho M_y^\dagger(\pm 1)\right)} \right)_{nn}^2 \right] \\ &= -1 = W(\rho). \end{aligned}$$

Therefore,  $Q_W(|m\rangle\langle m|, \pm 1) = 0$ .

## G. Proof of Proposition 2

Fix  $\rho \in \mathbb{D}_*$ . Since  $\text{Tr}(d(\mathbf{N})\rho) = \text{Tr}(d(\mathbf{N})\Delta\rho)$  and  $\rho_{nn} = (\Delta\rho)_{nn}$  for  $n \in \mathbb{N}$ , the first two assertions are immediate from the definitions. As for the third and fourth assertions, let  $|\psi\rangle = \sum_{m=0}^{\infty} \langle m|\psi\rangle |m\rangle \in \mathcal{H}$ . Note that  $\rho|m\rangle = \sum_{n=0}^{n_{\max}(\rho)} \rho_{mn} |n\rangle$ , for  $m \in \mathbb{N}$ . Using (2)–(4):

$$\begin{aligned} M_g(0)\rho M_g^\dagger(0)|\psi\rangle &= \sum_{m,n=0}^{n_{\max}(\rho)} \rho_{mn} \cos\left(\frac{\phi_0 m + \phi_R}{2}\right) \cos\left(\frac{\phi_0 n + \phi_R}{2}\right) \langle m|\psi\rangle |n\rangle, \\ M_e(0)\rho M_e^\dagger(0)|\psi\rangle &= \sum_{m,n=0}^{n_{\max}(\rho)} \rho_{mn} \sin\left(\frac{\phi_0 m + \phi_R}{2}\right) \sin\left(\frac{\phi_0 n + \phi_R}{2}\right) \langle m|\psi\rangle |n\rangle, \\ M_g(+1)\rho M_g^\dagger(+1)|\psi\rangle &= \sum_{m=1, n=0}^{n_{\max}(\rho)+1} \rho_{m-1, n} \sin\left(\frac{\theta_0}{2}\sqrt{m}\right) \sin\left(\frac{\theta_0}{2}\sqrt{n+1}\right) \langle m|\psi\rangle |n+1\rangle, \\ M_e(+1)\rho M_e^\dagger(+1)|\psi\rangle &= \sum_{m,n=0}^{n_{\max}(\rho)} \rho_{mn} \cos\left(\frac{\theta_0}{2}\sqrt{m+1}\right) \cos\left(\frac{\theta_0}{2}\sqrt{n+1}\right) \langle m|\psi\rangle |n\rangle, \\ M_g(-1)\rho M_g^\dagger(-1)|\psi\rangle &= \sum_{m,n=0}^{n_{\max}(\rho)} \rho_{mn} \cos\left(\frac{\theta_0}{2}\sqrt{m}\right) \cos\left(\frac{\theta_0}{2}\sqrt{n}\right) \langle m|\psi\rangle |n\rangle, \\ M_e(-1)\rho M_e^\dagger(-1)|\psi\rangle &= \sum_{m=0, n=1}^{n_{\max}(\rho)} \rho_{m+1, n} \sin\left(\frac{\theta_0}{2}\sqrt{m+1}\right) \sin\left(\frac{\theta_0}{2}\sqrt{n}\right) \langle m|\psi\rangle |n-1\rangle. \end{aligned}$$

Since  $\Delta\rho \in D_* \subset \mathbb{D}_*$ ,  $n_{\max}(\Delta\rho) = n_{\max}(\rho)$  and  $(\Delta\rho)_{nn} = \rho_{nn}$ , the proof is straightforward from (8)–(13) in Appendix B.

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