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Stabilization of photon-number states via single-photon corrections: a first convergence analysis under an ideal set-up

H. B. Silveira  P. S. Pereira da Silva  P. Rouchon

Abstract—This paper presents a first mathematical convergence analysis of a Fock states feedback stabilization scheme via single-photon corrections. This measurement-based feedback has been developed and experimentally tested in 2012 by the cavity quantum electrodynamics group of Serge Haroche and Jean-Michel Raimond. Here, we consider the infinite-dimensional Markov model corresponding to the ideal set-up where detection errors and feedback delays have been disregarded. In this ideal context, we show that any goal Fock state can be stabilized by a Lyapunov-based feedback for any initial quantum state belonging to the dense subset of finite rank density operators with support in a finite photon-number sub-space. Closed-loop simulations illustrate the performance of the feedback law.

I. INTRODUCTION

In [8], a photon-number states (Fock state) feedback stabilization scheme via single-photon corrections was described and experimentally tested. Such control problem is relevant for quantum information applications [6], [4]. The quantum state \( \rho \) corresponds to the density operator of a microwave field stored inside a superconducting cavity and described as a quantum harmonic oscillator. At each sample step \( k \in \mathbb{N} \), a probe atom is launched inside the cavity. The measurement outcome \( y_k \), detected by a sensor is the energy-state of this probe atom after its interaction with the microwave field. Each probe atom is considered as a two-level system: either it is detected in the lowest energy state \( |g\rangle \), or the highest energy state \( |e\rangle \). Consequently, the measurement outcomes corresponds to a discrete-valued output \( y_k \) with only two distinct possibilities: \( g \) or \( e \). Similarly, the control inputs \( u_k \) are also discrete-valued with 3 distinct possibilities: \(-1, 0, +1\). The open-loop value \( u_k = 0 \) corresponds to a dispersive atom/field interaction: it achieves in fact a Quantum Non-Demolition measurement of Fock states [2]. The two other values \( u_k = \pm 1 \) correspond to resonant atom/field interactions where the probe atom and the field exchange energy quanta: these values achieve single-photon corrections.

Although the feedback law proposed and implemented in [8] considered imperfect detections on \( y_k \) and delays in the control, here we focus on an ideal-set up, that is, detection errors and control delays have been disregarded. Theorem 2 shows that, by adding an arbitrarily small term to the Lyapunov function used in [8], one ensures almost sure stabilization of any goal Fock state for the closed-loop ideal set-up. This is achieved by relying on an infinite-dimensional Markov model of the ideal set-up that takes into account the back-action of the measurement outcome \( y_k \) on the quantum state \( \rho_{k+1} \).

Loosely speaking, in [8], the control value \( u_k \) at each sampling step \( k \) was chosen so as to minimize the conditional expectation of the Lyapunov function \( V(\rho_k) = \text{Tr}(d(N)\rho_k) \), where \( N \) is the photon-number operator, \( d(n) = (\bar{n} - \bar{n})^2 \) and \( \bar{n} = \langle \bar{n} \rangle \) is the goal Fock state. However, in closed-loop, the difference between such \( V \) and its conditional expectation is not strictly positive: such \( V \) does not become a strict Lyapunov function in closed-loop and additional arguments have to be considered to prove convergence. These additional arguments are related to Lasalle invariance. They are well established in a smooth context where the control \( u \) is a smooth function of the state \( \rho \). This cannot be the case here since \( u \) is a discrete-valued control. In order to overcome such technical difficulties, we propose, similarly to [1], to add the arbitrarily small term \( -\epsilon \sum_{n=0}^{\infty} \langle n|\rho_k|n \rangle^2 \) to \( V(\rho_k) \), where \( \epsilon > 0 \). This slightly modified control-Lyapunov function becomes then a strict-Lyapunov function in closed-loop that simplifies notably the convergence analysis. Moreover, the developed convergence analysis is done in the infinite-dimensional setting in the following sense: we show that, for any initial density operator \( \rho_0 \) with a finite photon-number support (\( \rho_0|n \rangle = 0 \) for \( n \) large enough), the closed-loop trajectory \( k \mapsto \rho_k \) remains also with a finite photon-number support with a uniform bound on the maximum photon-number. This almost finite-dimensional behavior simplifies the convergence analysis despite the fact that such condition on \( \rho_0 \) is met on a dense subset of density operators (Hilbert-Schmidt topology on the Banach space of Hilbert-Schmidt self-adjoint operators).

The paper is organized as follows. Section II presents the ideal Markov model of the experimental set-up of the controlled microwave super-conducting cavity reported in [8] and precisely formulates the Fock state stabilization problem here treated (see Definition 1). Section III establishes the proposed solution to the control problem in two distinct parts. Firstly, Section III-A considers the case where the initial condition \( \rho_0 \) is a diagonal density operator (see Theorem 1).
Only the main ideas of the convergence proof are outlined. The technical details are given in Section [IV]. Afterwards, in Section [III-B] the main result of the paper is presented: the general solution is obtained from Theorem [I] for $\rho_0$ belonging to a dense subset (see Theorem [2]). The simulation results are exhibited in Section [IV]. The proof of some intermediate results and computations required in Sections [III] and [V] are presented in Appendices [B,C]. Finally, the concluding remarks are given in Section [VI].

II. IDEAL MARKOV MODEL

Denote by $\mathcal{H}$ the separable complex Hilbert space $L_2(\mathbb{C})$ with orthonormal basis $\{ |n\rangle, n \in \mathbb{N} \}$ of Fock states (photons). Hence, $\mathcal{H} = \{ \sum_{n \in \mathbb{N}} \psi_n |n\rangle, \langle \psi_0, \psi_1, \ldots \} \in L_2(\mathbb{C})$. Let $\mathcal{D}$ be the set of all density operators on $\mathcal{H}$, that is, the set of trace-class, self-adjoint, non-negative operators. The almost sure convergence above is with respect to the density operators. It is clear that $M_g(u), M_e(u)$ are bounded operators on $\mathcal{H}$ with $M_g^\dagger(u) M_g(u) + M_e^\dagger(u) M_e(u) = I$ (identity operator), $M_e(-1) = M_e^\dagger(+1) = a \sin(\frac{\theta}{2}) \sqrt{N}/\sqrt{N}$, and $M_g(-1), M_g(0), M_e(0), M_e(+1)$ are self-adjoint. It is easy to see that if the initial condition $\rho_0$ is a density operator, then, for all realizations of the ideal Markov process $\rho_k$ is a density operator for $k \in \mathbb{N}$.

Notice that $\bar{\pi} = \|\pi\| \pi$ is a steady state of the Markov process $\{(u_k, \pi_k)\}$ with $u_k = 0$, where $\pi_k$ is an arbitrary control. The control problem here treated is given as follows:

Definition 1: For the ideal Markov process $\{u_k, \pi_k\}$, the control problem is to find a feedback law $u_k = f(\rho_k)$ such that, given an initial condition $\rho_0 \in \mathcal{D}$, the closed-loop trajectory $\rho_k$ converges almost surely towards the goal Fock state $\bar{\pi} = \|\pi\| \pi$ as $k \to \infty$.

The almost sure convergence above is with respect to the probabilities amplitudes $P_n(\rho) = \operatorname{Tr} (|n\rangle \langle n| \rho)$ of $\rho$ that is, $\lim_{k \to 0} P_n(\rho_k) = P_n(\bar{\pi})$ for each $n \in \mathbb{N}$. In other words, $\lim_{k \to 0} P_n(\rho_k) = 1$ and $\lim_{k \to 0} P_n(\rho_k) = 0$ when $n \neq \pi$. The solution proposed in this paper for the control problem above is developed in the next section.

III. STABILIZATION OF FOCK STATES

Given any operator $A: \mathcal{H} \rightarrow \mathcal{H}$, let $A_{mn} = \langle m | A | n \rangle$ for $m, n \in \mathbb{N}$. Hence, $A_{mn}$ is the $n$-th diagonal element of $A$, while $A_{mn}$ with $m \neq n$ correspond to its “off-diagonal” elements. One says that the operator $A$ is diagonal when $A_{mn} = 0$ for all $m, n \in \mathbb{N}$ with $m \neq n$. One shall begin by solving the control problem given in Definition [I] in the particular case where the initial condition $\rho_0$ is diagonal (see Theorem [I] in Section [III-A]). Afterwards, in Section [III-B] the solution to the general non-commutative case is presented (see Theorem [2]): its solution relies essentially on the diagonal case.

A. Diagonal case

For each $n^* \in \mathbb{N}$, define

$$D_{n^*} = \{ \rho \in \mathcal{D} : \rho \text{ is diagonal and } \rho |n\rangle = 0, \forall n > n^* \}.$$  

Consider the set $D_{n^*} = \bigcup_{n \in \mathbb{N}} D_{n^*}$, $\subset \mathbb{D}$. Note that $D_{n^*} \subset D_{n^*+1}$, and that each element $\rho$ of $D_{n^*}$ is “finite dimensional” in the following sense: $\rho \in \mathcal{D}$ is in $D_{n^*}$ if and only if $\rho = \sum_{n=0}^{n^*} \rho_{nn} |n\rangle \langle n|$, and $\rho \in D_{n^*}$ may be considered as an operator from $\mathcal{H}$ to the finite-dimensional space $\mathcal{H}_{n^*} = \text{span} \{|0\rangle, \ldots, |n^*\rangle\}$, or as a density matrix on $\mathcal{H}_{n^*}$. One defines the functions $n_{min}, D_{n^*} \rightarrow \mathbb{N}$, $n_{max}, D_{n^*} \rightarrow \mathbb{N}$ and $n_{length}, D_{n^*} \rightarrow \mathbb{N}$ respectively by:

- $n_{min}(\rho)$ is the smallest $n \in \mathbb{N}$ such that $\rho |n\rangle \neq 0$;
- $n_{max}(\rho)$ is the greatest $n \in \mathbb{N}$ such that $\rho |n\rangle \neq 0$;
- $n_{length}(\rho) = n_{max}(\rho) - n_{min}(\rho)$.

It is clear that, given $\rho \in D_{n^*}$, one has $\rho \in D_{n^*}$ if and only if $n_{max}(\rho) \leq n^*$. The next result exhibits the properties of the state $\rho_k$ of (1)-(4) with respect to these functions.

1As usual in quantum physics, it is here assumed that the measurement outcome $y_k = y$ cannot occur when $\operatorname{Tr} (M_g(u_k) \rho_0 M_g^\dagger(u_k)) = 0$, for $y = g, e$.

2For instance, $M_g(+1)|n\rangle = \left( \sin \frac{\phi}{2} \sqrt{N}/\sqrt{N} \right) \sqrt{n+1} |n+1\rangle = \sin \frac{\phi}{2} \sqrt{n+1} |n+1\rangle$. In order for the definition of $M_e(-1)$ to be consistent, it is assumed $\sin(0)/0 = 1$. Note that if $\rho = |n\rangle \langle n|$ for some $n \in \mathbb{N}$, then $\rho \in D_{n^*}$.
Proposition 1: For every realization of the ideal Markov process $\{1-4\}$ with initial condition $\rho_0 \in D_*$, one has that $\rho_k \in D_*$ for all $k \in \mathbb{N}$ with:

- If $u_k = 0$ or $u_k = -1$, then $n_{max}(\rho_{k+1}) \leq n_{max}(\rho_k)$ and $n_{length}(\rho_{k+1}) \leq n_{length}(\rho_k)$;
- If $u_k = +1$, then $n_{max}(\rho_{k+1}) \leq n_{max}(\rho_k) + 1$ and $n_{length}(\rho_{k+1}) \leq n_{length}(\rho_k)$.

Proof: See Appendix $\Box$

Take a goal photon-number $\pi \in \mathbb{N}$. As in [1], consider the following Lyapunov function $V_c: D_* \rightarrow \mathbb{R}$ defined as

$$V_c(\rho) = \text{Tr} \left( d(N)\rho \right) - \epsilon \sum_{n \in \mathbb{N}} \rho_{nn}^2, \quad \text{for } \rho \in D_*,$$

(5)

where $\epsilon > 0$ is a real number and $d(n) = (n-\pi)^2$ as defined in [8]. The feedback law $u: D_* \rightarrow \{-1, 0, 1\}$ is given by

$$u = f(\rho) \triangleq \text{Argmin}_{\nu \in \{-1, 0, 1\}} \mathbb{E}_n \left[ V_c(\rho_{k+1}) \mid \rho_k = \nu, u_k = \nu \right].$$

(6)

Note that for each $\rho \in D_*$ and $n^* \geq n_{max}(\rho)$, $d(N)\rho$ in (5) is a well-defined self-adjoint, nonnegative, trace-class operator on $\mathcal{H}$, considering $d(N)$ as an operator on $\mathcal{H}_{n^*}$ and $\rho$ as an operator from $\mathcal{H}$ to $\mathcal{H}_{n^*}$. Indeed, $d(N)\rho = \sum_{n=0}^{n^*} \rho_{nn}(n-\pi)^2/n$. Thus, (5) is well-defined. Moreover, since $\mathcal{H}_{n^*}$ is invariant under $\rho \in D_*$, for $n^* \geq n_{max}(\rho)$, it is clear that $\text{Tr}(d(N)\rho) = \text{Tr}_{\mathcal{H}_{n^*}}(d(N)\rho)$, where the right-hand side one considers $\rho$ as an operator on the finite-dimensional space $\mathcal{H}_{n^*}$ and the trace is taken over $\mathcal{H}_{n^*}$.

We have the following convergence result when $\rho_0 \in D_*$:

Theorem 1: Let $\pi \in \mathbb{N}$ and $\epsilon > 0$. In $\{2, 4\}$, assume that $\phi_0/\pi$ and $(\theta_0/\pi)^2$ are irrational numbers, and take $\phi_R = \pi/2 - \pi\phi_0$. Consider the closed-loop Markov process $\{1-4\}$ with $u_k = f(\rho_k)$, where the feedback law $f$ is as in (6). Then, given any initial condition $\rho_0 \in D_*$, one has that $\rho_k$ converges almost surely towards $\pi = [\pi/\pi]$, as $k \rightarrow \infty$.

Its proof is decomposed into two steps:

First Step. Choose $\pi \in \mathbb{N}$ and $\epsilon > 0$. Let $n_0 = n_{length}(\rho_0)$, $r_0 = n_{min}(\rho_0)$. Then, there exists an integer $m_0 > n_0 + r_0 + \pi + 1$ (depending on $n_0, r_0, \pi$ and $\epsilon$) such that, for all closed-loop realizations $\rho_k$, one has $\rho_k \in D_{m_0}$ for $k \in \mathbb{N}$.

Second Step. Choose irrational numbers $\phi_0/\pi$ and $(\theta_0/\pi)^2$ in $\{2, 4\}$, and take $\phi_R = \pi/2 - \pi\phi_0$. In $D_{m_0}$, $V_c$ is a strict super-martingale: for all density operators $\rho \in D_{m_0}$, one has

$$\mathbb{E}_n \left[ V_c(\rho_{k+1}) \mid \rho_k = \rho, u_k = f(\rho) \right] = V_c(\rho) - QV_c(\rho, f(\rho)),$$

where $QV_c(\rho, f(\rho)) \geq 0$, and $QV_c(\rho, f(\rho)) = 0$ if and only if $\rho = \pi$. The almost sure convergence follows then from usual results on strict super-martingales for Markov processes with compact state spaces.

The complete proof of the two steps above is presented in Section $\Box$. The general case where the initial condition $\rho_0$ is not necessarily diagonal is treated in the next subsection.

B. General case

Consider, for each $n^* \in \mathbb{N}$,

$$\mathbb{D}_{n^*} = \{ \rho \in \mathbb{D} \mid \rho |n\rangle = 0, \forall n > n^* \} \subset \mathbb{D}_{n^*+1},$$

and let $\mathbb{D}_* = \bigcup_{n^* \in \mathbb{N}} \mathbb{D}_{n^*} \supset \mathbb{D}_*$. It is clear that $\rho \in \mathbb{D}$ is in $\mathbb{D}_{n^*}$ if and only if $\rho = \sum_{m,n=0}^{n_{max}} \rho_{mn} |m\rangle |n\rangle$. Consequently, $\mathbb{D}_*$ is a dense subset of $\mathbb{D}$ when $\mathbb{D}$ is endowed with the subspace topology induced from the Hilbert-Schmidt norm. Indeed, let $\mathcal{J}_2$ be the complex Banach space of all Hilbert-Schmidt operators on $\mathcal{H}$ with the Hilbert-Schmidt norm $\|B\|_2 = (\sum_{m,n \in \mathbb{N}} |B_{mn}|^2)^{1/2}$, for $B \in \mathcal{J}_2$ [7], [3]. Since $\mathbb{D} \subset \mathcal{J}_2$ and $\rho \in \mathbb{D}_{n^*}$ has the form $\rho = \sum_{m,n=0}^{n_{max}} \rho_{mn} |m\rangle |n\rangle$, the density property of $\mathbb{D}_*$ in $\mathbb{D}$ is clear.

One has that $\rho \in \mathbb{D}_{n^*}$ may be considered as an operator from $\mathcal{H}$ to the finite-dimensional space $\mathcal{H}_{n^*}$, or as a density matrix on $\mathcal{H}_{n^*}$. Hence, $d(N)\rho$ is a well-defined trace-class operator on $\mathcal{H}$, by considering $d(N)$ as an operator on $\mathcal{H}_{n^*}$ and $\rho$ as an operator from $\mathcal{H}$ to $\mathcal{H}_{n^*}$. Indeed, $d(N)\rho = \sum_{n=0}^{n_{max}} \rho_{nn}(n-\pi)^2/n$. Thus, (5) is well-defined. Moreover, since $\mathcal{H}_{n^*}$ is invariant under $\rho \in D_*$, for $n_{max}(\rho) \geq n_{max}(\rho)$, it is clear that $\text{Tr}(d(N)\rho) = \text{Tr}_{\mathcal{H}_{n^*}}(d(N)\rho)$, where on the right-hand side one considers $\rho$ as an operator on the finite-dimensional space $\mathcal{H}_{n^*}$ and the trace is taken over $\mathcal{H}_{n^*}$.

We have the following convergence result when $\rho_0 \in D_*$:

Proposition 2: Let $\pi \in \mathbb{N}$, $u \in \{-1, 0, 1\}$, $y = g, e$. Take $\alpha = \text{Tr} \left( M_y(u)\rho M_y(u)^* \right)$. Then:

- $\text{Tr}(\Delta \rho) = \text{Tr}(\Delta \rho)$, for every diagonal bounded operator $A: \mathcal{H} \rightarrow \mathcal{H}$;
- $V_c(\rho) = V_c(\Delta \rho)$, for $\epsilon > 0$;
- $\alpha^{-1} M_y(u)\rho M_y(u)^*$ belongs to $\mathbb{D}_*$ with $\Delta (\alpha^{-1} M_y(u)\rho M_y(u)^*) = \alpha^{-1} M_y(u)(\Delta \rho) M_y(u)^*$;
- $[M_y(u)(\Delta \rho) M_y(u)^*]_{nn} = [M_y(u)\rho M_y(u)^*]_{nn}$, for all $n \in \mathbb{N}$. In particular, $\alpha = \text{Tr} \left( M_y(u)\rho M_y(u)^* \right)$.

Proof: See Appendix $\Box$

Now, let $\epsilon > 0$ and $\pi = [\pi/\pi], \pi \in \mathbb{N}$. Assume that $\rho_0 \in D_*$. Let $\rho_k, k \in \mathbb{N}$, be the corresponding closed-loop trajectory for a fixed realization of $\{1-4\}$ with feedback $u_k = f(\rho_k)$, where $f$ is as in (6). It is immediate from the proposition above that:

- $\rho_k \in \mathbb{D}_{n^*}$, for $k \in \mathbb{N}$;
- $\Delta \rho_k \in D_*$, $k \in \mathbb{N}$, is the corresponding closed-loop trajectory of $\{1-4\}$ for the initial condition $\Delta \rho_0$, the same realization (and with the same transition probabilities $p_{e,k} = p_{k,k})$, as well as the same feedback $u_k = f(\rho_k)$, where the feedback law $f$ is as in (6).

From these arguments, Theorem $\Box$ and the fact that $\Delta \pi = \pi$, one immediately obtains the following generic solution to the control problem, that is, when the initial condition $\rho_0$ belongs to the dense subset $\mathbb{D}_*$ of $\mathbb{D}$:

Theorem 2: Let $\pi \in \mathbb{N}$ and $\epsilon > 0$. In $\{2, 4\}$, assume that $\phi_0/\pi$ and $(\theta_0/\pi)^2$ are irrational numbers, and take $\phi_R = \pi/2 - \pi\phi_0$. Consider the closed-loop Markov process $\{1-4\}$ with $u_k = f(\rho_k)$, where the feedback law $f$ is as in (6). Then, given any initial condition $\rho_0 \in D_*$, one has that $\rho_k$ converges almost surely towards $\pi = [\pi/\pi]$ as $k \rightarrow \infty$. 
IV. SIMULATION RESULTS

This section presents the closed-loop simulation results concerning the application of Theorem 2 above to the ideal Markov process (1–4). The quantum experimental results exhibited in [8] used the following control parameter values in (2–4): \( \phi_0/\pi = 0.252 \) and \( \theta_0/\pi \approx 2/\sqrt{\pi + 1} \). However, according to the assumptions in Theorem 2, \( \phi_0/\pi \) and \( (\theta_0/\pi)^2 \) should be irrational numbers. Hence, here one chooses \( \phi_0/3.14 = 0.252 \) and \( \theta_0/3.14 = 2/\sqrt{\pi + 1} \). One takes \( \rho_0 = \sum_{n=0}^{15} |n\rangle\langle n|/16 \in \mathbb{D} \), as the initial condition, \( \pi = 10 \) for the goal Fock state \( \overline{\rho} = |\pi\rangle\langle \pi| \), and \( \epsilon = 10^3 \) as the gain for the feedback \( u_k = f(\rho_k) \) in (5)–(6). Figure 1 exhibits the simulation results for one closed-loop realization with such choices and a final sample step of 120. It shows: the dynamics of the populations of \( \rho_k \) (top), the controls \( u_k \) (middle) and the simulated outcomes \( y_k \) (bottom). The populations of \( \rho_k \) correspond to the following observables:

\[
A_1 = \sum_{n=0}^{14} |n\rangle\langle n| (n < \overline{n}), \quad A_2 = |\overline{n}\rangle\langle \overline{n}| (n = \overline{n}), \quad A_3 = \sum_{n=\overline{n}+1}^{\infty} |n\rangle\langle n| (n > \overline{n}).
\]

Therefore, one sees from the dynamics of the populations that \( \rho_k \) converges to \( \overline{\rho} \) as \( k \to \infty \), which is in accordance with Theorem 2.

Note that \( \langle |\overline{n}\rangle\langle \overline{n}| \rangle \approx 1 \) and \( u_k = 0 \) for all \( k > 45 \).

Recall that Theorem 2 assumes that \( \epsilon > 0 \). In order to further analyze the performance of the Lyapunov-based feedback law here proposed, we now make a comparison with the one used experimentally in [8], which corresponds to take \( \epsilon = 0 \) in (5), i.e. to disregard the term \( -\epsilon \sum_{n \in \mathbb{N}} \rho_{n \overline{n}}^2 \). Figure 2 presents the simulation results for one closed-loop realization of such case. The control parameters, \( \rho_0 \) and \( \pi = 10 \) are the same as above. Note that \( \langle |\overline{n}\rangle\langle \overline{n}| \rangle \approx 1 \) and \( u_k = 0 \) for all \( k > 78 \). In order to make a comparison in terms of the speed of convergence, define the settling time \( k_s \) to be the smallest \( k \in \mathbb{N} \) such that \( \langle |\overline{n}\rangle\langle \overline{n}| \rangle > 0.9 \) for all \( k \geq k_s \). One has \( k_s = 45 \) for the case \( \epsilon = 10^3 \) above, and \( k_s = 78 \) for \( \epsilon = 0 \). Therefore, in the two realizations here simulated, the choice of \( \epsilon = 10^3 \) reduced the settling time \( k_s \) by nearly 42\% with respect to \( \epsilon = 0 \). This behavior is typical on an average basis, thereby justifying the term \( -\epsilon \sum_{n \in \mathbb{N}} \rho_{n \overline{n}}^2 \) in (5).

Table I shows the average value \( \overline{k}_s \) and the standard deviation \( \sigma \) of \( k_s \) for \( \epsilon \in \{0,0.1,1,10,10^2,10^3,10^4,10^5\} \), of a total of 5000 realizations for each \( \epsilon \). Notice that when \( \epsilon \) is relatively large or relatively small in comparison to \( \epsilon = 10^3 \), the average settling time \( \overline{k}_s \) deteriorated. Furthermore, although for \( \epsilon = 10^5 \) one has that \( \overline{k}_s \) increased by nearly 22\% in comparison to \( \epsilon = 10^3 \), the standard deviation \( \sigma \) decreased by nearly 62\%. Computer simulations have suggested that a choice of \( \epsilon > 0 \) which may perhaps significantly improve \( \overline{k}_s \) generally depends on the initial condition \( \rho_0 \) and on the goal Fock state \( \overline{\rho} = |\overline{n}\rangle\langle \overline{n}| \), and it has to be determined heuristically.

V. PROOF OF THEOREM 1 (DIAGONAL CASE)

Proof of the First Step:

Let \( \epsilon > 0 \). Define \( V: D_+ \to \mathbb{R} \) and \( W: D_+ \to \mathbb{R} \) as

\[
V(\rho) = \text{Tr} (d(N)\rho), \quad W(\rho) = -\sum_{n \in \mathbb{N}} \rho_{n \overline{n}}^2, \quad (7)
\]

respectively. Note that \( V_0 = V + \epsilon W \). Define:

\[
Q_W(\rho, u) = W(\rho) - \mathbb{E} [W(\rho_{k+1}) | \rho_k = \rho, u_k = u],
\]

\[
Q_V(\rho, u) = V(\rho) - \mathbb{E} [V(\rho_{k+1}) | \rho_k = \rho, u_k = u],
\]

\[
Q_{V_\epsilon}(\rho, u) = V_\epsilon(\rho) - \mathbb{E} [V_\epsilon(\rho_{k+1}) | \rho_k = \rho, u_k = u],
\]

and

\[
Q_{\bar{V}}(\rho, u) = \mathbb{E} [\bar{V}(\rho_{k+1}) | \rho_k = \rho, u_k = u],
\]

where

\[
\bar{V}(\rho) = \text{Tr} (d(N)\rho) - \sum_{n \in \mathbb{N}} \rho_{n \overline{n}}^2.
\]
for $\rho \in D_s$ and $u \in \{-1, 0, 1\}$. The proof of Theorem 1 is a straightforward consequence of the next proposition:

**Proposition 3:** Let $\epsilon > 0$ and $n_0, r_0, \pi \in \mathbb{N}$. There exists an integer $n_0 > n_0 + r_0 + \pi + 1$ (depending on $\epsilon, n_0, r_0, \pi$) such that, for each $\rho \in D_s$ with $|\text{length}(\rho)| \leq n_0$, if $n_{\max}(\rho) = m_0$, then

$$Q_V(\rho, -1) > \max\{Q_V(\rho, 0), Q_V(\rho, +1)\}.$$ 

In fact, given $\rho_0 \in D_s$, let $n_0 = |\text{length}(\rho_0)|$ and $r_0 = n_{\min}(\rho_0)$. Note that $n_{\max}(\rho_0) = n_0 < m_0$. By Proposition 1 $\rho_k \in D_s$ with $|\text{length}(\rho_k)| \leq n_0$, for all $k \in \mathbb{N}$. Since $u = f(\rho)$ maximizes $Q_V(\rho, f(\rho))$, Proposition 3 implies that when $n_{\max}(\rho_k) = m_0$ for some $k \in \mathbb{N}$, then the input $u_k$ will be always be equal to $-1$, and hence Proposition 1 ensures that $n_{\max}(\rho_{k+1}) \leq n_{\max}(\rho_k) = m_0$. Therefore, $n_{\max}(\rho_k) \leq m_0$, $k \in \mathbb{N}$, showing the First Step.

The following two lemmas are instrumental for showing Proposition 3. Their proofs are given in Appendix D and Appendix E, respectively.

**Lemma 1:** Given an arbitrary nonzero $\theta_0 \in \mathbb{R}$, fix any $a \in \mathbb{R}$ such that $0 < a < 1/2$. For all nonzero $N, N \in \mathbb{N}$, there exists an integer $N > N$ big enough such that,

$$0 < 1/2 - a \leq \sin^2 \left(\frac{\theta_0}{2} \sqrt{N} \right) \leq 1/2 + a,$$

for $n = N, N + 1, \ldots, N + n_0 - 1$.

**Lemma 2:** Let $\rho \in D_s$. Then:

- $|Q_V(\rho, u)| \leq 1$, for each $u \in \{-1, 0, 1\}$;
- $Q_V(\rho, 0) = 0$;
- $Q_V(\rho, +1) = -\sum_{n \in \mathbb{N}} \rho_{\min}[2(n - \pi) + 1] \sin^2 \left(\frac{\theta_0}{2} \sqrt{n + 1} \right)$;
- $Q_V(\rho, -1) = \sum_{n \in \mathbb{N}} \rho_{\max}[2(n - \pi) - 1] \sin^2 \left(\frac{\theta_0}{2} \sqrt{n} \right)$.

The proof of Proposition 3 is shown in the sequel.

**Proof:** Let $\epsilon > 0$ and $n_0, r_0, \pi \in \mathbb{N}$. One has to show that there exists $m_0 > n_0 + r_0 + \pi + 1$ such that, if $\rho \in D_s$ with $|\text{length}(\rho)| \leq n_0$, then $u = -1$ always maximizes $Q_V(\rho, u)$ whenever $n_{\max}(\rho) = m_0$. From Lemma 2 and the fact that $Q_V = Q_V + cQ_W$, to complete the proof it suffices to show that:

- If $\rho \in D_s$ is such that $n_{\max}(\rho) \leq n_0$ and $n_{\min}(\rho) \geq n_0 + \pi$, then $Q_V(\rho, +1) \leq 0$;
- There exists $m_0 > n_0 + r_0 + \pi + 1$ such that $Q_V(\rho, -1) > 2\epsilon$, whenever $\rho \in D_s$ is such that $n_{\min}(\rho) \leq n_0$ and $n_{\max}(\rho) = m_0$.

Note that

$$Q_V(\rho, +1) = -\sum_{n = n_{\min}(\rho)}^{n_{\max}(\rho)} \rho_{n}[2(n - \pi) + 1] \sin^2 \left(\frac{\theta_0}{2} \sqrt{n + 1} \right),$$

for any $\rho \in D_s$. Thus, if $n_{\min}(\rho) \leq n_0$ and $n_{\min}(\rho) \geq \pi + n_0$, then $n_{\min}(\rho) \geq \pi$, and hence the first claim is shown.

Now, fix $0 < a < 1/2$ and let $N \geq \frac{2}{1/2 - a + 2\pi + 1}$. Applying Lemma 1 for $N_0 = n_0 + r_0 + 1$ and such choice of $N$, one gets $N > N$ in which $0 < 1/2 - a \leq \sin^2 \left(\frac{\theta_0}{2} \sqrt{n+1} \right)$.

As $N$ is an integer, it follows that $N \geq \pi + 1$.

4As $N$ is an integer, it follows that $N \geq \pi + 1$.
Theorem 3: [5, Theorem 1, p. 195] Let \( \Omega \) be a probability space and let \( W \) be a measurable space. Consider that \( X_k: \Omega \to W, k \in \mathbb{N} \), is a Markov chain with respect to the natural filtration. Let \( Q: W \to \mathbb{R} \) and \( V: W \to \mathbb{R} \) be measurable non-negative functions with \( V(X_k) \) integrable for all \( k \in \mathbb{N} \). If \( \mathbb{E}[V(X_{k+1}) | X_k] - V(X_k) = -Q(X_k) \), for \( k \in \mathbb{N} \), then \( \lim_{k \to \infty} Q(X_k) = 0 \) almost surely.

Indeed, let \( J_1 \) be the complex Banach space of all trace-class operators on \( \mathcal{H} \) with the trace norm \( \| \cdot \|_1 \), that is, \( \|B\|_1 = \text{Tr}(\|B\|) \), for \( B \in J_1 \). Recall that \( \{B\} \leq \|B\|_1 \). Every \( B \in J_1 \) and each bounded operator \( \mathcal{A}: \mathcal{H} \to \mathcal{H} \), where \( \|\cdot\| \) is the usual operator norm (sup norm of bounded operators) [7], [3]. Consider the subspace topology on \( D_{mo} \) with respect to \( J_1 \). One has that the closed-loop trajectory \( \rho_k, k \in \mathbb{N} \), is a Markov chain with phase space \( D_{mo} \) (with respect to the natural filtration and the Borel algebra on \( D_{mo} \)). It is clear that \( D_{mo} \) is compact, and that \( Q \) is continuous on \( D_{mo} \), for all \( \epsilon > 0 \), where \( \alpha \Delta \min_{\rho \in D_{mo}} V_\epsilon(\rho) \). The theorem above implies that \( \rho_k \) converges almost surely towards \( \overline{\rho} \) as \( k \to \infty \) (with respect to the trace norm). This completes the proof of Theorem 1.

VI. CONCLUDING REMARKS

This paper provided a convergence analysis of Fock states stabilization via single-photon corrections under an ideal set-up, that is, assuming perfect measurement detection and no control delays. In terms of convergence speed, the simulation results here presented have justified the inclusion of the term \( -\epsilon \sum_{n \in \mathbb{N}} f^2(n) \) in the Lyapunov-based feedback law (5)–(6). It is straightforward to verify that the convergence analysis developed in this paper remains valid for: (i) any other function \( d(n) \) in (5) satisfying \( d(n) = 0 \), \( d(n) \) is increasing for \( n > \overline{n} \) and \( d(n) \) is decreasing for \( n < \overline{n} \); and (ii) \( \epsilon > 0 \) dependent on \( n \), that is, to take the term \( -\sum_{n \in \mathbb{N}} \epsilon_n \rho_n^2 \). However, it is an open problem how to choose the function \( d(n) \) and the gains \( \epsilon_n > 0 \) so as to achieve the best convergence speed.

Finally, the feedback law used in [8], which corresponds to \( \epsilon = 0 \), was tailored for an experimental set-up with measurement imperfections and control delays. The convergence analysis of such realistic situation will be investigated in the future.

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APPENDIX

A. Basic properties of the operators \( \mathcal{N}, \alpha \) and \( \alpha^1 \)

Fix \( n^* \in \mathbb{N} \) and let \( \mathcal{H}_{n^*} = \text{span}\{0, \ldots, \overline{n^*}\} \). Consider the (linear) operators \( \mathcal{N}: \mathcal{H}_{n^*} \to \mathcal{H}_{n^*}, \alpha: \mathcal{H}_{n^*} \to \mathcal{H}_{n^*+1} \) defined respectively as \( \mathcal{N}(n) = n \mid n \rangle \), \( \alpha(0) = 0, \alpha(n) = \sqrt{n} \mid n - 1 \rangle \) for \( n \geq 1 \).

One also recalls that if \( A \) is a bounded operator on \( \mathcal{H} \) and \( B \in J_1 \), then \( AB, BA \in J_1 \) with \( Tr(AB) = Tr(BA) \).

B. Proof of Proposition 1

Fix any \( \rho \in D_\star \), and let \( n \in \mathbb{N} \). In particular, \( \rho(n) = \rho_{nn}[n] \). It then follows from (2)–(4) that:

\[
M_\theta(0) \rho M_\theta^\dagger(0) \rho \mid n \rangle \langle n \mid = \rho_{nn} \cos^2 \left( \frac{\phi_{N+\phi_0}}{2} \right) \mid n \rangle \langle n \mid, \\
M_\theta(0) \rho M_\theta^\dagger(0) \rho \mid n \rangle \langle n \mid = \rho_{nn} \sin^2 \left( \frac{\phi_{N+\phi_0}}{2} \right) \mid n \rangle \langle n \mid, \\
M_{\theta}(-1) \rho M_{\theta}^\dagger(-1) \rho \mid n \rangle \langle n \mid = \rho_{n-1,n-1} \sin^2 \left( \frac{\phi_{N+\phi_0}}{2} \right) \mid n \rangle \langle n \mid, \\
M_{\theta}(-1) \rho M_{\theta}^\dagger(-1) \rho \mid n \rangle \langle n \mid = \rho_{n-1,n-1} \sin^2 \left( \frac{\phi_{N+\phi_0}}{2} \right) \mid n \rangle \langle n \mid.
\]

Therefore:

\[
M_{\theta}(0) \rho M_{\theta}^\dagger(0) = \sum_{n=0}^{\overline{m}(n)} \rho_{nn} \cos^2 \left( \frac{\phi_{N+\phi_0}}{2} \right) \mid n \rangle \langle n \mid, \\
M_{\theta}(0) \rho M_{\theta}^\dagger(0) = \sum_{n=0}^{\overline{m}(n)} \rho_{nn} \sin^2 \left( \frac{\phi_{N+\phi_0}}{2} \right) \mid n \rangle \langle n \mid, \\
M_{\theta}(-1) \rho M_{\theta}^\dagger(-1) = \sum_{n=0}^{\overline{m}(n)} \rho_{n-1,n-1} \sin^2 \left( \frac{\phi_{N+\phi_0}}{2} \right) \mid n \rangle \langle n \mid, \\
M_{\theta}(-1) \rho M_{\theta}^\dagger(-1) = \sum_{n=0}^{\overline{m}(n)} \rho_{n-1,n-1} \sin^2 \left( \frac{\phi_{N+\phi_0}}{2} \right) \mid n \rangle \langle n \mid.
\]

By assumption, \( \rho_0 \in D_\star \). Then, (1), (8)–(13) above and induction on \( k \) show the assertions in Proposition 1.

C. Computation of \( Q(V, \rho, u) \)

Fix any \( \rho \in D_\star \) and \( \overline{n} \in \mathbb{N} \). Recall that \( V(\rho) = \text{Tr}(d(\mathcal{N})\rho) \), where \( d: \mathbb{N} \to \mathbb{R} \) be given by \( d(n) = (n - \overline{n})^2 \). Note that (1) implies that, for each \( u \in \{ -1, 0, 1 \} \),

\[
\mathbb{E}[V(\rho_k^+ \mid \rho_k = \rho, \eta_k = u)] = \text{Tr}(d(\mathcal{N}) M_{\theta}(\rho) \rho M_{\theta}^\dagger(\rho)) + \text{Tr}(d(\mathcal{N}) M_{\theta}(\rho) \rho M_{\theta}^\dagger(\rho)).
\]
Take $u = 0$. From (3)–(9) in Appendix B, one has
\[
E[V(\rho_{k+1})] | \rho_k = \rho, u_k = 0 \]
\[= \text{Tr } \left( d(N)M_{g}(0)\rho M_{g}^{\dagger}(0) \right) + \text{Tr } \left( d(N)M_{e}(0)\rho M_{e}^{\dagger}(0) \right) \]
\[= \text{Tr } \left( d(N) \left[ M_{g}(0)\rho M_{g}^{\dagger}(0) + M_{e}(0)\rho M_{e}^{\dagger}(0) \right] \right) \]
\[= \text{Tr } (d(N)\rho) = V(\rho). \]
In particular,
\[Q_V(\rho, 0) = 0. \tag{15} \]

Now, take $u = +1$. Then, (14) above and (10)–(11) in Appendix B provide that
\[
E[V(\rho_{k+1})] | \rho_k = \rho, u_k = +1 \]
\[= \text{Tr } (d(N)\rho) + \text{Tr } \left[ \sin^2 \left( \frac{\theta}{2} \sqrt{N + 1} \right) d(N + 1) - d(N) \right) \rho \]
\[= V(\rho) + \text{Tr } \left[ \sin^2 \left( \frac{\theta}{2} \sqrt{N + 1} \right) d(N + 1) - d(N) \right) \rho \]
In particular,
\[Q_V(\rho, +1) = -\text{Tr } \left[ \sin^2 \left( \frac{\theta}{2} \sqrt{N + 1} \right) d(N + 1) - d(N) \right) \rho \]
\[= -\sum_{n \in \mathbb{N}} \rho_{nn} \left[ 2(n - \overline{m}) + 1 \right] \sin^2 \left( \frac{\theta}{2} \overline{n} + \overline{1} \right). \tag{16} \]

Finally, take $u = -1$. Using (14) above and (12)–(13) in Appendix B, one has
\[E[V(\rho_{k+1})] | \rho_k = \rho, u_k = -1 \]
\[= \text{Tr } (d(N)\rho) + \text{Tr } \left[ \sin^2 \left( \frac{\theta}{2} \sqrt{N} \right) d(N - 1) - d(N) \right) \rho \]
\[= V(\rho) + \text{Tr } \left[ \sin^2 \left( \frac{\theta}{2} \sqrt{N} \right) d(N - 1) - d(N) \right) \rho \]
In particular,
\[Q_V(\rho, -1) = -\text{Tr } \left[ \sin^2 \left( \frac{\theta}{2} \sqrt{N} \right) d(N - 1) - d(N) \right) \rho \]
\[= \sum_{n \in \mathbb{N}} \rho_{nn} \left[ 2(n - \overline{m}) - 1 \right] \sin^2 \left( \frac{\theta}{2} \sqrt{n} \right). \tag{17} \]

D. Proof of Lemma 7

Assume that $N_0$ is even (otherwise one may take $N_0 + 1$ instead of $N_0$ in this proof). Define the function $\eta$: $\mathbb{N} \to \mathbb{R}$ by
\[\eta(\ell) = \left[ \frac{2}{\theta_0} \left( \frac{\ell \pi}{2} + \frac{\pi}{4} \right) \right]^2. \tag{18} \]
By definition, one has $\frac{\pi}{2} \sqrt{\eta(\ell)} = \ell \frac{\pi}{2} + \frac{\pi}{4}$ for all $\ell \in \mathbb{N}$. Let $h = \pi/4 - \arcsin \left( \sqrt{1/2 - a} \right)$. Using the definition of $h$ and the symmetric of the function $\sin^2(\cdot)$, it is easy to show that
\[1/2 < a \leq \sin^2(x + \pi/4) \leq a + 1/2, \quad \forall x \in [-h, h]. \tag{19} \]
Let $\overline{t} \in \mathbb{N}$ be even and big enough such that the following two conditions are simultaneously met:
\[\eta(\overline{t}) > N_0/2 + N, \quad \frac{1}{8} \theta_0 N_0 / \sqrt{\eta(\overline{t})} - N_0/2 \leq h. \tag{20} \]
Now, take $N = \lfloor \eta(\overline{t}) \rfloor - N_0/2 + 1 > N$, where $\lfloor \cdot \rfloor$ denotes the greatest integer which is less or equal to $\eta$. By construction, $\eta(\overline{t})$ is in-between the points $N_0/2 + 1$ and $N_0/2$, and hence it is in the interval $[N, N + N_0 - 1]$. Then, for $n = N, \ldots, N + N_0 - 1$, one has that $|n - \eta(\overline{t})| < N_0/2$. Consider the function $\phi(x) = \frac{\theta_0}{4} \sqrt{x}$. From the fact that $\phi'(x) = \frac{\theta_0}{4} \sqrt{x}$, by the mean value theorem applied to the function $\phi$ and the second inequality in (20), one obtains
\[\left| \frac{\theta_0}{2} \sqrt{n} - \frac{\theta_0}{2} \sqrt{\eta(\overline{t})} \right| < h, \quad \text{for } n = N, \ldots, N + N_0 - 1. \]

Then, the proof follows easily from (13), (19) and the fact that $\sin^2(x - \ell \pi/2) = \sin^2(x)$, for every even $\ell \in \mathbb{N}$.

E. Proof of Lemma 2

Proof of the first claim: Let $u \in \{-1, 0, 1\}, \rho \in D_n$. Recall that $W(\rho) = -\sum_{n \in \mathbb{N}} \rho_{nn}$. Then $\text{Tr } (\rho) = \sum_{n \in \mathbb{N}} \rho_{nn} = 1$, so $\text{Tr } (\rho) - \sum_{n \in \mathbb{N}} \rho_{nn} \leq W(\rho) \leq 0$. Now, by (14), $E[W(\rho_{k+1})] | \rho_k = \rho, u_k = u] = p_{g,k} W(\rho_{g,k}^{\dagger}) + p_{e,k} W(\rho_{e,k}^{\dagger})$, where $p_{g,k}, p_{e,k} \geq 0$ with $p_{g,k} + p_{e,k} = 1$. Thus $-1 \leq E[W(\rho_{k+1})] | \rho_k = \rho, u_k = u] \leq 0$. Since $Q_W(\rho, u)$ is the difference of two numbers that are in-between $-1$ and $0$, one concludes that $|Q_W(\rho, u)| \leq 1$.

The second, third and fourth claims, are immediate from (15), (16) and (17) in Appendix C respectively.

F. Proof of Lemma 3

Proof of the first claim: Let $\rho \in D_{m_0}$. By (3)–(9) in Appendix B, $M_g(0)\rho M_g^{\dagger}(0) + M_e(0)\rho M_e^{\dagger}(0) = \rho$. Taking $\rho_k = \rho$ in $u_k = 0$ in (14), define
\[\rho^y \triangleq \rho_{k+1} = \frac{M_g(0)\rho M_g^{\dagger}(0)}{\text{Tr}(M_g(0)\rho M_g^{\dagger}(0))}, \quad \text{for } y = g, e. \]
Hence, $\alpha \rho^y + (1 - \alpha) \rho^e = \rho$, where $\alpha \triangleq p_{g,k} = \text{Tr } (M_g(0)\rho M_g^{\dagger}(0))$. In particular, $\alpha \rho^y + (1 - \alpha) \rho^e = \rho_{nn}$, for $n \in \mathbb{N}$. Note that, if $\alpha = 0$, then $M_g(0)\rho M_g^{\dagger}(0) = 0$, and so $\rho^e = \rho$. Similarly, $\alpha = 1$ implies $\rho^y = \rho$. Thus, the identity $\alpha \rho^y + (1 - \alpha) \rho^e = \rho_{nn}$, for $n \in \mathbb{N}$, still holds when $\alpha = 0$ or $\alpha = 1$. From (14), $\mathcal{A} = p_{g,k}$, one has $Q_W(\rho, 0) = W(\rho) - [p_{g,k} W(\rho_{g,k}^{\dagger}) + p_{e,k} W(\rho_{e,k}^{\dagger})]$
\[= \sum_{n \in \mathbb{N}} (\alpha \rho_{nn})^2 + (1 - \alpha) (\rho_{nn})^2 - [\alpha \rho_{nn}^y + (1 - \alpha) \rho_{nn}^e]^2 \]
\[= \alpha (1 - \alpha) \sum_{n \in \mathbb{N}} [\rho_{nn}^y - \rho_{nn}^e]^2 \geq 0. \tag{21} \]

More precisely, $\sin^2(\pi/2 - x) = 1 - \cos^2(\pi/2 - x) = 1 - \sin^2(x).


thereby showing the first part of the first claim.

If \( \rho = |m\rangle\langle m| \) for some \( m \in \mathbb{N} \) with \( 0 < \alpha < 1 \), then (8) in Appendix B imply that \( \rho^\theta = \rho^\theta = \rho \), and so \( Q_W(\rho, 0) = 0 \). Now, one observes that \( \rho = |m\rangle\langle m| \) for some \( m \in \mathbb{N} \) whenever \( Q_W(\rho, 0) = 0 \). Suppose \( Q_W(\rho, 0) = 0 \). Then, (21) implies that \( \alpha = 0 \), or \( \alpha = 1 \), or \( \rho_{nn}^g = \rho_{nn}^\beta \) for all \( m \in \mathbb{N} \) with \( 0 < \alpha < 1 \). Assume that \( \alpha = 0 \). Hence, \( M_g(0)\rho M_g^\dagger(0) = \sum_{n,m \in \mathbb{N}} \rho_{mn} \cos^2(\frac{\phi_{n+m} + \phi_{n}}{2}) |n\rangle\langle m| = 0 \) by (8) in Appendix B. Suppose that \( \rho \neq |m\rangle\langle m| \) for every \( m \in \mathbb{N} \). Thus, there exists \( n_1, n_2 \in \mathbb{N} \) with \( n_1 \neq n_2 \), \( \rho_{n_1,n_1} > 0 \), \( \rho_{n_2,n_2} > 0 \). Recall that \( \sin(x_1) = \pm \sin(x_2) \) if and only if \( x_1 + x_2 = \ell \pi \) or \( x_2 - x_1 = \ell \pi \), where \( \ell \) is an integer. Therefore, \( \sin(\frac{\phi_{n_1} + \phi_{n_2}}{2}) = \pm \sin(\frac{\phi_{n_1} + \phi_{n_2}}{2}) \), which contradicts the assumptions that \( \phi_0 / \pi \) is an irrational number and \( \phi_\ell / \pi = 2 - \ell \phi_0 \). One has shown that \( \rho = |m\rangle\langle m| \) for some \( m \in \mathbb{N} \) whenever \( \alpha = 0 \). If \( \alpha = 1 \), or \( \rho_{nn}^g = \rho_{nn}^\beta \) for all \( m \in \mathbb{N} \) with \( 0 < \alpha < 1 \), then from similar arguments and computations one also concludes that \( \rho = |m\rangle\langle m| \) for some \( m \in \mathbb{N} \).

Proof of the second claim: Let \( m \in \mathbb{N} \) and take \( \rho = |m\rangle\langle m| \in D_\star \). It is clear that \( W(\rho) = -\sum_{n \in \mathbb{N}} \rho_{nn}^\beta = -1 \). From (10)–(13) in Appendix B one has that:

\[
\begin{align*}
(M_g(\pm 1)\rho M_g^\dagger(\pm 1))_{nn} &= \delta(n, m) \cos^2(\frac{\phi_{n+m}}{2}) \
(M_\ell(\pm 1)\rho M_\ell^\dagger(\pm 1))_{nn} &= \delta(n, m) \cos^2(\frac{\phi_{n+m}}{2}) \
(M_g(-1)\rho M_g^\dagger(-1))_{nn} &= \delta(n, m) \sin^2(\frac{\phi_{n+m}}{2}) \
(M_\ell(-1)\rho M_\ell^\dagger(-1))_{nn} &= \delta(n + 1, m) \sin^2(\frac{\phi_{n+m}}{2}),
\end{align*}
\]

(22)

where \( \delta(n, m) \) is the usual Kronecker delta: \( \delta(n, m) = 0 \) if \( n \neq m \), and \( \delta(n, m) = 1 \) if \( n = m \). In particular:

\[
\begin{align*}
\text{Tr} \left( M_g(\pm 1)\rho M_g^\dagger(\pm 1) \right) &= \frac{\phi_{n+m}}{2} \sqrt{m+1}, \\
\text{Tr} \left( M_\ell(\pm 1)\rho M_\ell^\dagger(\pm 1) \right) &= \frac{\phi_{n+m}}{2} \sqrt{m}, \\
\text{Tr} \left( M_g(-1)\rho M_g^\dagger(-1) \right) &= \frac{\phi_{n+m}}{2} \sqrt{m+1}, \\
\text{Tr} \left( M_\ell(-1)\rho M_\ell^\dagger(-1) \right) &= \frac{\phi_{n+m}}{2} \sqrt{m},
\end{align*}
\]

\[
\sum_{n \in \mathbb{N}} \left( \frac{M_g(u)\rho M_g^\dagger(u)}{\text{Tr} \left( M_g(u)\rho M_g^\dagger(u) \right)} \right)^2_{nn} = 1,
\]

for \( u = \pm 1 \), \( y = g, \ell \)

(assuming no division by 0). Now, using (11) and the above computations, one gets

\[
\mathbb{E} \left[ W(\rho_{k+1}) | \rho_k = \rho, \rho_{kk} = \pm 1 \right] = p_{g,k} W(\rho_{k+1}^g) + p_{\ell,k} W(\rho_{k+1}^\ell) = - \sum_{y = g, \ell} \left[ \text{Tr} \left( M_g(\pm 1)\rho M_g^\dagger(\pm 1) \right) \times \right]
\]

\[
\times \sum_{n \in \mathbb{N}} \left( \frac{M_g(\pm 1)\rho M_g^\dagger(\pm 1)}{\text{Tr} \left( M_g(\pm 1)\rho M_g^\dagger(\pm 1) \right)} \right)^2_{nn} = -1 = W(\rho).
\]

Therefore, \( Q_W(|m\rangle\langle m|, \pm 1) = 0 \).

---

**References**


