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Stability of continuous-time quantum filters with measurement imperfections

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Abstract

The fidelity between the state of a continuously observed quantum system and the state of its associated quantum filter, is shown to be always a submartingale. The observed system is assumed to be governed by a continuous-time Stochastic Master Equation (SME), driven simultaneously by Wiener and Poisson processes and that takes into account incompleteness and errors in measurements. This stability result is the continuous-time counterpart of a similar stability result already established for discrete-time quantum systems and where the measurement imperfections are modeled by a left stochastic matrix.

Keywords. Quantum filtering, stability, continuous-time stochastic master equations, Wiener process, Poisson process, quantum trajectories, measurement errors.

1 Introduction

Since the work of Davies [14], the time evolution of the state (density operator) ρ_t at time t of an observed quantum system can be described by a Stochastic Master Equation (SME) taking into account the back-action of the measurements on ρ_t . Such SMEs (see [5] for a modern exposure) have been the starting point of the seminal contributions of Belavkin to quantum filtering and control [7, 8, 4, 9]. Quantum filters are used to get an estimate ρ_t^e of ρ_t based on an initial guess ρ_0^e of ρ_0 and on the measurement outcomes between 0 and t . Quantum filtering is related to quantum trajectories [13, 12] and their original motivations for Monte-Carlo simulations. Roughly speaking, quantum filtering replaces the random numbers used at each time-step of a Monte-Carlo simulation by the measurement outcomes to update the estimate ρ_t^e . An important practical issue deals with the convergence: does ρ_t^e converge towards ρ_t when t tends to $+\infty$, even if $\rho_0^e \neq \rho_0$? Few convergence results are available up to now, except the sufficient conditions established in [24, 25] for diffusive SMEs. Particular results for quantum non demolition indirect measurements have been established in [6, 10] As far as we know, general and checkable necessary and sufficient convergence conditions do not exist yet.

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In this paper we do not investigate directly convergence issues. We focus on stability issues to ensure the absence of time divergence. We prove in Theorem 5 that for a large class of continuous-time SMEs driven simultaneously by Wiener and Poisson processes, associated quantum filters are stable: the fidelity between ρ and its estimate ρ^e ,

$$F(\rho, \rho^e) = \text{Tr}^2 \left(\sqrt{\sqrt{\rho} \rho^e \sqrt{\rho}} \right), \quad (1)$$

is a sub-martingale. In the Wiener case, Theorem 5 shows that, whatever the Hamiltonian H , the measurement operators V_ν and the detection efficiencies $\bar{\eta}_\nu \in [0, 1]$ are, the fidelity $F(\rho_t, \rho_t^e)$ is a sub-martingale where ρ_t obeys the following diffusive SME

$$\begin{aligned} d\rho_t = & -i[H, \rho_t] dt + \left(\sum_\nu V_\nu \rho_t V_\nu^\dagger - \frac{1}{2}(V_\nu^\dagger V_\nu \rho_t + \rho_t V_\nu^\dagger V_\nu) \right) dt \\ & + \sum_\nu \sqrt{\bar{\eta}_\nu} \left(V_\nu \rho_t + \rho_t V_\nu^\dagger - \text{Tr} \left((V_\nu + V_\nu^\dagger) \rho_t \right) \rho_t \right) dW_\nu(t) \end{aligned}$$

driven by the Wiener processes W_ν , and where the estimate ρ_t^e obeys the following non-linear stochastic equation

$$\begin{aligned} d\rho_t^e = & -i[H, \rho_t^e] dt + \left(\sum_\nu V_\nu \rho_t^e V_\nu^\dagger - \frac{1}{2}(V_\nu^\dagger V_\nu \rho_t^e + \rho_t^e V_\nu^\dagger V_\nu) \right) dt \\ & + \sum_\nu \sqrt{\bar{\eta}_\nu} \left(V_\nu \rho_t^e + \rho_t^e V_\nu^\dagger - \text{Tr} \left((V_\nu + V_\nu^\dagger) \rho_t^e \right) \rho_t^e \right) \left(dy_\nu(t) - \sqrt{\bar{\eta}_\nu} \text{Tr} \left((V_\nu + V_\nu^\dagger) \rho_t^e \right) dt \right). \end{aligned}$$

driven by the measures $dy_\nu(t) = dW_\nu(t) + \sqrt{\bar{\eta}_\nu} \text{Tr} \left((V_\nu + V_\nu^\dagger) \rho_t \right) dt$ and initialized to any density matrix ρ_0^e . In the Poisson case, Theorem 5 ensures, as for the Wiener case, the stability of the quantum filtering process. Additionally, it provides a new kind of SMEs taking into account incompleteness and errors in the jump detections:

$$\begin{aligned} d\rho_t = & -i[H, \rho_t] dt + \left(\sum_\mu V_\mu \rho_t V_\mu^\dagger - \frac{1}{2}(V_\mu^\dagger V_\mu \rho_t + \rho_t V_\mu^\dagger V_\mu) \right) dt \\ & + \sum_\mu \left(\frac{\bar{\theta}_\mu \rho_t + \sum_\nu \bar{\eta}_{\mu,\nu} V_\nu \rho_t V_\nu^\dagger}{\bar{\theta}_\mu + \sum_\nu \bar{\eta}_{\mu,\nu} \text{Tr} \left(V_\nu \rho_t V_\nu^\dagger \right)} - \rho_t \right) \left(dN_\mu(t) - \left(\bar{\theta}_\mu + \sum_\nu \bar{\eta}_{\mu,\nu} \text{Tr} \left(V_\nu \rho_t V_\nu^\dagger \right) \right) dt \right) \end{aligned}$$

driven by the the Poisson processes $N_\mu(t)$ with $\langle dN_\mu(t) \rangle = \left(\bar{\theta}_\mu + \sum_\nu \bar{\eta}_{\mu,\nu} \text{Tr} \left(V_\nu \rho_t V_\nu^\dagger \right) \right) dt$, where the detection imperfections are modeled through the parameters $\bar{\theta}_\mu \geq 0$ and $\bar{\eta}_{\mu,\nu} \geq 0$ with $\sum_\mu \bar{\eta}_{\mu,\nu} \leq 1$.

The proof of Theorem 5 is based on discrete-time approximations of continuous-time SMEs. Such approximations have already been investigated in [3, 15, 17, 18]. They rely on indirect measurements, originally introduced in [11] and well explained with suggestive physical systems in [16, 26]. For discrete-time SMEs, it is proved in [21, 23] that the fidelity between the quantum state and its estimate is always a submartingale. Theorem 5 is obtained by passing to the limit from discrete to continuous time. The fidelity between the quantum

state and its estimate remains a submartingale. The obtained continuous-time SMEs are slightly more general than the ones usually encountered in the literature. Such SMEs could be of some interest to derive quantum filters taking into account a larger class of incompleteness and errors in measurements and jump detections.

This paper is structured in two main sections. Section 2 is devoted to Theorem 2, a restrictive version of Theorem 5 to the diffusive SMEs with perfect measurements. For this simplified but representative case, the discrete-time approximation is presented and the passage to the continuous-time limit is detailed during the proof of Proposition 1. Section 3 is devoted to Theorem 5 and fully exploits the tools and methods developed in Section 2. We consider SMEs driven simultaneously by Poisson and Wiener processes. We recall first the structure, described in [23] and based on a left stochastic matrix, of discrete-time SMEs associated to imperfect measurements. We apply on the discrete-time approximations such left stochastic matrix modeling of imperfections and errors. Then we take, thanks to Theorem 4, the limit to get the continuous-time SMEs and its associated quantum filters with imperfections. Their structures are more general than the usual ones encountered in the literature. This leads to Theorem 5 ensuring the stability of the obtained quantum filters. Section 4 is a short conclusion proposing some connection with Petz characterization of monotone metrics on matrix spaces.

Some intermediate and partial results related to Theorems 5 can be founded in [2, 1].

2 Perfect measurements

Let us start this section by presenting the jump-diffusive SMEs describing the evolutions of quantum state ρ_t and its estimate ρ_t^e .

2.1 Continuous-time filters

We consider quantum systems of finite dimensions $1 < N < \infty$. The state space of such a system is given by the set of density matrices

$$\mathcal{D} := \{\rho \in \mathbb{C}^{N \times N} \mid \rho = \rho^\dagger, \quad \text{Tr}(\rho) = 1, \quad \rho \geq 0\}.$$

Formally a real quantum trajectory $\rho \in \mathcal{D}$ in Schrödinger picture can be described by the following SME (cf. [8, 9, 5])

$$d\rho_t = \left(-i[H, \rho_t] + \sum_{\xi} \mathcal{L}_{\xi}(\rho_t) \right) dt + \sum_{\nu} \Lambda_{\nu}(\rho_t) dW_{\nu}(t) + \sum_{\mu} \Upsilon_{\mu}(\rho_t) \left(dN_{\mu}(t) - \text{Tr}(V_{\mu} \rho_t V_{\mu}^{\dagger}) dt \right), \quad (2)$$

where

- the notation $[A, B]$ refers to $AB - BA$;
- $H = H^{\dagger}$ is a Hermitian operator corresponding to the total Hamiltonian of the system;
- dN_{μ} are the Poisson processes with $\mu \in \{1, \dots, m_P\}$ and dW_{ν} are the Wiener processes with $\nu \in \{m_P + 1, \dots, m_P + m_W\}$;

- The Lindblad superoperator \mathcal{L}_ξ ($\xi \in \{1, \dots, m_W + m_P\}$) is defined by

$$\mathcal{L}_\xi(\rho) := V_\xi \rho V_\xi^\dagger - \frac{1}{2}(V_\xi^\dagger V_\xi \rho + \rho V_\xi^\dagger V_\xi), \quad (3)$$

where V_ξ is an arbitrary matrix which determines the measurement process (typically the coupling to the probe field for quantum optic systems);

- The superoperators Λ_ν and Υ_μ are defined respectively by

$$\Lambda_\nu(\rho) := V_\nu \rho + \rho V_\nu^\dagger - \text{Tr}\left((V_\nu + V_\nu^\dagger)\rho\right)\rho \quad \text{and} \quad \Upsilon_\mu(\rho) := \frac{V_\mu \rho V_\mu^\dagger}{\text{Tr}\left(V_\mu \rho V_\mu^\dagger\right)} - \rho;$$

- The measurement outcomes are dN_μ and dy_ν where

$$N_\mu(t) - \int_0^t \text{Tr}\left(V_\mu \rho_s V_\mu^\dagger\right) ds \quad \text{is a martingale and} \quad dy_\nu(t) = dW_\nu(t) + \text{Tr}\left(V_\nu \rho_t V_\nu^\dagger\right) dt. \quad (4)$$

All the developments remain valid when H and V_ξ are deterministic time-varying matrices. For clarity sake, we do not recall below such possible time dependence.

In this paper, the notation ρ^e corresponds to the estimate filter associated to the filter ρ . This estimate filter is provided from the measurement outcomes dy_ν and dN_μ and depends on the real quantum trajectory ρ via the measurement outcomes (4). It has the following expression derived from (2) where $dW_\nu(t)$ is replaced by $dy_\nu(t) - \text{Tr}\left(V_\nu \rho_t^e V_\nu^\dagger\right) dt$ (see e.g., [9, 5]):

$$d\rho_t^e = \left(-i[H, \rho_t^e] + \sum_\xi \mathcal{L}_\xi(\rho_t^e) \right) dt + \sum_\nu \Lambda_\nu(\rho_t^e) \left(dy_\nu(t) - \text{Tr}\left(V_\nu \rho_t^e V_\nu^\dagger\right) dt \right) + \sum_\mu \Upsilon_\mu(\rho_t^e) \left(dN_\mu(t) - \text{Tr}\left(V_\mu \rho_t^e V_\mu^\dagger\right) dt \right). \quad (5)$$

When $\rho_0^e \neq \rho_0$, ρ_t^e and ρ_t do not coincide in general. However, we will see that the fidelity between ρ_t^e and ρ_t is a sub-martingale.

2.2 Discrete-time filters

First let us briefly remind the model of quantum repeated indirect measurement approach. The physics underlying such approach is well explained in [16, 26]. In [3, 15, 19], it was rigorously shown that such discrete-time approximations associated to the real state converges to the continuous model described in Equation (2).

2.2.1 Quantum repeated measurement approach

We consider the setup of quantum repeated interaction of \mathcal{H} (describing the Hilbert space of the system state) with an infinite chain $\bigotimes_k \mathcal{K}_k$ (describing the Hilbert space of the environment) with $\mathcal{K}_k = \mathcal{K}$ for all k . More precisely, the first copy $\mathcal{K}_1 = \mathcal{K}$ interacts with \mathcal{H} during a time δ and then disappear. Next, the second copy \mathcal{K}_2 comes to interact with \mathcal{H} and so on.

This setting is coupled with indirect measurement, that is, after each interaction between \mathcal{H} and \mathcal{K} , a measurement of an observable of \mathcal{K} is performed.

Here, we assume that $\mathcal{H} = \mathbb{C}^N$ and the environment (meter system) is composed of $m_P + m_W$ qubits, so $\mathcal{K} = (\mathbb{C}^2)^{\otimes m_P + m_W}$. Also, take $|\psi\rangle$ as the initial state of the system and $|G\rangle$ as the initial state of the qubits' meter which is defined as the ground state of the meter system: all qubits in the ground state $|g\rangle$. As a result, the initial state of the system coupled to its environment is described by $|\psi\rangle \otimes |G\rangle$.

For $\delta = 1/n$ with n large, assume that the Schrödinger evolution between time 0 to time $1/n$ is given by

$$H_{\text{tot}} = H \otimes I + \sqrt{n} \sum_{\xi} \left(iV_{\xi} \otimes \sigma_{+\xi} - iV_{\xi}^{\dagger} \otimes \sigma_{-\xi} \right), \quad (6)$$

where

- H is the Hamiltonian of the system used in (2) and the operators V_{ξ} are those appearing in the Lindblad superoperators (3)
- $\sigma_{+\xi} = (|e\rangle\langle g|)_{\xi}$ and $\sigma_{-\xi} = \sigma_{+\xi}^{\dagger} = (|g\rangle\langle e|)_{\xi}$ where the notation $(A)_{\xi}$ is an operator \mathcal{K} defined by $(A)_{\xi} = \bigotimes_{1 \leq i < \xi} I \otimes A \otimes \bigotimes_{\xi < i \leq m_W + m_P} I$. Note that $\sigma_{-\xi} |G\rangle = 0$.

In the sequel, symbol \otimes will be removed for compact formulae.

Take the observables $X_{\nu} = (|g\rangle\langle e|)_{\nu} + (|e\rangle\langle g|)_{\nu}$ for $\nu \in \{m_P + 1, \dots, m_W + m_P\}$ and $Z_{\mu} = (|e\rangle\langle e|)_{\mu} - (|g\rangle\langle g|)_{\mu}$ for $\mu \in \{1, \dots, m_P\}$. For n large, the measurement of all qubits at final time $t = 1/n$ according to the observables X_{ν} and Z_{μ} , yields an approximation for the Wiener and Poisson processes, respectively. This results from a development versus $1/n$ of the measurement operators associated to the associated discrete-time stochastic evolution.

We have $e^{-iH_{\text{tot}}/n} = \mathbb{I} - \frac{i}{n} H_{\text{tot}} - \frac{1}{2n^2} H_{\text{tot}}^2 + \mathcal{O}(1/n^{3/2})$, where \mathbb{I} is the identity operator. Now replacing H_{tot} by its expression given in (6), we find

$$e^{-iH_{\text{tot}}/n} \approx \mathbb{I} + \frac{1}{\sqrt{n}} \sum_{\xi} \left(V_{\xi} \sigma_{+\xi} - V_{\xi}^{\dagger} \sigma_{-\xi} \right) - \frac{i}{n} H - \frac{1}{2n} \sum_{\xi} \left(V_{\xi}^{\dagger} V_{\xi} P_{g_{\xi}} + V_{\xi} V_{\xi}^{\dagger} P_{e_{\xi}} \right),$$

where \approx means up to $\mathcal{O}(1/n^{3/2})$ terms. Here, $P_{g_{\xi}} = (|g\rangle\langle g|)_{\xi}$ and $P_{e_{\xi}} = (|e\rangle\langle e|)_{\xi}$. Note that

$$P_{g_{\xi}} |G\rangle = |G\rangle \quad \text{and} \quad P_{e_{\xi}} |G\rangle = 0.$$

Thus the system coupled to its environment evolves as follows

$$e^{-iH_{\text{tot}}/n} |\psi\rangle \otimes |G\rangle \approx \left(\mathbb{I} - \frac{i}{n} H - \frac{1}{2n} \sum_{\xi} V_{\xi}^{\dagger} V_{\xi} \right) |\psi\rangle \otimes |G\rangle + \frac{1}{\sqrt{n}} \sum_{\xi} V_{\xi} |\psi\rangle \otimes |G_{\xi}\rangle.$$

Let us consider the qubits measurements. The measurement outcomes for X_{ν} are stored in $x_{\nu - m_P} \in \{-1, 1\}$, $x = (x_{\nu - m_P}) \in \{-1, 1\}^{m_W}$, and those of Z_{μ} in $z_{\mu} \in \{0, 1\}$, $z = (z_{\mu}) \in \{0, 1\}^{m_P}$, as follows (all these observables commute):

- if during the measure of X_{ν} , the corresponding qubit collapses to $(|g\rangle + |e\rangle)/\sqrt{2}$ (resp. $(|g\rangle - |e\rangle)/\sqrt{2}$) set $x_{\nu - m_P} = +1$ (resp. $x_{\nu - m_P} = -1$).

- if during the measure of Z_μ , the corresponding qubit collapses to $|g\rangle$ (resp. $|e\rangle$), set $z_\mu = 0$ (resp. $z_\mu = 1$).

The probability to get, for two different μ and μ' , $z_\mu = z_{\mu'} = 1$, is in order of $\mathcal{O}(1/n^{3/2})$. Thus for z , we need only consider the following cases: either all z_μ are equal to 0 or only a single one is equal to 1, the other ones being 0.

Consider the measurement outcomes $z = (z_\mu)$ and $x = (x_{\nu-m_P})$. The associated wave packet collapse of $e^{-iH_{\text{tot}}/n} |\psi\rangle \otimes |G\rangle$ yields the un-normalized state $M_{x,z} |\psi\rangle \otimes |x, z\rangle$, where the measurement operator is denoted by $M_{x,z}$ and where $|x, z\rangle$ is the normalized state of the qubits characterized by $X_\nu |x, z\rangle = x_{\nu-m_P} |x, z\rangle$ and $Z_\mu |x, z\rangle = (2z_\mu - 1) |x, z\rangle$. We have to consider two situations: either $\sum z_\mu = 0$ denoted by $z = 0$ or $\sum_\mu z_\mu = 1$. When $z_\mu = 0$ for all μ , some computations yield

$$M_{x,0} \approx \left(\prod_\nu \frac{1}{\sqrt{2}} \right) \left(\mathbb{I} - \frac{i}{n} H - \frac{1}{2n} \sum_\xi V_\xi^\dagger V_\xi + \frac{1}{\sqrt{n}} \sum_\nu x_{\nu-m_P} V_\nu \right). \quad (7)$$

When $z_\mu = 1$ and $z_{\mu'} = 0$ for all $\mu' \neq \mu$, similar computations give

$$M_{x,\mu} \approx \left(\prod_\nu \frac{1}{\sqrt{2}} \right) \frac{V_\mu}{\sqrt{n}} \quad (8)$$

where we have denoted $M_{x,z}$ by $M_{x,\mu}$ for such z .

For an arbitrary state ρ at $t = 0$, not necessarily pure as $\rho = |\psi\rangle \langle \psi|$, the state ρ_1 at time $t = 1/n$ is given by

$$\rho_1 = \frac{M_{x,z} \rho M_{x,z}^\dagger}{p_{x,z}(\rho)}, \quad (9)$$

which happens with probability $p_{x,z}(\rho) = \text{Tr} \left(M_{x,z} \rho M_{x,z}^\dagger \right)$.

The expression of ρ_1 is obtained by neglecting the terms of orders strictly greater than 1 versus $1/n$. Some usual calculations yield

$$M_{x,0} \rho M_{x,0}^\dagger = c_W \left(\rho + \frac{1}{\sqrt{n}} \sum_\nu x_{\nu-m_P} (V_\nu \rho + \rho V_\nu^\dagger) + \frac{1}{n} \left(-i[H, \rho] - \sum_\mu \frac{\{\rho, V_\mu^\dagger V_\mu\}}{2} + \sum_\nu \mathcal{L}_\nu(\rho) \right) \right) \quad (10)$$

with probability

$$p_{x,0}(\rho) = c_W \left(1 + \frac{1}{\sqrt{n}} \sum_\nu x_{\nu-m_P} \text{Tr} \left((V_\nu + V_\nu^\dagger) \rho \right) - \frac{1}{n} \sum_\mu \text{Tr} \left(\rho V_\mu^\dagger V_\mu \right) \right). \quad (11)$$

The notation $\{A, B\}$ used in (10) corresponds to $AB + BA$ and $c_W = (\frac{1}{2})^{m_W}$.

Similarly, we have

$$M_{x,\mu} \rho M_{x,\mu}^\dagger = \frac{c_W}{n} V_\mu \rho V_\mu^\dagger, \quad \text{with probability } p_{x,\mu}(\rho) = \frac{c_W}{n} \text{Tr} \left(V_\mu \rho V_\mu^\dagger \right). \quad (12)$$

Now we can obtain the asymptotic description of the transition for all possible observations. Indeed, the expression of the whole quantum trajectory ρ_k can be obtained by replacing ρ_1 by ρ_{k+1} and ρ by ρ_k in Equation (9).

As a conclusion, we find

$$\rho_{k+1} = \frac{M_{x,\mu}\rho_k M_{x,\mu}^\dagger}{\text{Tr}\left(M_{x,\mu}\rho_k M_{x,\mu}^\dagger\right)} \quad \text{with probability } p_{x,\mu}(\rho_k) \quad (13)$$

$$(14)$$

where $p_{x,\mu}(\rho_k)$ is given by (12,11).

2.2.2 Stability with respect to initial condition

Consider the Markov chain (13). Assume that we do not know precisely the initial state ρ_0 and we have at our disposal an estimate ρ_0^e . Assume also that we know the measurement result at step k $(x_k, \mu_k) \in \{-1, 1\}^{mw} \times \{0, 1\}$. It is then natural to consider the following recursive update of our estimation ρ_{k+1}^e using the knowledge of measurement result at step k and the previous estimate ρ_k^e (see e.g., [26]):

$$\rho_{k+1}^e = \frac{M_{x_k, \mu_k} \rho_k^e M_{x_k, \mu_k}^\dagger}{\text{Tr}\left(M_{x_k, \mu_k} \rho_k^e M_{x_k, \mu_k}^\dagger\right)} \quad (15)$$

Note that the probability $p_{x_k, \mu_k}(\rho_k)$ given by (11,12) to get (x_k, μ_k) depends on the hidden state ρ_k and not on ρ_k^e .

Remark 1. Let us stress that the above description is not always valid. Indeed, the normalization $\text{Tr}\left(M_{x_k, \mu_k} \rho_k^e M_{x_k, \mu_k}^\dagger\right)$ can vanish and the formula (15) is then not defined. This problem does not appear when describing the true evolution since the normalization describing the true state corresponds to the probability of apparition (then if this vanishes this means that the corresponding state can not appear). This problem of non-definition for the discrete-time estimate filter is related to the problem underlined in the definition of (5). In general, this question has been taken into account in [21, 23] for the discrete-time filter. In our context focusing on asymptotic evolution, such a problem will not appear.

Theorem 1 ([21]). Consider any arbitrary Markov chain (ρ_k, ρ_k^e) satisfying respectively Equations (13) and (15):

$$(\rho_{k+1}, \rho_{k+1}^e) = \left(\frac{M_{x,\mu}\rho_k M_{x,\mu}^\dagger}{\text{Tr}\left(M_{x,\mu}\rho_k M_{x,\mu}^\dagger\right)}, \frac{M_{x,\mu}\rho_k^e M_{x,\mu}^\dagger}{\text{Tr}\left(M_{x,\mu}\rho_k^e M_{x,\mu}^\dagger\right)} \right) \quad \text{with probability } p_{x,\mu}(\rho_k)$$

where $p_{x,\mu}(\rho_k)$ is given by (12) and (11). Then the fidelity $(F(\rho_k, \rho_k^e))$ defined in Equation (1) is a (\mathcal{F}_k) submartingale where $\mathcal{F}_k = \sigma\{(\rho_l, \rho_l^e) | l \leq k\}$. In particular, we have

$$\mathbb{E}(F(\rho_l, \rho_l^e) | \mathcal{F}_k) = \mathbb{E}(F(\rho_l, \rho_l^e) | (\rho_k, \rho_k^e)) \geq F(\rho_k, \rho_k^e),$$

for all $l > k$.

2.3 Stability of continuous-time filters

From the description of the quantum trajectory (ρ_k) and its associated quantum filter (ρ_k^e) with the time parameter n , we can define the associated continuous-time stochastic processes denoted by $(\rho_n(t))$ and $(\rho_n^e(t))$ with

$$\rho_n(t) = \rho_{[nt]}, \quad \rho_n^e(t) = \rho_{[nt]}^e.$$

It is clear that (ρ_n, ρ_n^e) is a Markov process as (ρ_k, ρ_k^e) is a Markov chain.

Before announcing the main result of this section, first let us show the following proposition.

Proposition 1. *The Markov process $(\rho_n(t), \rho_n^e(t))$ with ρ_n and ρ_n^e satisfying respectively Equations (13) and (15) converges in distribution, when n goes to infinity, to the Markov process (ρ_t, ρ_t^e) with ρ_t and ρ_t^e satisfying respectively Equations (2) and (5).*

Proof. The approach to prove this proposition is usual in probability theory: we show the convergence of the generator associated to the Markov process (ρ_n, ρ_n^e) towards the one associated to the Markov process (ρ_t, ρ_t^e) . Also, we need to prove the tightness property of the sequence (ρ_n, ρ_n^e) .

The tightness of the sequence $(\rho_n(t), \rho_n^e(t))$ is guaranteed if, for any $T > 0$, exists $M > 0$ such that for all $0 \leq t_1 \leq t_2 \leq T$ and $\forall t \in [t_1, t_2]$,

$$\mathbb{E} \left[\|(\rho_n(t_2), \rho_n^e(t_2)) - (\rho_n(t), \rho_n^e(t))\|^2 \|(\rho_n(t), \rho_n^e(t)) - (\rho_n(t_1), \rho_n^e(t_1))\|^2 \right] \leq M(t_2 - t_1)^2.$$

We do not develop the arguments for showing the tightness property. They can be obtained directly with the same arguments as [19].

Thus here, we just focus on the convergence of the generators. Take \mathcal{A}_n and \mathcal{A} as the generators associated respectively to the sequences (ρ_n, ρ_n^e) and (ρ, ρ^e) . We have to prove that, for any C^2 -real valued function $(\rho, \rho^e) \mapsto f(\rho, \rho^e)$,

$$\lim_{n \rightarrow +\infty} \sup_{(\rho, \rho^e) \in \mathcal{D}^2} |\mathcal{A}_n f(\rho, \rho^e) - \mathcal{A} f(\rho, \rho^e)| = 0.$$

We have the following expression for the generator $\mathcal{A}f(\rho, \rho^e)$

$$\begin{aligned} \mathcal{A}f(\rho, \rho^e) = & D_{(\rho, \rho^e)} f \cdot \left(-i[H, \rho] + \sum_{\xi} \mathcal{L}_{\xi}(\rho), -i[H, \rho^e] + \sum_{\xi} \mathcal{L}_{\xi}(\rho^e) + K(\rho, \rho^e) \right) \\ & + \frac{1}{2} \sum_{\nu} D_{(\rho, \rho^e)}^2 f \cdot (\Lambda_{\nu}(\rho), \Lambda_{\nu}(\rho^e); \Lambda_{\nu}(\rho), \Lambda_{\nu}(\rho^e)) \\ & + \sum_{\mu} \text{Tr} \left(V_{\mu} \rho V_{\mu}^{\dagger} \right) \left(f \left(\frac{V_{\mu} \rho V_{\mu}^{\dagger}}{\text{Tr}(V_{\mu} \rho V_{\mu}^{\dagger})}, \frac{V_{\mu} \rho^e V_{\mu}^{\dagger}}{\text{Tr}(V_{\mu} \rho^e V_{\mu}^{\dagger})} \right) - f(\rho, \rho^e) - D_{(\rho, \rho^e)} f \cdot (\Upsilon_{\mu}(\rho), \Upsilon_{\mu}(\rho^e)) \right), \end{aligned} \tag{16}$$

where

$$K(\rho, \rho^e) := \sum_{\mu} \Upsilon_{\mu}(\rho^e) \left(\text{Tr} \left(V_{\mu} \rho V_{\mu}^{\dagger} \right) - \text{Tr} \left(V_{\mu} \rho^e V_{\mu}^{\dagger} \right) \right) + \sum_{\nu} \Lambda_{\nu}(\rho^e) \left(\text{Tr} \left(V_{\nu} \rho V_{\nu}^{\dagger} \right) - \text{Tr} \left(V_{\nu} \rho^e V_{\nu}^{\dagger} \right) \right)$$

Now let us calculate the expression of \mathcal{A}_n . We have, up to $\mathcal{O}(n^{-1/2})$ terms,

$$\begin{aligned} \mathcal{A}_n f(\rho, \rho^e) &\approx \sum_x n p_{x,0}(\rho) \left(f \left(\frac{M_{x,0}\rho M_{x,0}^\dagger}{\text{Tr}(M_{x,0}\rho M_{x,0}^\dagger)}, \frac{M_{x,0}\rho^e M_{x,0}^\dagger}{\text{Tr}(M_{x,0}\rho^e M_{x,0}^\dagger)} \right) - f(\rho, \rho^e) \right) \\ &\quad + \sum_{x,\mu} n p_{x,\mu}(\rho) \left(f \left(\frac{M_{x,\mu}\rho M_{x,\mu}^\dagger}{\text{Tr}(M_{x,\mu}\rho M_{x,\mu}^\dagger)}, \frac{M_{x,\mu}\rho^e M_{x,\mu}^\dagger}{\text{Tr}(M_{x,\mu}\rho^e M_{x,\mu}^\dagger)} \right) - f(\rho, \rho^e) \right). \end{aligned}$$

Since $\frac{M_{x,0}\rho M_{x,0}^\dagger}{p_{x,0}(\rho)} = \rho + \mathcal{O}(n^{-1/2})$ and also, by using Taylor's formula, we have

$$\begin{aligned} f \left(\frac{M_{x,0}\rho M_{x,0}^\dagger}{p_{x,0}(\rho)}, \frac{M_{x,0}\rho^e M_{x,0}^\dagger}{p_{x,0}(\rho^e)} \right) - f(\rho, \rho^e) &= D_{(\rho, \rho^e)} f \cdot \left(\frac{M_{x,0}\rho M_{x,0}^\dagger}{p_{x,0}(\rho)} - \rho, \frac{M_{x,0}\rho^e M_{x,0}^\dagger}{p_{x,0}(\rho^e)} - \rho^e \right) \\ + \frac{1}{2} D_{(\rho, \rho^e)}^2 f \cdot \left(\frac{M_{x,0}\rho M_{x,0}^\dagger}{p_{x,0}(\rho)} - \rho, \frac{M_{x,0}\rho^e M_{x,0}^\dagger}{p_{x,0}(\rho^e)} - \rho^e; \frac{M_{x,0}\rho M_{x,0}^\dagger}{p_{x,0}(\rho)} - \rho, \frac{M_{x,0}\rho^e M_{x,0}^\dagger}{p_{x,0}(\rho^e)} - \rho^e \right) &+ \mathcal{O}(n^{-3/2}). \end{aligned}$$

Consequently,

$$\begin{aligned} n \sum_x p_{x,0}(\rho) \left(f \left(\frac{M_{x,0}\rho M_{x,0}^\dagger}{p_{x,0}(\rho)}, \frac{M_{x,0}\rho^e M_{x,0}^\dagger}{p_{x,0}(\rho^e)} \right) - f(\rho, \rho^e) \right) \\ = n D_{(\rho, \rho^e)} f \cdot \left(\sum_x M_{x,0}\rho M_{x,0}^\dagger - p_{x,0}(\rho)\rho, \sum_x \left(M_{x,0}\rho^e M_{x,0}^\dagger - p_{x,0}(\rho^e)\rho^e \right) \frac{p_{x,0}(\rho)}{p_{x,0}(\rho^e)} \right) \\ + \frac{1}{2} \sum_x n p_{x,0}(\rho) D_{(\rho, \rho^e)}^2 f \cdot \left(\frac{M_{x,0}\rho M_{x,0}^\dagger}{p_{x,0}(\rho)} - \rho, \frac{M_{x,0}\rho^e M_{x,0}^\dagger}{p_{x,0}(\rho^e)} - \rho^e; \frac{M_{x,0}\rho M_{x,0}^\dagger}{p_{x,0}(\rho)} - \rho, \frac{M_{x,0}\rho^e M_{x,0}^\dagger}{p_{x,0}(\rho^e)} - \rho^e \right) + \mathcal{O}(n^{-1/2}), \end{aligned} \quad (17)$$

where according to (10) and (11), up to $\mathcal{O}(n^{-1/2})$ terms, we have

$$\begin{aligned} n \sum_x \left(M_{x,0}\rho M_{x,0}^\dagger - p_{x,0}(\rho)\rho \right) &\approx -i[H, \rho] + \sum_\mu \left(\text{Tr}(\rho V_\mu^\dagger V_\mu) \rho - \frac{\{\rho, V_\mu^\dagger V_\mu\}}{2} \right) + \sum_\nu \mathcal{L}_\nu(\rho) \\ &= -i[H, \rho] + \sum_\xi \mathcal{L}_\xi(\rho) - \sum_\mu \Upsilon_\mu(\rho) \text{Tr}(V_\mu \rho V_\mu^\dagger), \end{aligned} \quad (18)$$

since, for any ν , $\sum_x x_{\nu-m_P} \Lambda_\nu(\rho) = 0$ and $\sum_x c_W = 1$. Also, we have

$$\begin{aligned} M_{x,0}\rho^e M_{x,0}^\dagger - p_{x,0}(\rho^e)\rho^e \\ = c_W \left(\frac{1}{\sqrt{n}} \sum_\nu x_{\nu-m_P} \Lambda_\nu(\rho^e) + \frac{1}{n} \left(-i[H, \rho^e] + \sum_\xi \mathcal{L}_\xi(\rho^e) - \sum_\mu \Upsilon_\mu(\rho^e) \text{Tr}(V_\mu \rho^e V_\mu^\dagger) \right) \right). \end{aligned}$$

Therefore, we find

$$\begin{aligned} n \sum_x \left(M_{x,0}\rho^e M_{x,0}^\dagger - p_{x,0}(\rho^e)\rho^e \right) \frac{p_{x,0}(\rho)}{p_{x,0}(\rho^e)} \\ = -i[H, \rho^e] + \sum_\xi \mathcal{L}_\xi(\rho^e) - \sum_\mu \Upsilon_\mu(\rho^e) \text{Tr}(V_\mu \rho^e V_\mu^\dagger) \\ + \sum_\nu \Lambda_\nu(\rho^e) \left(\text{Tr}(V_\nu \rho V_\nu^\dagger) - \text{Tr}(V_\nu \rho^e V_\nu^\dagger) \right) + \mathcal{O}(n^{-\frac{1}{2}}) \end{aligned} \quad (19)$$

since $\frac{p_{x,0}(\rho)}{p_{x,0}(\rho^e)} = 1 + \frac{1}{\sqrt{n}} \sum_{\nu} x_{\nu-m_P} \left(\text{Tr} \left((V_{\nu} + V_{\nu}^{\dagger}) \rho \right) - \text{Tr} \left((V_{\nu} + V_{\nu}^{\dagger}) \rho^e \right) \right) + \mathcal{O}(n^{-1/2})$. Moreover, we have used the fact that

$$\begin{aligned} \sum_x \sum_{\nu} c_W x_{\nu-m_P} \Lambda_{\nu}(\rho^e) \sum_{\nu'} x_{\nu'-m_P} \left(\text{Tr} \left((V_{\nu'} + V_{\nu'}^{\dagger}) \rho \right) - \text{Tr} \left((V_{\nu'} + V_{\nu'}^{\dagger}) \rho^e \right) \right) \\ = \sum_{\nu} \Lambda_{\nu}(\rho^e) \left(\text{Tr} \left((V_{\nu} + V_{\nu}^{\dagger}) \rho \right) - \text{Tr} \left((V_{\nu} + V_{\nu}^{\dagger}) \rho^e \right) \right), \end{aligned}$$

since $x_{\nu-m_P}^2 = 1$, $\sum_x c_W x_{\nu-m_P}^2 = 1$ and

$$\forall \nu \neq \nu' : \sum_x c_W x_{\nu-m_P} x_{\nu'-m_P} \Lambda_{\nu}(\rho^e) \left(\text{Tr} \left((V_{\nu'} + V_{\nu'}^{\dagger}) \rho \right) - \text{Tr} \left((V_{\nu'} + V_{\nu'}^{\dagger}) \rho^e \right) \right) = 0.$$

Equation (19) can be rewritten as follows

$$\begin{aligned} n \sum_x \left(M_{x,0} \rho^e M_{x,0}^{\dagger} - p_{x,0}(\rho^e) \rho^e \right) \frac{p_{x,0}(\rho)}{p_{x,0}(\rho^e)} \\ = -i[H, \rho^e] + \sum_{\xi} \mathcal{L}_{\xi}(\rho^e) + K(\rho, \rho^e) - \sum_{\mu} \Upsilon_{\mu}(\rho^e) \text{Tr} \left(V_{\mu} \rho V_{\mu}^{\dagger} \right) + \mathcal{O}(n^{-\frac{1}{2}}). \end{aligned}$$

As a result, the first righthand side term in Equation (17) can be written as follows

$$\begin{aligned} n D_{(\rho, \rho^e)} f \cdot \left(\sum_x M_{x,0} \rho M_{x,0}^{\dagger} - p_{x,0}(\rho) \rho, \sum_x \left(M_{x,0} \rho^e M_{x,0}^{\dagger} - p_{x,0}(\rho^e) \rho^e \right) \frac{p_{x,0}(\rho)}{p_{x,0}(\rho^e)} \right) \\ = D_{(\rho, \rho^e)} f \cdot \left(-i[H, \rho] + \sum_{\xi} \mathcal{L}_{\xi}(\rho), -i[H, \rho^e] + \sum_{\xi} \mathcal{L}_{\xi}(\rho^e) + K(\rho, \rho^e) \right) \\ - \sum_{\mu} \text{Tr} \left(V_{\mu} \rho V_{\mu}^{\dagger} \right) D_{(\rho, \rho^e)} f \cdot \left(\Upsilon_{\mu}(\rho), \Upsilon_{\mu}(\rho^e) \right) + \mathcal{O}(n^{-1/2}). \end{aligned}$$

Now let us calculate the second righthand side term in Equation (17). To get the zero order terms of $\sum_x n p_{x,0}(\rho) D_{(\rho, \rho^e)}^2 f \cdot \left(\frac{M_{x,0} \rho M_{x,0}^{\dagger}}{p_{x,0}(\rho)} - \rho, \frac{M_{x,0} \rho^e M_{x,0}^{\dagger}}{p_{x,0}(\rho^e)} - \rho^e; \frac{M_{x,0} \rho M_{x,0}^{\dagger}}{p_{x,0}(\rho)} - \rho, \frac{M_{x,0} \rho^e M_{x,0}^{\dagger}}{p_{x,0}(\rho^e)} - \rho^e \right)$,

we just need to combine the terms of order $n^{-1/2}$ in $\frac{M_{x,0} \rho M_{x,0}^{\dagger}}{p_{x,0}(\rho)} - \rho$ and in $M_{x,0} \rho M_{x,0}^{\dagger} - p_{x,0}(\rho) \rho$. Since

$$M_{x,0} \rho M_{x,0}^{\dagger} - p_{x,0}(\rho) \rho = \frac{c_W}{\sqrt{n}} \left(\sum_{\nu} x_{\nu-m_P} \Lambda_{\nu}(\rho) \right) + \mathcal{O}(n^{-1/2})$$

and

$$\frac{M_{x,0} \rho M_{x,0}^{\dagger}}{p_{x,0}(\rho)} - \rho \approx \frac{\rho + \frac{1}{\sqrt{n}} \sum_{\nu} x_{\nu-m_P} (V_{\nu} \rho + \rho V_{\nu}^{\dagger})}{1 + \frac{1}{\sqrt{n}} \sum_{\nu} x_{\nu-m_P} \text{Tr} \left((V_{\nu} + V_{\nu}^{\dagger}) \rho \right)} - \rho \approx \frac{1}{\sqrt{n}} \left(\sum_{\nu} x_{\nu-m_P} \Lambda_{\nu}(\rho) \right) + \mathcal{O}(n^{-1/2}),$$

we get, up to $\mathcal{O}(n^{-1/2})$ terms

$$\begin{aligned} \sum_x n p_{x,0}(\rho) D_{(\rho, \rho^e)}^2 f \cdot \left(\frac{M_{x,0} \rho M_{x,0}^{\dagger}}{p_{x,0}(\rho)} - \rho, \frac{M_{x,0} \rho^e M_{x,0}^{\dagger}}{p_{x,0}(\rho^e)} - \rho^e; \frac{M_{x,0} \rho M_{x,0}^{\dagger}}{p_{x,0}(\rho)} - \rho, \frac{M_{x,0} \rho^e M_{x,0}^{\dagger}}{p_{x,0}(\rho^e)} - \rho^e \right) \approx \\ \sum_{x, \nu, \nu'} c_W x_{\nu-m_P} x_{\nu'} D_{(\rho, \rho^e)}^2 f \cdot (\Lambda_{\nu}(\rho), \Lambda_{\nu}(\rho^e); \Lambda_{\nu'}(\rho), \Lambda_{\nu'}(\rho^e)) = \sum_{\nu} D_{(\rho, \rho^e)}^2 f \cdot (\Lambda_{\nu}(\rho), \Lambda_{\nu}(\rho^e); \Lambda_{\nu}(\rho), \Lambda_{\nu}(\rho^e)), \end{aligned}$$

where for the above equality, we have used the following facts

$$x_{\nu-m_P}^2 = 1, \quad \sum_x c_W = 1 \quad \text{and} \quad \forall \nu \neq \nu' : \quad \sum_x c_W x_{\nu-m_P} x_{\nu'-m_P} D_\rho^2 f(\rho) \cdot (\Lambda_\nu(\rho), \Lambda_{\nu'}(\rho)) = 0.$$

Thus, we find

$$\begin{aligned} & \sum_x p_{x,0}(\rho) \left(f \left(\frac{M_{x,0}\rho M_{x,0}^\dagger}{p_{x,0}(\rho)}, \frac{M_{x,0}\rho^e M_{x,0}^\dagger}{p_{x,0}(\rho^e)} \right) - f(\rho, \rho^e) \right) \\ &= D_{(\rho, \rho^e)} f \cdot \left(-i[H, \rho] + \sum_\xi \mathcal{L}_\xi(\rho), -i[H, \rho^e] + \sum_\xi \mathcal{L}_\xi(\rho^e) + K(\rho, \rho^e) \right) \\ & \quad - \sum_\mu \text{Tr} \left(V_\mu \rho V_\mu^\dagger \right) D_{(\rho, \rho^e)} f \cdot \left(\Upsilon_\mu(\rho), \Upsilon_\mu(\rho^e) \right) \\ & \quad + \frac{1}{2} \sum_\nu D_\rho^2 f(\rho) \cdot (\Lambda_\nu(\rho), \Lambda_\nu(\rho^e); \Lambda_\nu(\rho), \Lambda_\nu(\rho^e)) + \mathcal{O}(n^{-1/2}). \end{aligned} \quad (20)$$

According to (12), we have also

$$\begin{aligned} & \sum_{x,\mu} n p_{x,\mu}(\rho) \left(f \left(\frac{M_{x,\mu}\rho M_{x,\mu}^\dagger}{\text{Tr}(M_{x,\mu}\rho M_{x,\mu}^\dagger)}, \frac{M_{x,\mu}\rho^e M_{x,\mu}^\dagger}{\text{Tr}(M_{x,\mu}\rho^e M_{x,\mu}^\dagger)} \right) - f(\rho, \rho^e) \right) \\ &= \sum_\mu \text{Tr} \left(V_\mu \rho V_\mu^\dagger \right) \left(f \left(\frac{V_\mu \rho V_\mu^\dagger}{\text{Tr}(V_\mu \rho V_\mu^\dagger)}, \frac{V_\mu \rho^e V_\mu^\dagger}{\text{Tr}(V_\mu \rho^e V_\mu^\dagger)} \right) - f(\rho, \rho^e) \right). \end{aligned} \quad (21)$$

Finally, by Equations (20) and (21), we find that

$$\mathcal{A}_n f(\rho, \rho^e) = \mathcal{A} f(\rho, \rho^e) + \mathcal{O}(n^{-1/2}),$$

which finishes the proof of Proposition 1. \square

Now we are in the state to announce the main result of this section.

Theorem 2. *Consider the Markov process (ρ_t, ρ_t^e) satisfying respectively Equations (2) and (5). Then the fidelity $(F(\rho_t, \rho_t^e))$ defined in Equation (1) is a (\mathcal{F}_t) -submartingale, where $\mathcal{F}_t = \sigma\{(\rho_\tau, \rho_\tau^e) | \tau \leq t\}$. In particular, we have,*

$$\mathbb{E}(F(\rho_\tau, \rho_\tau^e) | \mathcal{F}_t) = \mathbb{E}(F(\rho_\tau, \rho_\tau^e) | (\rho_t, \rho_t^e)) \geq F(\rho_t, \rho_t^e),$$

for all $\tau \geq t$.

Proof. By Theorem 1, we know that the fidelity $F(\rho_n, \rho_n^e)$ is a submartingale with respect to the natural filtration of (ρ_k, ρ_k^e) . In terms of $(\rho_n(t), \rho_n^e(t))$, it follows that

$$\mathbb{E}(F(\rho_n(\tau), \rho_n^e(\tau)) | (\rho_n(t), \rho_n^e(t))) \geq F(\rho_n(t), \rho_n^e(t)), \quad (22)$$

for all $\tau \geq t$. In Proposition 1, we showed that $(\rho_n(t), \rho_n^e(t))$ converges in distribution to (ρ_t, ρ_t^e) . This implies that for any continuous real function $f(\rho, \rho^e)$ we have

$$\forall \tau \geq t, \quad \lim_{n \rightarrow +\infty} \mathbb{E}(f(\rho_n(\tau), \rho_n^e(\tau)) | (\rho_n(t), \rho_n^e(t))) = \mathbb{E}(f(\rho_\tau, \rho_\tau^e) | (\rho_t, \rho_t^e)).$$

As F is continuous, the limit of (22) for n tending to ∞ yields $\mathbb{E}(F(\rho_\tau, \rho_\tau^e) | (\rho_t, \rho_t^e)) \geq F(\rho_t, \rho_t^e)$ for all $\tau \geq t$. Since (ρ_t, ρ_t^e) is a Markov process, the result follows. \square

3 Imperfect measurements

By imperfect measurements, we mean both unread measurements performed by the environment (decoherence) and active measurements performed by non-ideal detectors. Starting from the discrete-time case experimentally used in [22] and detailed in [23], we derive the continuous-time optimal filters driven either by Poisson, Wiener processes or both of them. We obtain the SMEs and their associated quantum filters.

3.1 Discrete-time filters

The presentation here is very much inspired from [23] and adapted to (13) for n large. The jump-events labelled by (x, z) are of type (x, μ) with $\mu \in \{0, 1, \dots, m_P\}$. The effectively measured events are labelled by $s \in \{1, \dots, m\}$, with m denoting the number of distinct experimental detector outcomes. Suppose that we know the correlation between the jump-events (x, μ) and the experimental detection s . These correlations are modeled here by classical probabilities through a stochastic matrix η : $\eta_{s,(x,\mu)}^n$ that gives the probability of experimental detection s knowing that the effective jump-event is (x, μ) . Since $\eta_{s,(x,\mu)}^n \geq 0$ and for each (x, μ) , $\sum_{s=1}^m \eta_{s,(x,\mu)}^n = 1$, the matrix $\eta^n = (\eta_{s,(x,\mu)}^n)$ is a left stochastic matrix. We assume the following asymptotic for η^n ,

$$\eta^n = \eta^\infty + \frac{\tilde{\eta}}{n} + \mathcal{O}(n^{-2}), \quad (23)$$

where η^∞ is a left stochastic matrix.

We still denote by ρ the state associated to these experimental detections which is the best estimation of the system state knowing the initial state and all previous experimental detections. Following [23], it obeys to the following Markov process:

$$\rho_{k+1} = \frac{\mathcal{M}_s^n(\rho_k)}{\text{Tr}(\mathcal{M}_s^n(\rho_k))} \quad \text{with probability } p_s(\rho_k) = \text{Tr}(\mathcal{M}_s^n(\rho_k)) \quad (24)$$

where $\mathcal{M}_s^n(\rho) \triangleq \sum_{x,\mu} \eta_{s,(x,\mu)}^n M_{x,\mu} \rho M_{x,\mu}^\dagger$, $M_{x,0}$ and $M_{x,\mu}$ are given respectively by Equations (7) and (8) depending also on n .

Suppose that the initial state of dynamics (24) is not well known. Let ρ_0^e be an arbitrary initial estimate, the estimate discrete-time filter satisfies the following dynamics

$$\rho_{k+1}^e = \frac{\mathcal{M}_{s_k}^n(\rho_k^e)}{\text{Tr}(\mathcal{M}_{s_k}^n(\rho_k^e))} \quad (25)$$

where s_k corresponds to the experimental detection at time-step k .

Remark 2. For the same reason given in Remark 1, the above description is not always valid. Since the normalization $\text{Tr}(\mathcal{M}_s^n(\rho_k^e))$ can vanish and the formula (25) is then not well defined. Such a problem will not be appeared when ρ_k^e is full rank. (When $\text{Tr}(\mathcal{M}_s^n(\rho_k^e)) = 0$, we can still define the value of ρ_{k+1}^e , see more details in [23].) Again, as we consider the asymptotic evolution, such a problem will not appear.

We now state a theorem ensuring the stability of such estimation procedure whatever the initial state ρ_0^e is.

Theorem 3 ([21, 23]). Consider the Markov chain (ρ_k, ρ_k^e) satisfying (24) and (25):

$$(\rho_{k+1}, \rho_{k+1}^e) = \left(\frac{\mathcal{M}_s^n(\rho_k)}{\text{Tr}(\mathcal{M}_s^n(\rho_k))}, \frac{\mathcal{M}_s^n(\rho_k^e)}{\text{Tr}(\mathcal{M}_s^n(\rho_k^e))} \right) \quad \text{with probability } p_s(\rho_k) = \text{Tr}(\mathcal{M}_s^n(\rho_k)).$$

Then the fidelity $(F(\rho_k, \rho_k^e))$ is a (\mathcal{F}_k) -submartingale where $\mathcal{F}_k = \sigma\{(\rho_l, \rho_l^e) | l \leq k\}$. In particular, we have

$$\mathbb{E}(F(\rho_l, \rho_l^e) | \mathcal{F}_k) = \mathbb{E}(F(\rho_l, \rho_l^e) | (\rho_k, \rho_k^e)) \geq F(\rho_k, \rho_k^e),$$

for all $l > k$.

As far as we know, the continuous-time asymptotic versions of the discrete-time dynamics (24) and (25) associated to a left stochastic matrix η^n with asymptotics (23), have not been established up to now. In the following, we derive such continuous-time SMEs which are the limits of these discrete-time dynamics and prove their stability.

3.2 Continuous-time filters as limit of discrete-time filters

Take n large and consider the piece-wise constant continuous-time stochastic processes denoted by $\rho_n(t)$ and $\rho_n^e(t)$ with

$$\rho_n(t) = \rho_{[nt]}, \quad \rho_n^e(t) = \rho_{[nt]}^e. \quad (26)$$

where the discrete-time process (ρ_k, ρ_k^e) obeys to (24) with $k = [nt]$, the entire part of nt . It is clear that $(\rho_n(t), \rho_n^e(t))$ is a Markov process, as (ρ_k, ρ_k^e) is the Markov chain of Theorem 3.

Now suppose that ρ_t and ρ_t^e be respectively the solutions of the continuous-time dynamics of the true filter and its estimate at time t . Let \mathcal{A}_n and \mathcal{A} be respectively the Markov generators of (ρ_n, ρ_n^e) and (ρ, ρ^e) . Then for all C^2 -real valued function f , we are looking for the continuous-time processes ρ and ρ^e such that the following limit holds.

$$\lim_{n \rightarrow \infty} \sup_{(\rho, \rho^e) \in \mathcal{D}^2} |\mathcal{A}_n f(\rho, \rho^e) - \mathcal{A} f(\rho, \rho^e)| = 0.$$

We will see that such continuous-time limit depends essentially on the structure of $\eta^\infty = \lim_{n \rightarrow \infty} \eta^n$ described in Lemma 1 and yields a generalization of the usual stochastic master equations driven by Wiener and Poisson processes.

Lemma 1. Take the left stochastic matrix $\eta_{s,(x,\mu)}^\infty$ defined by (23) with line index $s \in \{1, \dots, m\}$ and column index $(x, \mu) \in \{-1, 1\}^{m_W} \times \{0, 1, \dots, m_P\}$. Consider the following partition (S^W, S^P) of $\{1, \dots, m\}$ labeling the number of experimental detections:

$$S^P = \left\{ s \in \{1, \dots, m\} \mid \sum_{x \in \{-1, 1\}^{m_W}} \eta_{s,(x,0)}^\infty = 0 \right\}, \quad S^W = \left\{ s \in \{1, \dots, m\} \mid \sum_{x \in \{-1, 1\}^{m_W}} \eta_{s,(x,0)}^\infty > 0 \right\}.$$

Then we have

- The $\bar{\eta}_{s,\mu}$'s defined by

$$\bar{\eta}_{s,\mu} \triangleq 2^{-m_W} \sum_{x \in \{-1, 1\}^{m_W}} \eta_{s,(x,\mu)}^\infty, \quad (s, \mu) \in S^P \times \{1, \dots, m_P\}, \quad (27)$$

satisfy

$$\forall (s, \mu) \in S^P \times \{1, \dots, m_P\}, \quad 0 \leq \bar{\eta}_{s,\mu} \leq 1 \quad \text{and} \quad \forall \mu \in \{1, \dots, m_P\} \quad \sum_{s \in S^P} \bar{\eta}_{s,\mu} \leq 1. \quad (28)$$

- The singular values of the matrix E with entries

$$\forall (s, \nu) \in S^W \times \{m_P, \dots, m_P + m_W\}, E_{s, \nu} \triangleq 2^{-m_W/2} \frac{\sum_x x_\nu - m_P \eta_{s, (x, 0)}^\infty}{\sqrt{\sum_x \eta_{s, (x, 0)}^\infty}} \quad (29)$$

belong to $[0, 1]$.

Proof. Inequality (28) is a direct consequence of S^P definition, because for any (x, μ) , $\sum_s \eta_{s, (x, \mu)}^\infty = 1$ and x belongs to a set of cardinal 2^{m_W} .

The matrix E is well defined because, by definition of S^W , the denominators in (29) are all strictly positive. The singular values of E do not exceed 1, if and only if, for any unitary vector $z \in \mathbb{R}^{m_W}$, the Euclidian norm of Ez does not exceed 1. With $z = (z_j)$ we have

$$\|Ez\|^2 = 2^{-m_W} \sum_{s \in S^W} \frac{\left(\sum_{x, j} x_j z_j \eta_{s, (x, 0)}^\infty \right)^2}{\sum_x \eta_{s, (x, 0)}^\infty}$$

where (x, j) varies in $\{-1, 1\}^{m_W} \times \{1, \dots, m_W\}$. With $\vartheta_s = \sum_x \eta_{s, (x, 0)}^\infty$, $\vartheta_{s, x} = \frac{\eta_{s, (x, 0)}^\infty}{\vartheta_s}$ and $\langle x|z \rangle = \sum_j x_j z_j$ we have $\|Ez\|^2 = 2^{-m_W} \sum_{s \in S^W} \vartheta_s (\sum_x \langle x|z \rangle \vartheta_{s, x})^2$. By convexity of $\alpha \mapsto \alpha^2$, we have

$$\forall s \in S^W, \left(\sum_x \vartheta_{s, x} \langle x|z \rangle \right)^2 \leq \sum_x \vartheta_{s, x} (\langle x|z \rangle)^2$$

since $\sum_x \vartheta_{s, x} = 1$ and $\vartheta_{s, x} \geq 0$. Thus

$$\|Ez\|^2 \leq 2^{-m_W} \sum_{s \in S^W} \eta_{s, (x, 0)}^\infty \sum_x (\langle x|z \rangle)^2 = 2^{-m_W} \sum_x (\langle x|z \rangle)^2$$

since $\sum_s \eta_{s, (x, 0)}^\infty = 1$ for any x . We have

$$\sum_x (\langle x|z \rangle)^2 = \sum_{x, j, j'} x_j x_{j'} z_j z_{j'} = \sum_{j, j'} z_j z_{j'} \left(\sum_x x_j x_{j'} \right) = \sum_j z_j^2 \left(\sum_x x_j^2 \right) = 2^{m_W} \sum_j z_j^2$$

since x in $\{-1, 1\}^{m_W}$ implies that $\sum_x x_j x_{j'} = 0$ for $j \neq j'$. Thus $\|Ez\|^2 \leq 1$ when $\|z\| = 1$. \square

Next theorem provides a generalization of usual SME driven by Wiener processes with detection errors to SME driven simultaneously by Wiener and Poisson processes with detections errors.

Theorem 4. Consider $(\rho_n(t), \rho_n^e(t))$ defined by (26) and associated to (24,25) with a left stochastic matrix η^n verifying (23). Then, for $n \rightarrow +\infty$, the process $(\rho_n(t), \rho_n^e(t))$ converges in distribution to the unique solutions of

$$\begin{aligned} d\rho_t = & -i[H, \rho_t] dt + \left(\sum_{\xi} V_{\xi} \rho_t V_{\xi}^{\dagger} - \frac{1}{2} (V_{\xi}^{\dagger} V_{\xi} \rho_t + \rho_t V_{\xi}^{\dagger} V_{\xi}) \right) dt \\ & + \sum_{s \in S^P} \left(\frac{\bar{\theta}_s \rho_t + \sum_{\mu} \bar{\eta}_{s, \mu} V_{\mu} \rho_t V_{\mu}^{\dagger}}{\bar{\theta}_s + \sum_{\mu} \bar{\eta}_{s, \mu} \text{Tr}(V_{\mu} \rho_t V_{\mu}^{\dagger})} - \rho_t \right) \left(dN_s(t) - (\bar{\theta}_s + \sum_{\mu} \bar{\eta}_{s, \mu} \text{Tr}(V_{\mu} \rho_t V_{\mu}^{\dagger})) dt \right) \\ & + \sum_{s \in S^W} \sqrt{\bar{\eta}_s} \left(\sum_{\nu} c_{s, \nu} (V_{\nu} \rho_t + \rho_t V_{\nu}^{\dagger} - \text{Tr}((V_{\nu} + V_{\nu}^{\dagger}) \rho_t) \rho_t) \right) dW_s(t) \quad (30) \end{aligned}$$

and

$$\begin{aligned}
d\rho_t^e = & -i[H, \rho_t^e] + \left(\sum_{\xi} V_{\xi} \rho_t^e V_{\xi}^{\dagger} - \frac{1}{2} (V_{\xi}^{\dagger} V_{\xi} \rho_t^e + \rho_t^e V_{\xi}^{\dagger} V_{\xi}) \right) dt \\
& + \sum_{s \in S^P} \left(\frac{\bar{\theta}_s \rho_t^e + \sum_{\mu} \bar{\eta}_{s,\mu} V_{\mu} \rho_t^e V_{\mu}^{\dagger}}{\bar{\theta}_s + \sum_{\mu} \bar{\eta}_{s,\mu} \text{Tr}(V_{\mu} \rho_t^e V_{\mu}^{\dagger})} - \rho_t^e \right) \left(dN_s(t) - \left(\bar{\theta}_s + \sum_{\mu} \bar{\eta}_{s,\mu} \text{Tr}(V_{\mu} \rho_t^e V_{\mu}^{\dagger}) \right) dt \right) \\
& + \sum_{s \in S^W} \sqrt{\bar{\eta}_s} \left(\sum_{\nu} c_{s,\nu} (V_{\nu} \rho_t^e + \rho_t^e V_{\nu}^{\dagger} - \text{Tr}((V_{\nu} + V_{\nu}^{\dagger}) \rho_t^e) \rho_t^e) \right) \times \dots \\
& \dots \times \left(dy_s(t) - \sqrt{\bar{\eta}_s} \text{Tr} \left(\sum_{\nu} c_{s,\nu} (V_{\nu} + V_{\nu}^{\dagger}) \rho_t^e \right) dt \right) \quad (31)
\end{aligned}$$

with partition (S_P, S_W) of $\{1, \dots, m\}$ defined in Lemma 1, with, in the above sums, $\mu \in \{1, \dots, m_P\}$, $\nu \in \{m_P + 1, \dots, m_P + m_W\}$ and $\xi \in \{1, \dots, m_P + m_W\}$, and where

- $s \in S^P$ is related to the Poisson process $dN_s(t)$ characterized by

$$\langle dN_s(t) \rangle = \left(\bar{\theta}_s + \sum_{\mu} \bar{\eta}_{s,\mu} \text{Tr}(V_{\mu} \rho_t^e V_{\mu}^{\dagger}) \right) dt$$

with $\bar{\eta}_{s,\mu}$ given by (27) and $\bar{\theta}_s = 2^{-m_W} \sum_x \tilde{\eta}_{s,(x,0)} \geq 0$ ($\tilde{\eta}$ defined in (23)).

- $s \in S^W$ is related to the continuous signal y_s related to the Wiener process dW_s via

$$dy_s(t) = dW_s(t) + \sqrt{\bar{\eta}_s} \text{Tr} \left(\sum_{\nu} c_{s,\nu} (V_{\nu} + V_{\nu}^{\dagger}) \rho_t^e \right) dt. \quad (32)$$

The efficiencies $\bar{\eta}_s$ belong to $[0, 1]$ and correspond to the eigenvalues of EE^{\dagger} with matrix E defined in Lemma 1. The real coefficients $c_{s,\nu}$ are given by the entries of the orthogonal matrix C appearing in the singular value decomposition of $E = RDC$ with R and C orthogonal and D the rectangular diagonal matrix with diagonal entries $\sqrt{\bar{\eta}_s}$:

$$\forall s, s' \in S^W, \quad \sum_{\nu} c_{s,\nu} c_{s',\nu} = \delta_{s,s'}$$

The proof of this theorem is given in the next subsection. It admits the same structure as the proof of Proposition 1 but with slightly more complicated computations for the Markov generators. The stability of such quantum filters is ensured in the following theorem.

Theorem 5. Consider the Markov process (ρ_t, ρ_t^e) , satisfying (30,31) where the positive integers m_P, m_W and m are arbitrary, where the $m_P + m_W$ square matrices V_{ξ} are arbitrary, where the partition (S^W, S^P) of $\{1, \dots, m\}$ is arbitrary, where $\forall (s, \mu) \in S^P \times \{1, \dots, m_P\}$, $\bar{\theta}_s \geq 0$, $\bar{\eta}_{s,\mu} \in [0, 1]$ and $\sum_{s' \in S^P} \bar{\eta}_{s',\mu} \leq 1$, where

$$\forall (s, s') \in S^W, \quad \bar{\eta}_s \in [0, 1] \quad \text{and} \quad \sum_{\nu \in \{m_P+1, \dots, m_P+m_W\}} c_{s,\nu} c_{s',\nu} = \delta_{s,s'}$$

Then, the fidelity $F(\rho_t, \rho_t^e)$ defined in Equation (1) is a (\mathcal{F}_t) -submartingale, where $\mathcal{F}_t = \sigma\{(\rho_\tau, \rho_\tau^e) | \tau \leq t\}$. In particular, we have,

$$\mathbb{E}(F(\rho_\tau, \rho_\tau^e) | \mathcal{F}_t) = \mathbb{E}(F(\rho_\tau, \rho_\tau^e) | (\rho_t, \rho_t^e)) \geq F(\rho_t, \rho_t^e),$$

for all $\tau \geq t$.

Proof. The proof is similar to the one given for Theorem 2. It relies on a direct application of Theorem 3 and Theorem 4. The assumptions made on the real coefficients $\bar{\theta}_s, \bar{\eta}_{s,mu}, \bar{\eta}_s$ and $c_{s,\nu}$ implies the existence of a family of stochastic process $(\rho_n(t), \rho_n^e(t))$ defined in (26) converging towards (ρ_t, ρ_t^e) for n large. This implication relies on manipulations based on Lemma 1 and providing a family of stochastic matrices $\eta^n = \eta^\infty + \tilde{\eta}/n + O(1/n)$ such that these coefficients $(\bar{\theta}_s, \bar{\eta}_{s,mu}, \bar{\eta}_s, c_{s,\nu})$ are related to η^∞ and $\tilde{\eta}$ according to Theorem 4. These manipulations are simple and not detailed here. \square

3.3 Proof of Theorem 4

The tightness property of $(\rho_n(t), \rho_n^e(t))$ can be concluded by the similar argument appeared in [19]. We give here the convergence proof of the Markov generators \mathcal{A}_n of $(\rho_n(t), \rho_n^e(t))$ towards the Markov generator \mathcal{A} of (ρ_t, ρ_t^e) . We detail the convergence proof for $S^P = \emptyset$ with $S^W = \{1, \dots, m\}$ and for $S^P = \{1, \dots, m\}$ with $S^W = \emptyset$. The general case where both S^P and S^W are not empty is just a concatenation of the two previous ones.

3.3.1 Proof of Theorem 4 when $S^P = \emptyset$ and $S^W = \{1, \dots, m\}$

Consider the singular value decomposition of matrix E defined in Lemma 1: $E = RDC$ with $R \in O(m)$, $C \in O(m_W)$ and D the rectangular diagonal matrix with diagonal formed by $\sqrt{\bar{\eta}_s}$. Notice that the entries of C coincide with the $c_{s,\nu}$. Set

$$\bar{p}_s \triangleq 2^{-m_W} \sum_x \eta_{s,(x,0)}^\infty, \quad \bar{p}_{s,\nu}^+ \triangleq 2^{-m_W} \sum_{x|x_\nu - m_P = 1} \eta_{s,(x,0)}^\infty, \quad \bar{p}_{s,\nu}^- \triangleq 2^{-m_W} \sum_{x|x_\nu - m_P = -1} \eta_{s,(x,0)}^\infty.$$

Then $E_{s,\nu} = \sum_\nu \frac{\bar{p}_{s,\nu}^+ - \bar{p}_{s,\nu}^-}{\bar{p}_s}$. Since R is an orthogonal matrix, $dW = (dW_s)_{s \in \{1, \dots, m\}}$ and $R^\dagger dW$ define similar Wiener processes. Thus, with replacing dW by $R^\dagger dW$ in (30,31), we have

$$d\rho_t = -i[H, \rho_t] dt + \sum_\xi \mathcal{L}_\xi(\rho_t) dt + \sum_s \left(\sum_\nu \frac{\bar{p}_{s,\nu}^+ - \bar{p}_{s,\nu}^-}{\sqrt{\bar{p}_s}} \Lambda_\nu(\rho_t) \right) dW_s(t), \quad (33)$$

$$d\rho_t^e = -i[H, \rho_t^e] dt + \sum_\xi \mathcal{L}_\xi(\rho_t^e) dt + \mathcal{K}(\rho_t, \rho_t^e) dt + \sum_s \sum_\nu \frac{\bar{p}_{s,\nu}^+ - \bar{p}_{s,\nu}^-}{\sqrt{\bar{p}_s}} \Lambda_\nu(\rho_t^e) dW_s(t), \quad (34)$$

with

$$\mathcal{K}(\rho, \rho^e) := \sum_s \left(\sum_\nu \frac{\bar{p}_{s,\nu}^+ - \bar{p}_{s,\nu}^-}{\bar{p}_s} \left(\text{Tr} \left((V_\nu + V_\nu^\dagger)(\rho - \rho^e) \right) \right) \right) \left(\sum_{\nu'} (\bar{p}_{s,\nu'}^+ - \bar{p}_{s,\nu'}^-) \Lambda_{\nu'}(\rho^e) \right).$$

The infinitesimal generator associated to the Markov process (ρ_n, ρ_n^e) satisfying dynamics (24) and (25), can be written as follows

$$\mathcal{A}_n f(\rho, \rho^e) \approx n \sum_s p_s(\rho) \left(f \left(\frac{\mathcal{M}_s^n(\rho)}{p_s(\rho)}, \frac{\mathcal{M}_s^n(\rho^e)}{p_s(\rho^e)} \right) - f(\rho, \rho^e) \right).$$

We can approximate the generator \mathcal{A}_n via the second order expansion of f around ρ and ρ^e .

$$\begin{aligned} \mathcal{A}_n f(\rho, \rho^e) &= nD_{(\rho, \rho^e)} f \cdot \left(\sum_s \mathcal{M}_s^n(\rho) - p_s(\rho)\rho, \sum_s \left(\mathcal{M}_s^n(\rho^e) - p_s(\rho^e)\rho^e \right) \frac{p_s(\rho)}{p_s(\rho^e)} \right) \\ &+ n \sum_s p_s(\rho) \frac{1}{2} D_{(\rho, \rho^e)}^2 f \cdot \left(\frac{\mathcal{M}_s^n(\rho)}{p_s(\rho)} - \rho, \frac{\mathcal{M}_s^n(\rho^e)}{p_s(\rho^e)} - \rho^e; \frac{\mathcal{M}_s^n(\rho)}{p_s(\rho)} - \rho, \frac{\mathcal{M}_s^n(\rho^e)}{p_s(\rho^e)} \right) + \mathcal{O}(n^{-1/2}). \end{aligned}$$

Let us first calculate the first term in above. By using (10), (11) and (12), we find

$$\begin{aligned} \mathcal{M}_s^n(\rho) - p_s(\rho)\rho &= \sum_x \eta_{s,(x,0)}^n \left(M_{x,0}\rho M_{x,0}^\dagger - p_{x,0}(\rho)\rho \right) + \sum_{x,\mu} \eta_{s,(x,\mu)}^n \left(M_{x,\mu}\rho M_{x,\mu}^\dagger - p_{x,\mu}(\rho)\rho \right) \\ &= \sum_x \frac{\eta_{s,(x,0)}^{n,cw}}{\sqrt{n}} \sum_\nu x_{\nu-m_P} \Lambda_\nu(\rho) \\ &+ \sum_x \frac{\eta_{s,(x,0)}^{n,cw}}{n} \left(-i[H, \rho] + \sum_\xi \mathcal{L}_\xi(\rho) - \sum_\mu \Upsilon_\mu(\rho) \text{Tr} \left(V_\mu \rho V_\mu^\dagger \right) \right) \\ &+ \sum_{x,\mu} \frac{\eta_{s,(x,\mu)}^{n,cw}}{n} \Upsilon_\mu(\rho) \text{Tr} \left(V_\mu \rho V_\mu^\dagger \right). \end{aligned} \quad (35)$$

We have

$$\sum_s \mathcal{M}_s^n(\rho) - p_s(\rho)\rho = \frac{1}{n} \left(-i[H, \rho] + \sum_\xi \mathcal{L}_\xi(\rho) \right), \quad (36)$$

since for all $(x, 0)$ and (x, μ) : $\sum_s \eta_{s,(x,0)}^n = \sum_s \eta_{s,(x,\mu)}^n = 1$. Also note that for all ν :

$$\sum_x x_{\nu-m_P} \Lambda_\nu(\rho) = 0.$$

Equation (35) can be rewritten as follows,

$$\begin{aligned} \mathcal{M}_s^n(\rho) - p_s(\rho)\rho &= \sum_x \frac{\eta_{s,(x,0)}^{n,cw}}{\sqrt{n}} \sum_\nu x_{\nu-m_P} \Lambda_\nu(\rho) + \mathcal{O}(n^{-1/2}) \\ &= \sum_\nu \frac{\bar{p}_{s,\nu}^+ - \bar{p}_{s,\nu}^-}{\sqrt{n}} \Lambda_\nu(\rho) + \mathcal{O}(n^{-1/2}). \end{aligned} \quad (37)$$

Thus $\mathcal{M}_s^n(\rho) - p_s(\rho)\rho = \mathcal{O}(n^{-1/2})$. Now using (11) and (12), we have

$$\begin{aligned} p_s(\rho) &= \sum_x \eta_{s,(x,0)}^n p_{x,0}(\rho) + \sum_{x,\mu} \eta_{s,(x,\mu)}^n p_{x,\mu}(\rho) = \\ &\sum_x \eta_{s,(x,0)}^{n,cw} \left(1 + \frac{1}{\sqrt{n}} \sum_\nu x_{\nu-m_P} \text{Tr} \left((V_\nu + V_\nu^\dagger)\rho \right) - \frac{1}{n} \sum_\mu \text{Tr} \left(\rho V_\mu^\dagger V_\mu \right) \right) \\ &+ \sum_{x,\mu} \frac{\eta_{s,(x,\mu)}^{n,cw}}{n} \text{Tr} \left(V_\mu \rho V_\mu^\dagger \right). \end{aligned}$$

By equation in above, the approximate value of $\frac{p_s(\rho)}{p_s(\rho^e)}$ is given by

$$\begin{aligned}\frac{p_s(\rho)}{p_s(\rho^e)} &= 1 + \frac{c_W}{\sqrt{n\bar{p}_s}} \sum_x \eta_{s,(x,0)}^n \left(\sum_{\nu} x_{\nu-m_P} \left(\text{Tr} \left((V_{\nu} + V_{\nu}^{\dagger}) \rho \right) - \text{Tr} \left((V_{\nu} + V_{\nu}^{\dagger}) \rho^e \right) \right) \right) + \mathcal{O}(n^{-1/2}) \\ &= 1 + \frac{1}{\sqrt{n\bar{p}_s}} \left(\sum_{\nu} (\bar{p}_{s,\nu}^+ - \bar{p}_{s,\nu}^-) \left(\text{Tr} \left((V_{\nu} + V_{\nu}^{\dagger}) \rho \right) - \text{Tr} \left((V_{\nu} + V_{\nu}^{\dagger}) \rho^e \right) \right) \right) + \mathcal{O}(n^{-1/2}).\end{aligned}$$

Finally, we find the following

$$\sum_s n \left(\mathcal{M}_s^n(\rho^e) - p_s(\rho^e) \rho^e \right) \frac{p_s(\rho)}{p_s(\rho^e)} = -i[H, \rho^e] + \sum_{\xi} \mathcal{L}_{\xi}(\rho^e) + \mathcal{K}(\rho, \rho^e) + \mathcal{O}(n^{-1/2}).$$

Since $p_s(\rho) = \bar{p}_s + \mathcal{O}(n^{-1/2})$ and $\frac{\mathcal{M}_s^n(\rho)}{p_s(\rho)} - \rho = \mathcal{O}(n^{-1/2})$, the zero order terms of

$$n \sum_s p_s(\rho) \frac{1}{2} D_{(\rho, \rho^e)}^2 f \cdot \left(\frac{\mathcal{M}_s^n(\rho)}{p_s(\rho)} - \rho, \frac{\mathcal{M}_s^n(\rho^e)}{p_s(\rho^e)} - \rho^e; \frac{\mathcal{M}_s^n(\rho)}{p_s(\rho)} - \rho, \frac{\mathcal{M}_s^n(\rho^e)}{p_s(\rho^e)} - \rho^e \right)$$

is given by the following computations:

$$\begin{aligned}& n \sum_s p_s(\rho) D_{(\rho, \rho^e)}^2 f \cdot \left(\frac{\mathcal{M}_s^n(\rho)}{p_s(\rho)} - \rho, \frac{\mathcal{M}_s^n(\rho^e)}{p_s(\rho^e)} - \rho^e; \frac{\mathcal{M}_s^n(\rho)}{p_s(\rho)} - \rho, \frac{\mathcal{M}_s^n(\rho^e)}{p_s(\rho^e)} - \rho^e \right) \\ &= n \sum_s \frac{1}{\bar{p}_s} D_{(\rho, \rho^e)}^2 f \cdot \left(\mathcal{M}_s^n(\rho) - p_s(\rho) \rho, \mathcal{M}_s^n(\rho^e) - p_s(\rho^e) \rho^e; \mathcal{M}_s^n(\rho) - p_s(\rho) \rho, \mathcal{M}_s^n(\rho^e) - p_s(\rho^e) \rho^e \right) + \mathcal{O}(n^{-1/2}) \\ &= \sum_s \frac{1}{\bar{p}_s} D_{(\rho, \rho^e)}^2 f \cdot \left(\sum_{\nu} (\bar{p}_{s,\nu}^+ - \bar{p}_{s,\nu}^-) \Lambda_{\nu}(\rho), \sum_{\nu} (\bar{p}_{s,\nu}^+ - \bar{p}_{s,\nu}^-) \Lambda_{\nu}(\rho^e); \sum_{\nu'} (\bar{p}_{s,\nu'}^+ - \bar{p}_{s,\nu'}^-) \Lambda_{\nu'}(\rho), \sum_{\nu'} (\bar{p}_{s,\nu'}^+ - \bar{p}_{s,\nu'}^-) \Lambda_{\nu'}(\rho^e) \right) \\ &+ \mathcal{O}(n^{-1/2}),\end{aligned}$$

where we have used Equation (37) and the fact that we have to keep only the zero order terms of $\frac{1}{p_s(\rho)}$ and $\frac{p_s(\rho)}{p_s(\rho^e)^2}$ which is equal to $\frac{1}{\bar{p}_s}$.

We get finally the following expression for the generator \mathcal{A} which is given as the limit of \mathcal{A}_n when n goes to infinity.

$$\begin{aligned}\mathcal{A}f(\rho, \rho^e) &= D_{\rho} f \cdot \left(-i[H, \rho] + \sum_{\xi} \mathcal{L}_{\xi}(\rho), -i[H, \rho^e] + \sum_{\xi} \mathcal{L}_{\xi}(\rho^e) + \mathcal{K}(\rho, \rho^e) \right) \\ &+ \frac{1}{2} \sum_s D_{(\rho, \rho^e)}^2 f \cdot \left(\sum_{\nu} \frac{\bar{p}_{s,\nu}^+ - \bar{p}_{s,\nu}^-}{\sqrt{\bar{p}_s}} \Lambda_{\nu}(\rho), \sum_{\nu} \frac{\bar{p}_{s,\nu}^+ - \bar{p}_{s,\nu}^-}{\sqrt{\bar{p}_s}} \Lambda_{\nu}(\rho^e); \sum_{\nu'} \frac{\bar{p}_{s,\nu'}^+ - \bar{p}_{s,\nu'}^-}{\sqrt{\bar{p}_s}} \Lambda_{\nu'}(\rho), \sum_{\nu'} \frac{\bar{p}_{s,\nu'}^+ - \bar{p}_{s,\nu'}^-}{\sqrt{\bar{p}_s}} \Lambda_{\nu'}(\rho^e) \right).\end{aligned}$$

The expression given in above corresponds well to the infinitesimal generator associated to the Markov process (ρ, ρ^e) satisfying dynamics (33) and (34).

Comparing this equation to the one given in (2) and (5) for perfect measurements, we observe that the Poisson processes have completely disappeared. It just remains the ensemble average $\sum_{\mu} \mathcal{L}_{\mu}(\rho)$ in the deterministic part. Also, the original m_W Wiener processes indexed by ν have been reorganized in m Wiener processes indexed by s .

3.3.2 Proof of Theorem 4 when $S^P = \{1, \dots, m\}$ and $S^W = \emptyset$

In this case, we have

$$d\rho_t = -i[H, \rho_t] dt + \sum_{\xi} \mathcal{L}_{\xi}(\rho_t) dt + \sum_s \left(\frac{\bar{\theta}_s \rho_t + \sum_{\mu} \bar{\eta}_{s,\mu} V_{\mu} \rho_t V_{\mu}^{\dagger}}{\bar{\theta}_s + \sum_{\mu} \bar{\eta}_{s,\mu} \text{Tr}(V_{\mu} \rho_t V_{\mu}^{\dagger})} - \rho_t \right) \left(dN_s(t) - \left(\bar{\theta}_s + \sum_{\mu} \bar{\eta}_{s,\mu} \text{Tr}(V_{\mu} \rho_t V_{\mu}^{\dagger}) \right) dt \right), \quad (38)$$

and

$$d\rho_t^e = -i[H, \rho_t^e] dt + \sum_{\xi} \mathcal{L}_{\xi}(\rho_t^e) dt + \sum_s \left(\frac{\bar{\theta}_s \rho_t^e + \sum_{\mu} \bar{\eta}_{s,\mu} V_{\mu} \rho_t^e V_{\mu}^{\dagger}}{\bar{\theta}_s + \sum_{\mu} \bar{\eta}_{s,\mu} \text{Tr}(V_{\mu} \rho_t^e V_{\mu}^{\dagger})} - \rho_t^e \right) \left(dN_s(t) - \left(\bar{\theta}_s + \sum_{\mu} \bar{\eta}_{s,\mu} \text{Tr}(V_{\mu} \rho_t^e V_{\mu}^{\dagger}) \right) dt \right). \quad (39)$$

The infinitesimal generator associated to the Markov process (ρ_n, ρ_n^e) is given by the following

$$\mathcal{A}_n f(\rho, \rho^e) \approx n \sum_s p_s(\rho) \left(f \left(\frac{\mathcal{M}_s^n(\rho)}{p_s(\rho)}, \frac{\mathcal{M}_s^n(\rho^e)}{p_s(\rho^e)} \right) - f(\rho, \rho^e) \right).$$

The equation in above can be rewritten as follows

$$\begin{aligned} \mathcal{A}_n f(\rho, \rho^e) \approx & n \sum_s p_s(\rho) \left(f \left(\frac{\mathcal{M}_s^n(\rho)}{p_s(\rho)}, \frac{\mathcal{M}_s^n(\rho^e)}{p_s(\rho^e)} \right) - f(\rho, \rho^e) \right) \\ & + n D_{(\rho, \rho^e)} f \cdot \left(\sum_s (\mathcal{M}_s^n(\rho) - p_s(\rho)\rho), \sum_s (\mathcal{M}_s^n(\rho^e) - p_s(\rho^e)\rho^e) + \mathcal{K}_2(\rho, \rho^e) \right) \\ & - n \sum_s p_s(\rho) D_{(\rho, \rho^e)} f \cdot \left(\frac{\mathcal{M}_s^n(\rho)}{p_s(\rho)} - \rho, \frac{\mathcal{M}_s^n(\rho^e)}{p_s(\rho^e)} - \rho^e \right), \quad (40) \end{aligned}$$

with $\mathcal{K}_2(\rho, \rho^e) := \sum_s \left(\frac{\mathcal{M}_s^n(\rho^e)}{p_s(\rho^e)} - \rho^e \right) (p_s(\rho) - p_s(\rho^e))$.

Now let us give the expression of $\mathcal{M}_s^n(\rho)$

$$\begin{aligned} \mathcal{M}_s^n(\rho) &= \sum_x \eta_{s,(x,0)}^n M_{x,0} \rho M_{x,0}^{\dagger} + \sum_{x,\mu} \eta_{s,(x,\mu)}^n M_{x,\mu} \rho M_{x,\mu}^{\dagger} \\ &= \frac{1}{n} \left(\bar{\theta}_s \rho + \sum_{\mu} \bar{\eta}_{s,\mu} V_{\mu} \rho V_{\mu}^{\dagger} \right) + \mathcal{O}(n^{-\frac{3}{2}}), \quad (41) \end{aligned}$$

where for the last term in above, we have used (10), (12), (23), $\bar{\theta}_s = c_W \sum_x \tilde{\eta}_{s,(x,0)}$ and $\bar{\eta}_{s,\mu} = c_W \sum_x \eta_{s,(x,\mu)}^{\infty}$.

Consequently, the probability $p_s(\rho)$ has the following form

$$p_s(\rho) = \frac{1}{n} \left(\bar{\theta}_s + \sum_{\mu} \bar{\eta}_{s,\mu} \text{Tr}(V_{\mu} \rho V_{\mu}^{\dagger}) \right) + \mathcal{O}(n^{-\frac{3}{2}}). \quad (42)$$

Now by using Equations (36), (41) and (42), we find the following expression for the limit of the generator \mathcal{A}_n expressed in (40)

$$\begin{aligned}
\mathcal{A}f(\rho, \rho^e) &= D_{(\rho, \rho^e)}f \cdot \left(-i[H, \rho] + \sum_{\xi} \mathcal{L}_{\xi}(\rho), -i[H, \rho^e] + \sum_{\xi} \mathcal{L}_{\xi}(\rho^e) + \mathcal{K}_2(\rho, \rho^e) \right) \\
&+ \sum_s \left(\bar{\theta}_s + \sum_{\mu} \bar{\eta}_{s, \mu} \text{Tr}(V_{\mu} \rho V_{\mu}^{\dagger}) \right) \times \\
&\left(f \left(\frac{\bar{\theta}_s \rho + \sum_{\mu} \bar{\eta}_{s, \mu} V_{\mu} \rho V_{\mu}^{\dagger}}{\bar{\theta}_s + \sum_{\mu} \bar{\eta}_{s, \mu} \text{Tr}(V_{\mu} \rho V_{\mu}^{\dagger})}, \frac{\bar{\theta}_s \rho^e + \sum_{\mu} \bar{\eta}_{s, \mu} V_{\mu} \rho^e V_{\mu}^{\dagger}}{\bar{\theta}_s + \sum_{\mu} \bar{\eta}_{s, \mu} \text{Tr}(V_{\mu} \rho^e V_{\mu}^{\dagger})} \right) - f(\rho, \rho^e) \right) \\
&- \sum_s \left(\bar{\theta}_s + \sum_{\mu} \bar{\eta}_{s, \mu} \text{Tr}(V_{\mu} \rho V_{\mu}^{\dagger}) \right) \times \\
&D_{(\rho, \rho^e)}f \cdot \left(\frac{\bar{\theta}_s \rho + \sum_{\mu} \bar{\eta}_{s, \mu} V_{\mu} \rho V_{\mu}^{\dagger}}{\bar{\theta}_s + \sum_{\mu} \bar{\eta}_{s, \mu} \text{Tr}(V_{\mu} \rho V_{\mu}^{\dagger})} - \rho, \frac{\bar{\theta}_s \rho^e + \sum_{\mu} \bar{\eta}_{s, \mu} V_{\mu} \rho^e V_{\mu}^{\dagger}}{\bar{\theta}_s + \sum_{\mu} \bar{\eta}_{s, \mu} \text{Tr}(V_{\mu} \rho^e V_{\mu}^{\dagger})} - \rho^e \right).
\end{aligned}$$

In fact, we have

$$\mathcal{A}_n f(\rho, \rho^e) = \mathcal{A}f(\rho, \rho^e) + \mathcal{O}(n^{-\frac{1}{2}}).$$

We observe that the expression of $\mathcal{A}f(\rho, \rho^e)$ corresponds well to the generator associated to the Markov process (ρ, ρ^e) satisfying SMEs (38) and (39). Now use the tightness property of the sequence (ρ_n, ρ_n^e) to conclude the convergence of the processes (ρ_n, ρ_n^e) towards (ρ, ρ^e) .

3.3.3 Proof of Theorem 4 when $S^P \neq \emptyset$ and $S^W \neq \emptyset$

In this general case, the Markov generator of (ρ_n, ρ_n^e) is decomposed into two different sums:

$$\begin{aligned}
\mathcal{A}_n f(\rho, \rho^e) &\approx n \sum_{s \in \{1, \dots, m\}} p_s(\rho) \left(f \left(\frac{\mathcal{M}_s^n(\rho)}{p_s(\rho)}, \frac{\mathcal{M}_s^n(\rho^e)}{p_s(\rho^e)} \right) - f(\rho, \rho^e) \right) \\
&= n \sum_{s \in S^W} p_s(\rho) \left(f \left(\frac{\mathcal{M}_s^n(\rho)}{p_s(\rho)}, \frac{\mathcal{M}_s^n(\rho^e)}{p_s(\rho^e)} \right) - f(\rho, \rho^e) \right) + n \sum_{s \in S^P} p_s(\rho) \left(f \left(\frac{\mathcal{M}_s^n(\rho)}{p_s(\rho)}, \frac{\mathcal{M}_s^n(\rho^e)}{p_s(\rho^e)} \right) - f(\rho, \rho^e) \right).
\end{aligned}$$

The limit for n large of each sum versus $s \in S^W$ and $s \in S^P$ is then obtained following exactly the same calculations as the ones given in subsections 3.3.1 and 3.3.2. Since (30,31) are just the concatenation of (33,34) for the Wiener part and (38,39) for the Poisson part, we get directly the convergence of the processes $(\rho_n(t), \rho_n^e(t))$ towards (ρ_t, ρ_t^e) .

4 Conclusion

For a large class of SMEs driven by Wiener and Poisson processes, Theorem 5 shows that $\text{Tr}^2(\sqrt{\sqrt{\rho}\rho^e\sqrt{\rho}})$ between the quantum state ρ_t and its estimate ρ_t^e is a submartingale. Thus the "metric" $1 - \text{Tr}^2(\sqrt{\sqrt{\rho}\rho^e\sqrt{\rho}})$ is a non-negative super-martingale that vanishes only when $\rho = \rho^e$. A natural question is the following: do there exist other "metrics" $D(\rho, \rho^e)$ that are super-martingales for such a large class of quantum systems?

Any such "metric" D must be contractive for all Lindblad equations: in Theorem 5, there is no restriction on the degree of incompleteness of the measurements. Thus we can assume

$\bar{\eta}_s = 0$, $S^W = \{1, \dots, m\}$ and $S^P = \emptyset$. In this case, ρ and ρ^e obey the same ordinary Lindblad differential equation

$$\frac{d}{dt}\rho = -i[H, \rho] + \sum_{\xi} V_{\xi}\rho V_{\xi}^{\dagger} - \frac{1}{2}(V_{\xi}^{\dagger}V_{\xi}\rho + \rho V_{\xi}^{\dagger}V_{\xi}),$$

where H and V_{ξ} are arbitrary. In [20], Petz has given, via the theory of operator monotone functions, a complete characterization of distance that are contractive for all Lindblad evolutions. Could we exploit Petz results to characterize "metrics" $D(\rho, \rho^e)$ that are supermartingale for all the quantum filtering processes considered in this paper?

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