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# Estimation d'erreur d'hyper-réduction de problèmes élastoviscoplastiques

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## Résumé :

*Nous proposons un indicateur d'erreur pour les prévisions réalisées par hyper-réduction de modèle dans le cadre de simulations élastoviscoplastiques. Ces problèmes sont décrits par des équations aux dérivées partielles en espace et en temps, non linéaires. Nous considérons ici les modèles standards généralisés. Le caractère fortement non linéaire de ces problèmes rend difficile l'utilisation de calculs hors ligne permettant de faciliter des calculs en base réduite. Il est alors nécessaire de proposer des méthodes de réduction de l'ordre des modèles qui soient peu coûteuses en opération d'assemblage ou d'intégration en espace. L'hyper-réduction a été proposée afin de restreindre ces opérations à un sous-domaine spatial appelé Domaine d'Intégration Réduit (RID). Un modèle hyper-réduit consiste à résoudre une forme faible des équations sur une partie du domaine, en exploitant une représentation en base réduite des déplacements. Il en résulte une prévision partielle de l'état mécanique. Nous proposons d'exploiter une base de contraintes en équilibre pour appliquer la méthode de l'Erreur en Relation de Comportement. Nous exploitons une formulation incrémentale variationnelle pour obtenir une borne supérieure de l'erreur d'approximation. Cette approche est mise en œuvre dans le cadre d'une étude de sensibilité à des paramètres d'une loi de comportement élastoviscoplastique. La borne supérieure de l'erreur exacte permet d'estimer un intervalle d'erreur sur les sorties du modèle, si l'on admet que ces sorties sont des fonctions lipschitziennes des déplacements.*

## Abstract :

*We propose an error indicator related to the hyper-reduction of elastoviscoplastic problems. The mechanical variables are solution of partial derivative equations in space and in time. Here we restrict attention to generalized standard formulations. These equations are highly nonlinear. Therefore it is not possible to introduce off-line computation in order to reduce the computational complexity of reduced-basis predictions. It turns out that both assembly procedure and residual computations must have their complexity reduced by using a convenient approach. By introducing a Reduced Integration Domain (RID), the hyper-reduction method aims to restrain the assembly procedure to a sub-domain of the spatial domain. Internal variables and stress fields are computed only over the RID. We propose to introduce a reduced basis of admissible stresses, fulfilling the equilibrium equation, in order to apply the Constitutive Relation Error method. The incremental variational formalism is considered. This enables to obtain an upper bound of the approximation error. A sensitivity analysis is presented as numerical application. The parameter of concern are the material coefficients of the constitutive equation. The output error-deviation is estimated thanks to the error indicator, providing that outputs are Lipschitz functions of displacements.*

**Mots clefs :** POD, Reduced Order Model, Constitutive Relation Error, Incremental Variational Formulation

## 1 Introduction

In this paper, we present a method to solve reduced-order equations related to mechanical models involving internal variables. The proposed approach is based on the classical snapshot Proper Orthogonal Decomposition (POD) [1] and on an original Petrov Galerkin formulation of the reduced governing equations [2]. The Petrov Galerkin formulation aims to restrain predictions only to a subdomain, named the Reduced Integration Domain (RID) [2,3]. The novelty of this work is the error estimation by using partial predictions, since they are only available over the RID. We propose a set of assumptions related to the formalism of the Constitutive

Relation Error [4,5]. If these assumptions are fulfilled, then the proposed error indicator is an upper bound of the approximation error. In case of model outputs extracted from the solution by Lipschitz functions, this upper bound of the error provides bounds on the estimated outputs. These bounds on outputs are useful when considering results of sensitivity analyses. They help to solve the following contradiction. When the parameter variations are too small the output variations are too small compared to the output error. But, if the parameter variations are too large, the reduced basis approximation becomes irrelevant.

The problem setting of the elastoviscoplastic problem is carefully chosen to get an upper bound of the approximation error. A time integration scheme is introduced prior to the weak formulation of the equations. It is an implicate Euler scheme. Then we state the general initial-boundary value problem of a body undergoing quasi static loading conditions and infinitesimal strains. The constitutive laws are described by using an incremental potential in the framework of the irreversible thermodynamic processes. Error estimator and incremental variational formulations were introduced in [6] for mechanical problem of bodies undergoing large dynamic deformations. Extensions of this approach were proposed in [7,8] for effective response predictions of heterogeneous materials. The strain history is taken into account by using internal variables denoted by  $\alpha$ . These variables are the lump sum of the history of material changes. This approach has proven its ability to cover a broad spectrum of models in viscoelasticity, viscoplasticity, plasticity and also continuum damage mechanics.

The hyper-reduced equations are a Petrov-Galerin formulation of the equilibrium equations, obtained by using truncated test functions having zero values outside the RID. The vector form of the reduced equations is similar to the one obtained by the Missing Point Estimation method [9] proposed for the Finite Volume Method. The strength of hyper-reduction is its ability to reduce mechanical models in material science while keeping unchanged the formulation of the constitutive equations [3]. The internal variables and the stresses are predicted over the RID only. The smaller the RID, the lower the computational complexity and the higher the approximation errors. These points have been developed in previous papers : for elastoplastic problems [10], finite strain elasto-plastic models [3], hyperelastic viscoplastic simulations of damping [11], crystal plasticity of microscopic aggregates [12], viscoelastic-viscoplastic unit cell homogenization problems [13], sintering simulations [14]. In case of mechanical problems involving internal variables, these variables can be recovered over the full domain  $\Omega$  after being predicted in the RID. This is performed by using the gappy POD [15] as explained in [11].

## 2 Incremental variational formulation

The continuous medium is occupying a domain  $\Omega$ . The displacement field at time  $t_{n+1}$  is defined on  $\Omega$  and is denoted by  $\mathbf{u}_{n+1}$ . The boundary  $\partial\Omega$  of  $\Omega$  is denoted by  $\partial_U\Omega \cup \partial_F\Omega$ . On  $\partial_U\Omega$ , there is the Dirichlet condition  $\mathbf{u}_{n+1} = \mathbf{u}_c$ . On  $\partial_F\Omega$ , there is a given force field  $\mathbf{F}$ . For the sake of simplicity, the dependence on  $n$  is omitted for  $\mathbf{u}_c$  and  $\mathbf{F}$ . The displacement field belongs to a function space  $\mathbf{u}_c + \mathcal{V}$ , where  $\mathcal{V}$  is a vector space defined by :

$$\mathcal{V} = \{ \mathbf{u} \in H^1(\Omega) \mid \mathbf{u}|_{\partial_U\Omega} = 0 \} \quad (1)$$

The statement of the mechanical problem is the following. We seek an estimation of the displacement field  $\mathbf{u}_{n+1} \in \mathbf{u}_c + \mathcal{V}$  defined by the constitutive equations and the principle of virtual work :

$$\int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}^*) : \boldsymbol{\sigma}_{n+1} d\Omega - \int_{\partial_F\Omega} \mathbf{u}^* \cdot \mathbf{F} d\Gamma = 0 \quad \forall \mathbf{u}^* \in \mathcal{V} \quad (2)$$

$$\boldsymbol{\sigma}_{n+1} = \frac{\partial w_{\Delta}}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}(\mathbf{u}_{n+1})) \quad \forall \mathbf{x} \in \Omega \quad (3)$$

where  $\mathbf{u}^*$  is a test function, the product between two second-order tensor  $\boldsymbol{\varepsilon}$  and  $\boldsymbol{\sigma}$  is understood to be  $\boldsymbol{\varepsilon} : \boldsymbol{\sigma} = \varepsilon_{ij}\sigma_{ij}$ , and  $w_{\Delta}$  is a condensed incremental potential. The convexity of  $w_{\Delta}$  has been proved in [7] under the assumption that the free energy and the dissipation potential are convex functions.  $w_{\Delta}$  depends also on internal variables. These variables are updated as functions of  $\boldsymbol{\varepsilon}(\mathbf{u}_{n+1})$ . In the sequel we restrict attention to the Finite Element (FE) approximation, denoted by  $\mathbf{u}_{n+1/FE}$ , and the hyper-reduced approximation denoted by  $\mathbf{u}_{n+1/ROM}$ . The error approximation of interest is the distance between  $\mathbf{u}_{n+1/FE} \in \mathbf{u}_c + \mathcal{V}$  and  $\mathbf{u}_{n+1/ROM} \in$

$\mathbf{u}_c + \mathcal{V}$ . The equations of the Finite Element model are obtained by substituting  $\mathcal{V}_h$ , the space of the FE ansatz functions, to  $\mathcal{V}$ . The model output is denoted by  $y$ . It is provided by the Lipschitz function  $\ell$  :

$$y = \ell(\mathbf{u}_h; \boldsymbol{\mu}) \quad (4)$$

Here  $\boldsymbol{\mu}$  is a vector of parameters of the condensed incremental potential. The purpose of the parametric study is the prediction of the following response surface, related to the sensitivity of the output  $y$  to parameter variations :

$$\boldsymbol{\mu} = \boldsymbol{\mu}_1 \pm 30\% \rightarrow \delta y = \ell(\mathbf{u}_{FE}; \boldsymbol{\mu}) - \ell(\mathbf{u}_{FE}; \boldsymbol{\mu}_1) \quad (5)$$

The Finite Element simulation related to  $\boldsymbol{\mu} = \boldsymbol{\mu}_1$  is performed to generate the reduced basis for the displacements and the stresses. The error on the output prediction reads :

$$|\delta y_{ROM} - \delta y| = |\ell(\mathbf{u}_{ROM}; \boldsymbol{\mu}) - \ell(\mathbf{u}_{FE}; \boldsymbol{\mu})| \leq \beta \|\mathbf{u}_{ROM} - \mathbf{u}_{FE}\|_{L^2(\Omega)} \quad (6)$$

where  $\ell$  is assumed to be a Lipschitz function and  $\beta$  is the Lipschitz constant of  $\ell$ .

### 3 Hyper-reduced prediction

We claim that the rank of the reduced system of equation can be preserved when the weak formulation is retrained to a subdomain of  $\Omega$ . Therefore, unique solutions of well-posed hyper-reduced problems can be forecasted. This subdomain is the Reduced Integration Domain (RID). It is denoted by  $\Omega_Z$ . As proposed in [2], the RID is reduced-basis dependent and it is constructed by off-line algebraic operations. In presence of internal variables, the RID receives also the contributions of the modes dedicated to the internal variables, as proposed in [3]. In a sense, these contributions make the RID construction physics dependent. The hyper-reduced formulation reads, find the displacement field  $\mathbf{u}_{n+1/ROM} \in \mathbf{u}_c + \mathcal{V}_{ROM}$  defined by the constitutive equations and the principle of virtual work :

$$\int_{\Omega_Z} \boldsymbol{\varepsilon}(\boldsymbol{\psi}_k) : \boldsymbol{\sigma}_{n+1} d\Omega - \int_{\partial_F \Omega} \boldsymbol{\psi}_k \cdot \mathbf{F} d\Gamma = 0 \quad \forall k \in \{1, \dots, N\} \quad (7)$$

$$\boldsymbol{\sigma}_{n+1} = \frac{\partial w_\Delta}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}(\mathbf{u}_{n+1/ROM})) \quad \forall \mathbf{x} \in \Omega \quad (8)$$

where  $\mathcal{V}_{ROM} = \text{span}\{\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_N\} \subset \mathcal{V}_h$ . The truncated test function  $\boldsymbol{\psi}_k$  have the same nodal values than the reduced vector  $\boldsymbol{\phi}_k$  excepted for nodes connected to the RID. For these nodes, the test function are set to zero. We refer the reader to [3] for more details about the hyper-reduction method.

As proposed in [16] for constitutive relation error estimation in the framework of model reduction, we introduce a reduced basis, denoted by  $(\boldsymbol{\phi}_k^\sigma)_{k=1, \dots, N^\sigma}$ , dedicated to stress fields fulfilling the following equilibrium condition :

$$\int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}^*) : \boldsymbol{\phi}_k^\sigma d\Omega = 0 \quad \forall \mathbf{u}^* \in \mathcal{V}_h \quad (9)$$

A linear elastic solution of the equilibrium equation provide a stress field denoted  $\boldsymbol{\sigma}_N$  that is not parameter dependent, such that :

$$\mathbf{u}^e \in \mathcal{V}_h \quad (10)$$

$$\int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}^*) : \boldsymbol{\sigma}_N d\Omega - \int_{\partial_F \Omega} \mathbf{u}^* \cdot \mathbf{F} d\Gamma = 0 \quad \forall \mathbf{u}^* \in \mathcal{V}_h \quad (11)$$

$$\boldsymbol{\sigma}_N = \mathbf{C} \boldsymbol{\varepsilon}(\mathbf{u}^e) \quad \forall \mathbf{x} \in \Omega \quad (12)$$

The approximation errors are transferred to discrepancies on the constitutive equation (8). Hence, the following projection of the stresses forecasted by hyper-reduction provides stresses, denoted by  $\hat{\boldsymbol{\sigma}}_{n+1}$ , fulfilling the FE equilibrium equation :

$$\hat{\boldsymbol{\sigma}}_{n+1} = \boldsymbol{\sigma}_N + \arg \min_{\boldsymbol{\sigma}^* \in \text{span}\{\boldsymbol{\phi}_1^\sigma, \dots, \boldsymbol{\phi}_{N^\sigma}^\sigma\}} \|\boldsymbol{\sigma}_{n+1} - \boldsymbol{\sigma}_N - \boldsymbol{\sigma}^*\|_{L^2(\Omega_Z)} \quad (13)$$

It is a gappy-POD reconstruction of an admissible stress field. The RID must be large enough to get a well-posed minimization problem, with unique solution.

## 4 Convexity and Constitutive Relation Error

The Legendre transformation provides a Constitutive Relation Error [4] :

$$\eta_{\Omega_Z}(\mathbf{u}_{ROM}, \hat{\boldsymbol{\sigma}}) = \sum_{n=1}^m \int_{\Omega_Z} w_{\Delta}(\boldsymbol{\varepsilon}(\mathbf{u}_{n+1/ROM})) + w_{\Delta}^*(\hat{\boldsymbol{\sigma}}_{n+1}) - \boldsymbol{\varepsilon}(\mathbf{u}_{n+1/ROM}) : \hat{\boldsymbol{\sigma}}_{n+1} d\Omega \geq 0, \quad (14)$$

with

$$w_{\Delta}^*(\hat{\boldsymbol{\sigma}}_{n+1}) = \sup_{\boldsymbol{\varepsilon}^*} (\boldsymbol{\varepsilon}^* : \hat{\boldsymbol{\sigma}}_{n+1} - w_{\Delta}(\boldsymbol{\varepsilon}^*)) \quad (15)$$

The following properties hold :

$$\eta_{\Omega_Z}(\mathbf{u}_{ROM}, \hat{\boldsymbol{\sigma}}) = 0 \quad \Leftrightarrow \quad \hat{\boldsymbol{\sigma}}_{n+1} = \frac{\partial w_{\Delta}}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}(\mathbf{u}_{n+1/ROM})) \quad \forall \mathbf{x} \in \Omega_Z \quad \forall n \quad (16)$$

$$\Leftrightarrow \quad \boldsymbol{\varepsilon}(\mathbf{u}_{n+1/ROM}) = \frac{\partial w_{\Delta}^*}{\partial \boldsymbol{\sigma}}(\hat{\boldsymbol{\sigma}}_{n+1}) \quad \forall \mathbf{x} \in \Omega_Z \quad \forall n \quad (17)$$

Thanks to convexity of  $w_{\Delta}^*$ , a pseudo distance between  $\hat{\boldsymbol{\sigma}}$  and  $\boldsymbol{\sigma}$  can be proposed. This pseudo distance is denoted by  $d_{\sigma}(\hat{\boldsymbol{\sigma}}, \boldsymbol{\sigma})$ . It reads :

$$d_{\sigma}(\hat{\boldsymbol{\sigma}}, \boldsymbol{\sigma}) = \sum_{n=1}^m \int_{\Omega_Z} w_{\Delta}^*(\hat{\boldsymbol{\sigma}}) - w_{\Delta}^*(\boldsymbol{\sigma}) - \frac{\partial w_{\Delta}^*}{\partial \boldsymbol{\sigma}}(\boldsymbol{\sigma}) : (\hat{\boldsymbol{\sigma}} - \boldsymbol{\sigma}) d\Omega \geq 0 \quad (18)$$

It is not a symmetric bilinear form. Similarly, a pseudo distance between  $\hat{\boldsymbol{\varepsilon}}$  and  $\boldsymbol{\varepsilon}$  is also proposed :

$$d_{\varepsilon}(\hat{\boldsymbol{\varepsilon}}, \boldsymbol{\varepsilon}) = \sum_{n=1}^m \int_{\Omega_Z} w_{\Delta}(\hat{\boldsymbol{\varepsilon}}) - w_{\Delta}(\boldsymbol{\varepsilon}) - \frac{\partial w_{\Delta}}{\partial \boldsymbol{\varepsilon}}(\boldsymbol{\varepsilon}) : (\hat{\boldsymbol{\varepsilon}} - \boldsymbol{\varepsilon}) d\Omega \geq 0 \quad (19)$$

A schematic representation of the pseudo distance  $d_{\sigma}$  is shown in Figure 1.

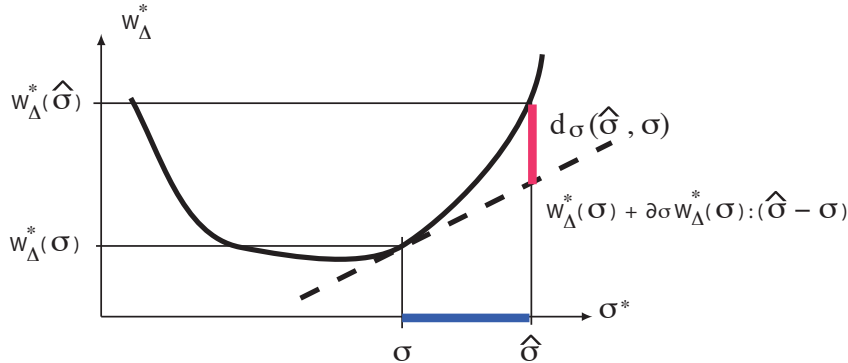


FIGURE 1 – Schematic view of the pseudo distance  $d_{\sigma}$ .

## 5 Upper bound of the approximation error

The pseudo distance  $d_{\varepsilon}(\boldsymbol{\varepsilon}(\mathbf{u}_{ROM}), \boldsymbol{\varepsilon}(\mathbf{u}_{FE}))$  is substituted for the norm of the approximation error  $\|\mathbf{u}_{ROM} - \mathbf{u}_{FE}\|_{L^2(\Omega_Z)}$  restricted to  $\Omega_Z$ . Therefore, the upper bound of the approximation error rely on the following assumption :

$$\left| \sum_{n=1}^m \int_{\partial_R \Omega_Z} (\mathbf{u}_{n+1/ROM} - \mathbf{u}_{n+1/FE})(\boldsymbol{\sigma}_{n+1/FE} - \hat{\boldsymbol{\sigma}}) \mathbf{n} d\Gamma \right| \quad (20)$$

$$\leq c_{\varepsilon} d_{\varepsilon}(\boldsymbol{\varepsilon}(\mathbf{u}_{ROM}), \boldsymbol{\varepsilon}(\mathbf{u}_{FE})) \quad (21)$$

$$+ c_{\sigma} d_{\sigma}(\hat{\boldsymbol{\sigma}}, \boldsymbol{\sigma}_{FE}) \quad (22)$$

with  $0 \leq c_{\varepsilon} < 1$ ,  $0 \leq c_{\sigma} < 1$  and  $\partial \Omega_Z = \partial_u \Omega_Z \cup \partial_F \Omega_Z \cup \partial_R \Omega_Z$ . The upper bound reads :

$$d_{\varepsilon}(\boldsymbol{\varepsilon}(\mathbf{u}_{ROM}), \boldsymbol{\varepsilon}(\mathbf{u}_{FE})) \leq \frac{1}{1 - c_{\varepsilon}} \eta_{\Omega_Z}(\mathbf{u}_{ROM}, \hat{\boldsymbol{\sigma}}) \quad \forall \mathbf{u}_{ROM} \in \mathbf{u}_c + \mathcal{V}, \quad \forall \hat{\boldsymbol{\sigma}} \in \boldsymbol{\sigma}_N + \text{span}(\boldsymbol{\phi}_k^{\sigma})_{k=1}^{N_{\sigma}} \quad (23)$$

The proof reads :

$$\begin{aligned}
\eta_{\Omega_Z}(\mathbf{u}_{ROM}, \hat{\boldsymbol{\sigma}}) &= \sum_{n=1}^m \int_{\Omega_Z} w_{\Delta}(\boldsymbol{\varepsilon}(\mathbf{u}_{n+1/ROM})) + w_{\Delta}^*(\hat{\boldsymbol{\sigma}}_{n+1}) - \boldsymbol{\varepsilon}(\mathbf{u}_{n+1/ROM}) : \hat{\boldsymbol{\sigma}}_{n+1} d\Omega \\
&= \sum_{n=1}^m \int_{\Omega_Z} w_{\Delta}(\boldsymbol{\varepsilon}(\mathbf{u}_{n+1/ROM})) + w_{\Delta}^*(\boldsymbol{\sigma}_{n+1/FE}) + w_{\Delta}^*(\hat{\boldsymbol{\sigma}}_{n+1}) - w_{\Delta}^*(\boldsymbol{\sigma}_{n+1/FE}) - \boldsymbol{\varepsilon}(\mathbf{u}_{n+1/ROM}) : \hat{\boldsymbol{\sigma}}_{n+1} d\Omega \\
&= d_{\varepsilon}(\boldsymbol{\varepsilon}(\mathbf{u}_{ROM}), \boldsymbol{\varepsilon}(\mathbf{u}_{FE})) + d_{\sigma}(\hat{\boldsymbol{\sigma}}, \boldsymbol{\sigma}_{FE}) \\
&+ \sum_{n=1}^m \int_{\Omega_Z} \boldsymbol{\sigma}_{n+1/FE} : \boldsymbol{\varepsilon}(\mathbf{u}_{n+1/ROM} - \mathbf{u}_{n+1}) + \boldsymbol{\sigma}_{n+1/FE} : \boldsymbol{\varepsilon}(\mathbf{u}_{n+1}) + \boldsymbol{\varepsilon}(\mathbf{u}_{n+1}) : (\hat{\boldsymbol{\sigma}}_{n+1} - \boldsymbol{\sigma}_{n+1/FE}) \\
&- \boldsymbol{\varepsilon}(\mathbf{u}_{n+1/ROM}) : \hat{\boldsymbol{\sigma}}_{n+1} d\Omega \\
&= d_{\varepsilon}(\boldsymbol{\varepsilon}(\mathbf{u}_{ROM}), \boldsymbol{\varepsilon}(\mathbf{u}_{FE})) + d_{\sigma}(\hat{\boldsymbol{\sigma}}, \boldsymbol{\sigma}_{FE}) \\
&+ \sum_{n=1}^m \int_{\Omega_Z} \boldsymbol{\sigma}_{n+1/FE} : \boldsymbol{\varepsilon}(\mathbf{u}_{n+1/ROM} - \mathbf{u}_{n+1}) + \boldsymbol{\varepsilon}(\mathbf{u}_{n+1}) : \hat{\boldsymbol{\sigma}}_{n+1} - \boldsymbol{\varepsilon}(\mathbf{u}_{n+1/ROM}) : \hat{\boldsymbol{\sigma}}_{n+1} d\Omega \\
&= d_{\varepsilon}(\boldsymbol{\varepsilon}(\mathbf{u}_{ROM}), \boldsymbol{\varepsilon}(\mathbf{u}_{FE})) + d_{\sigma}(\hat{\boldsymbol{\sigma}}, \boldsymbol{\sigma}_{FE}) \\
&+ \sum_{n=1}^m \int_{\Omega_Z} (\boldsymbol{\sigma}_{n+1/FE} - \hat{\boldsymbol{\sigma}}_{n+1}) : \boldsymbol{\varepsilon}(\mathbf{u}_{n+1/ROM} - \mathbf{u}_{n+1}) d\Omega
\end{aligned}$$

Finally, since  $\mathbf{u}_{n+1/ROM} - \mathbf{u}_{n+1/FE} \in \mathcal{V}_h$  and  $\boldsymbol{\sigma}_{n+1/FE} - \hat{\boldsymbol{\sigma}}_{n+1} \in \text{span}(\boldsymbol{\phi}_k^{\sigma})_{k=1}^{N_{\sigma}}$  then :

$$\begin{aligned}
&\int_{\Omega_Z} (\boldsymbol{\sigma}_{n+1/FE} - \hat{\boldsymbol{\sigma}}_{n+1}) : \boldsymbol{\varepsilon}(\mathbf{u}_{n+1/ROM} - \mathbf{u}_{n+1/FE}) d\Omega \\
&= \int_{\partial_R \Omega_Z} (\mathbf{u}_{n+1/ROM} - \mathbf{u}_{n+1/FE}) (\boldsymbol{\sigma}_{n+1/FE} - \hat{\boldsymbol{\sigma}}_{n+1}) \mathbf{n} d\Gamma,
\end{aligned}$$

where  $\mathbf{n}$  is the outward normal on the boundary  $\partial\Omega_Z$ .

## 6 Output bounds and conclusion

A last assumption upon the approximation error is proposed in order to obtain bounds on the output forecasted by the hyper-reduced model. This assumption reads :  $\exists \tilde{\beta} \geq 0$  such that,

$$\|\mathbf{u}_{ROM} - \mathbf{u}_{FE}\|_{L^2(\Omega)} \leq \tilde{\beta} d_{\varepsilon}(\boldsymbol{\varepsilon}(\mathbf{u}_{ROM}), \boldsymbol{\varepsilon}(\mathbf{u}_{FE})) \quad (24)$$

It can be interpreted by the following sentence. The approximation error over the RID is a representative estimate of the approximation error over the full domain. If the FE solution belongs to the subspace spanned by the reduced basis vectors, and if Equations (7) is not rank deficient, then the solution of the hyper-reduced equation is unique and it is  $\mathbf{u}_{FE}$ . Hence,  $\|\mathbf{u}_{ROM} - \mathbf{u}_{FE}\|_{L^2(\Omega)} = 0$  and  $d_{\varepsilon}(\boldsymbol{\varepsilon}(\mathbf{u}_{ROM}), \boldsymbol{\varepsilon}(\mathbf{u}_{FE})) = 0$ . If the reduced basis related to the stress can represent  $\boldsymbol{\sigma}_{n+1/FE} - \boldsymbol{\sigma}_N$ , then  $\hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma}_{FE}$ . It turns out that  $\eta_{\Omega_Z}(\mathbf{u}_{ROM}, \hat{\boldsymbol{\sigma}}) = 0$ .

Finally we obtain the following bounds :

$$\ell(\mathbf{u}_{FE}; \boldsymbol{\mu}) \in [\ell(\mathbf{u}_{ROM}; \boldsymbol{\mu}) - \beta \tilde{\beta} \frac{1}{1-c_{\varepsilon}} \eta_{\Omega_Z}(\mathbf{u}_{ROM}, \hat{\boldsymbol{\sigma}}), \ell(\mathbf{u}_{ROM}; \boldsymbol{\mu}) + \beta \tilde{\beta} \frac{1}{1-c_{\varepsilon}} \eta_{\Omega_Z}(\mathbf{u}_{ROM}, \hat{\boldsymbol{\sigma}})] \quad (25)$$

where  $\beta$ ,  $\tilde{\beta}$  and  $c_{\varepsilon}$  are not parameter dependent. Therefore, the product  $\gamma = \beta \tilde{\beta} \frac{1}{1-c_{\varepsilon}}$  can be estimated by using one additional FE solution related to  $\boldsymbol{\mu} = \boldsymbol{\mu}_2 \neq \boldsymbol{\mu}_1$ .

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