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► **To cite this version:**

Jesus Angulo, Santiago Velasco-Forero. Riemannian mathematical morphology. *Pattern Recognition Letters*, 2014, 47, pp.93-101. 10.1016/j.patrec.2014.05.015 . hal-00877144v3

**HAL Id: hal-00877144**

**<https://minesparis-psl.hal.science/hal-00877144v3>**

Submitted on 17 Jan 2016

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# Riemannian Mathematical Morphology<sup>★</sup>

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## Abstract

This paper introduces mathematical morphology operators for real-valued images whose support space is a Riemannian manifold. The starting point consists in replacing the Euclidean distance in the canonic quadratic structuring function by the Riemannian distance used for the adjoint dilation/erosion. We then extend the canonic case to a most general framework of Riemannian operators based on the notion of admissible Riemannian structuring function. An alternative paradigm of morphological Riemannian operators involves an external structuring function which is parallel transported to each point on the manifold. Besides the definition of the various Riemannian dilation/erosion and Riemannian opening/closing, their main properties are studied. We show also how recent results on Lasry–Lions regularization can be used for non-smooth image filtering based on morphological Riemannian operators. Theoretical connections with previous works on adaptive morphology and manifold shape morphology are also considered. From a practical viewpoint, various useful image embedding into Riemannian manifolds are formalized, with some illustrative examples of morphological processing real-valued 3D surfaces.

*Keywords:* mathematical morphology, manifold nonlinear image processing, Riemannian images, Riemannian image embedding, Riemannian structuring function, morphological processing of surfaces

## 1. Introduction

Pioneered for Boolean random sets (Matheron, 1975), extended latter to grey-level images (Serra, 1982) and more generally formulated in the framework of complete lattices (Serra, 1988; Heijmans, 1994), mathematical morphology is a nonlinear image processing methodology useful for solving efficiently many image analysis tasks (Soille, 1999). Our motivation in this paper is to formulate morphological operators for scalar functions on curved spaces.

Let  $E$  be the Euclidean  $\mathbb{R}^d$  or discrete space  $\mathbb{Z}^d$  (support space) and let  $\mathcal{T}$  be a set of grey-levels (space of values). It is assumed that  $\mathcal{T} = \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ . A grey-level image is represented by a function  $f : E \rightarrow \mathcal{T}$ ,  $f \in \mathcal{F}(E, \mathcal{T})$ , i.e.,  $f$  maps each pixel  $x \in E$  into a grey-level value in  $\mathcal{T}$ . Given a grey-level image, the two basic morphological mappings  $\mathcal{F}(E, \mathcal{T}) \rightarrow \mathcal{F}(E, \mathcal{T})$  are the dilation and the erosion given respectively by

$$\begin{cases} \delta_b(f)(x) = (f \oplus b)(x) = \sup_{y \in E} \{f(y) + b(y - x)\}, \\ \varepsilon_b(f)(x) = (f \ominus b)(x) = \inf_{y \in E} \{f(y) - b(y + x)\}, \end{cases}$$

where  $b \in \mathcal{F}(E, \mathcal{T})$  is the structuring function which determines the effect of the operator. By allowing infinity values, the further convention for ambiguous expressions should be considered:  $f(y) + b(x - y) = -\infty$  when  $f(y) = -\infty$  or  $b(x - y) = -\infty$ ,

and that  $f(y) - b(y + x) = +\infty$  when  $f(y) = +\infty$  or  $b(y + x) = -\infty$ . We easily note that both are invariant under translations in the spatial (“horizontal”) space  $E$  and in the grey-level (“vertical”) space  $\mathcal{T}$ , i.e.,  $f(x) \mapsto f_{h,\alpha}(x) = f(x - h) + \alpha$ ,  $x \in E$  and  $\alpha \in \mathbb{R}$ , then  $\delta_b(f_{h,\alpha})(x) = \delta_b(f)(x - h) + \alpha$ . The other morphological operators, such as the opening and the closing, are obtained by composition of dilation/erosion (Serra, 1982; Heijmans, 1994).

The structuring function is usually a parametric multi-scale family (Jackway and Deriche, 1996)  $b_\lambda(x)$ , where  $\lambda > 0$  is the scale parameter such that  $b_\lambda(x) = \lambda b(x/\lambda)$  and which satisfies the semi-group property  $(b_\lambda \oplus b_\mu)(x) = b_{\lambda+\mu}(x)$ . It is well known in the state-of-the-art of Euclidean morphology that the canonic family of structuring functions is the quadratic (or parabolic) one (Maragos, 1995; van den Boomgaard and Dorst, 1997); i.e.,

$$b_\lambda(x) = q_\lambda(x) = -\frac{\|x\|^2}{2\lambda}.$$

The most commonly studied framework, which additionally presents better properties of invariance, is based on flat structuring functions, called structuring elements. More precisely, let  $B$  be a Boolean set defined at the origin, i.e.,  $B \subseteq E$  or  $B \in \mathcal{P}(E)$ , which defines the “shape” of the structuring element, the associated structuring function is given by

$$b(x) = \begin{cases} 0 & \text{if } x \in B \\ -\infty & \text{if } x \in B^c \end{cases}$$

where  $B^c$  is the complement set of  $B$  in  $\mathcal{P}(E)$ . Hence, the flat dilation and flat erosion can be computed respectively by the moving local maxima and minima filters.

<sup>★</sup>This is an extended version of a paper that appeared at the 13th International Symposium of Mathematical Morphology held in May 27-29 in Uppsala, Sweden

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24 **Aim of the paper.** Let us consider now that the support space is  
 25 not Euclidean, see Fig. 1(a). This is the case for instance if we  
 26 deal with a smooth 3D surface, or more generally if the support  
 27 space is a Riemannian manifold  $\mathcal{M}$ . In all this paper, we consider  
 28 that  $\mathcal{M}$  is a finite dimensional compact manifold. Starting  
 29 point of this work is based on a Riemannian sup/inf-convolution  
 30 where the Euclidean distance in the canonic quadratic structur-  
 31 ing function is replaced by the Riemannian distance (Section 3).  
 32 Besides the definition of Riemannian dilation/erosion and Riemannian  
 33 opening/closing, we explore their properties and in particular the  
 34 associated granulometric scale-space. We also show how some theo-  
 35 retical results on Lasry–Lions regularization are useful for image  
 36 Lipschitz regularization using quadratic Riemannian dilation/erosion.  
 37 We then extend the canonic case to the most general framework of  
 38 Riemannian dilation/erosion and subsequent operators in Section 4,  
 39 by introducing the notion of admissible Riemannian structuring  
 40 function. Section 5 introduces a different paradigm of morphological  
 41 operators on Riemannian supported images, where the structuring  
 42 function is an external datum which is parallel transported to each  
 43 point on the manifold. We consider theoretically various useful case  
 44 studies of image manifolds in Section 7, but due to the limited paper  
 45 length, we only illustrate some cases of real-valued 3D surfaces.

46 **Related work.** Generalizations of Euclidean translation-invariant  
 47 morphology have followed three main directions. On the one hand,  
 48 adaptive morphology (Debayle and Pinoli, 2005; Lerallut et al.,  
 49 2007; Welk et al., 2011; Verdú et al., 2011; Ćurić et al., 2012;  
 50 Angulo, 2013; Landström and Thurley, 2013; Velasco-Forero and  
 51 Angulo, 2013), where the structuring function becomes dependent  
 52 on the position or the input image itself. Section 6 explores the  
 53 connections of our framework with such kind of approaches. On  
 54 the second hand, group morphology (Roerdink, 2000), where the  
 55 translation invariance is replaced by other group invariance  
 56 (similarity, affine, spherical, projective, etc.). Related to that,  
 57 we have also the morphology for binary manifolds (Roerdink, 1994),  
 58 whose relationship with our formulation is deeply studied in  
 59 Section 5. Finally, we should cite also the classical notion of  
 60 geodesic dilation (Lantuejoul and Beucher, 1981) as the basic  
 61 operator for (connective) geodesic reconstruction (Soille, 1999),  
 62 where the marker image is dilated according to the metric yielded  
 63 by the reference image (see also Section 6).

## 67 2. Basics on Riemannian manifold geometry

68 Let us remind in this section some basics on differential geometry  
 69 for Riemannian manifolds (Berger and Gostiaux, 1987), see Fig. 1(b)  
 70 for an explanatory diagram.

71 The *tangent space* of the manifold  $\mathcal{M}$  at a point  $p \in \mathcal{M}$ ,  
 72 denoted by  $T_p\mathcal{M}$ , is the set of all vectors tangent to  $\mathcal{M}$  at  $p$ .  
 73 The first issue to consider is how to transport vectors from one  
 74 point of  $\mathcal{M}$  to another. Let  $p, q \in \mathcal{M}$  and let  $\gamma : [a, b] \rightarrow \mathcal{M}$   
 75 be a parameterized curve (or path) from  $\gamma(a) = p$  to  $\gamma(b) = q$ .  
 76 For  $\mathbf{v} \in T_p\mathcal{M}$ , let  $\mathbf{V}$  be the unique parallel vector field along  
 77  $\gamma$  with  $\mathbf{V}(a) = \mathbf{v}$ . The map  $P_\gamma : T_p\mathcal{M} \rightarrow T_q\mathcal{M}$  determined  
 78 by  $P_\gamma(\mathbf{v}) = \mathbf{V}(b)$  is called *parallel transport from  $p$  to  $q$  along*

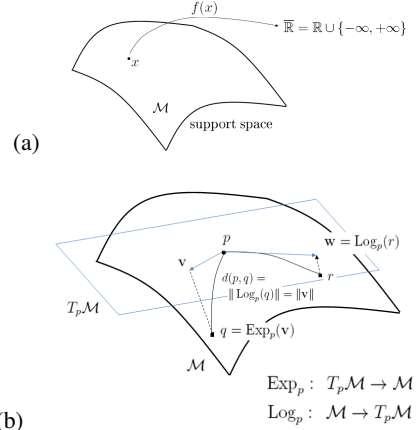


Figure 1: (a) Real-valued Riemannian image. (b) Riemannian manifold at tangent space a given point.

$\gamma$ , and  $P_\gamma(\mathbf{v})$  the parallel translate of  $\mathbf{v}$  along  $\gamma$  to  $q$ . Note that  
 parallel transport from  $p$  to  $q$  is path dependent: the difference  
 between two paths is a rotation around the normal to  $\mathcal{M}$  at  $q$ .  
 The *Riemannian distance* between two points  $p, q \in \mathcal{M}$ , denoted  
 $d(p, q)$ , is defined as the minimum length over all possible  
 smooth curves between  $p$  and  $q$ . A *geodesic*  $\gamma : [0, 1] \rightarrow \mathcal{M}$   
 connecting two points  $p, q \in \mathcal{M}$  is the shortest path on  $\mathcal{M}$  having  
 elements  $p$  and  $q$  as endpoints. The geodesic curve  $\gamma(t)$  can  
 be specified in terms of a starting point  $p \in \mathcal{M}$  and a tangent  
 vector (initial constant velocity)  $\mathbf{v} \in T_p\mathcal{M}$  as it represents the  
 solution of Christoffel differential equation with boundary condi-  
 tions  $\gamma(0) = p$  and  $\dot{\gamma}(0) = \mathbf{v}$ . The idea behind *exponential*  
*map*  $\text{Exp}_p$  is to parameterize a Riemannian manifold  $\mathcal{M}$ , locally  
 near any  $p \in \mathcal{M}$ , in terms of a mapping from the tangent  
 space  $T_p\mathcal{M}$  into a point in  $\mathcal{M}$ . The exponential map is injective  
 on a zero-centered ball  $B$  in  $T_p\mathcal{M}$  of some non-zero (possibly  
 infinity) radius. Thus for a point  $q$  in the image of  $B$  under  
 $\text{Exp}_p$  there exists a unique vector  $\mathbf{v} \in T_p\mathcal{M}$  corresponding to  
 a minimal length path under the exponential map from  $p$  to  $q$ .  
 Exponential maps may be associated to a manifold by the help  
 of geodesic curves. The exponential map  $\text{Exp}_p : T_p\mathcal{M} \rightarrow \mathcal{M}$   
 associated to any geodesic  $\gamma_{\mathbf{v}}$  emanating from  $p$  with tangent at  
 the origin  $\mathbf{v} \in T_p\mathcal{M}$  is defined as  $\text{Exp}_p(\mathbf{v}) = \gamma_{\mathbf{v}}(1)$ , where  
 the geodesic is given by  $\gamma_{\mathbf{v}}(t) = \text{Exp}_p(t\mathbf{v})$ . The geodesic has  
 constant speed equal to  $\|d\gamma_{\mathbf{v}}/dt\|(t) = \|\mathbf{v}\|$ , and thus the  
*exponential map preserves distances* for the initial point:  $d(p, \text{Exp}_p(\mathbf{v})) = \|\mathbf{v}\|$ .  
 A Riemannian manifold is geodesically complete if and only if  
 the exponential map  $\text{Exp}_p(\mathbf{v})$  is defined  $\forall p \in \mathcal{M}$  and  $\forall \mathbf{v} \in T_p\mathcal{M}$ .  
 The inverse operator, named *logarithm map*,  $\text{Exp}_p^{-1} = \text{Log}_p$   
 maps a point of  $q \in \mathcal{M}$  into to their associated tangent vectors  
 $\mathbf{v} \in T_p\mathcal{M}$ . The exponential map is in general only invertible  
 for a sufficient small neighbourhood of the origin in  $T_p\mathcal{M}$ ,  
 although on some manifolds the inverse exists for arbitrary  
 neighbourhoods. For a point  $q$  in the domain of  $\text{Log}_p$  the  
*geodesic distance* between  $p$  and  $q$  is given by  $d(p, q) = \|\text{Log}_p(q)\|$ .

### 115 3. Canonic Riemannian dilation and erosion

116 Let us start by a formal definition of the two basic canonic  
117 morphological operators for images supported on a Riemannian  
118 manifold.

**Definition 1.** Let  $\mathcal{M}$  a complete Riemannian manifold and  
 $d_{\mathcal{M}} : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}^+$ ,  $(x, y) \mapsto d_{\mathcal{M}}(x, y)$ , is the geodesic dis-  
tance on  $\mathcal{M}$ , for any image  $f : \mathcal{M} \rightarrow \overline{\mathbb{R}}$ ,  $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ ,  
so  $f \in \mathcal{F}(\mathcal{M}, \overline{\mathbb{R}})$  and for  $\lambda > 0$  we define for every  $x \in \mathcal{M}$  the  
canonic Riemannian dilation of  $f$  of scale parameter  $\lambda$  as

$$\delta_{\lambda}(f)(x) = \sup_{y \in \mathcal{M}} \left\{ f(y) - \frac{1}{2\lambda} d_{\mathcal{M}}(x, y)^2 \right\} \quad (1)$$

and the canonic Riemannian erosion of  $f$  of parameter  $\lambda$  as

$$\varepsilon_{\lambda}(f)(x) = \inf_{y \in \mathcal{M}} \left\{ f(y) + \frac{1}{2\lambda} d_{\mathcal{M}}(x, y)^2 \right\} \quad (2)$$

119 An obvious property of the canonic Riemannian dilation and  
120 erosion is the duality by the involution  $f(x) \mapsto \mathbb{C}f(x) = -f(x)$ ,  
121 i.e.,  $\delta_{\lambda}(f) = \mathbb{C}\varepsilon_{\lambda}(\mathbb{C}f)$ . As in classical Euclidean morphology,  
122 the adjunction relationship is fundamental for the construction  
123 of the rest of morphological operators.

**Proposition 2.** For any two real-valued images defined on the  
same Riemannian manifold  $\mathcal{M}$ , i.e.,  $f, g : \mathcal{M} \rightarrow \overline{\mathbb{R}}$ , the pair  
 $(\varepsilon_{\lambda}, \delta_{\lambda})$  is called the canonic Riemannian adjunction

$$\delta_{\lambda}(f)(x) \leq g(x) \Leftrightarrow f(x) \leq \varepsilon_{\lambda}(g)(x) \quad (3)$$

Hence, we have an adjunction if both images  $f$  and  $g$  are  
defined on the same Riemannian manifold  $\mathcal{M}$ , or in other terms,  
when the same ‘‘quadratic geodesic structuring function’’:

$$q_{\lambda}(x; y) = -\frac{1}{2\lambda} d_{\mathcal{M}}(x, y)^2, \quad (4)$$

is considered for pixel  $x \mapsto q_{\lambda}(x; y)$ ,  $y \in \mathcal{M}$  in both  $f$  and  
 $g$ . This result implies in particular that the canonic Riemannian  
dilation commutes with the supremum and the dual erosion  
with the infimum, i.e., for a given collection of images  
 $f_i \in \mathcal{F}(\mathcal{M}, \overline{\mathbb{R}})$ ,  $i \in I$ , we have

$$\delta_{\lambda} \left( \bigvee_{i \in I} f_i \right) = \bigvee_{i \in I} \delta_{\lambda}(f_i); \quad \varepsilon_{\lambda} \left( \bigwedge_{i \in I} f_i \right) = \bigwedge_{i \in I} \varepsilon_{\lambda}(f_i).$$

124 In addition, using the classical result on adjunctions in complete  
125 lattices (Heijmans, 1994), we state that the composition prod-  
126 ucts of the pair  $(\varepsilon_{\lambda}, \delta_{\lambda})$  lead to the adjoint opening and adjoint-  
127 closing if and only the field of geodesic structuring functions is  
128 computed on a common manifold  $\mathcal{M}$ .

**Definition 3.** Given an image  $f \in \mathcal{F}(\mathcal{M}, \overline{\mathbb{R}})$ , the canonic Rie-  
mannian opening and canonic Riemannian closing of scale pa-  
rameter  $\lambda$  are respectively given by

$$\gamma_{\lambda}(f)(x) = \sup_{z \in \mathcal{M}} \inf_{y \in \mathcal{M}} \left\{ f(y) + \frac{1}{2\lambda} d_{\mathcal{M}}(z, y)^2 - \frac{1}{2\lambda} d_{\mathcal{M}}(z, x)^2 \right\}, \quad (5)$$

and

$$\varphi_{\lambda}(f)(x) = \inf_{z \in \mathcal{M}} \sup_{y \in \mathcal{M}} \left\{ f(y) - \frac{1}{2\lambda} d_{\mathcal{M}}(z, y)^2 + \frac{1}{2\lambda} d_{\mathcal{M}}(z, x)^2 \right\}. \quad (6)$$

This technical point is very important since in some image  
manifold embedding the Riemannian manifold support  $\mathcal{M}$  of  
image  $f$  depends itself on  $f$ . If  $\mathcal{M}$  does not depends on  $f$ , the  
canonic Riemannian opening and closing are respectively given  
by  $\gamma_{\lambda}(f) = \delta_{\lambda}(\varepsilon_{\lambda}(f))$ , and  $\varphi_{\lambda}(f) = \varepsilon_{\lambda}(\delta_{\lambda}(f))$ . We notice that  
this issue was already considered by Roerdink (2009) for the  
case of adaptive neighbourhood morphology.

Having the canonic Riemannian opening and closing, all the  
other morphological filters defined by composition of them are  
easily obtained.

#### 3.1. Properties of $\delta_{\lambda}(f)$ and $\varepsilon_{\lambda}(f)$

Classical properties of Euclidean dilation and erosion have  
also the equivalent for Riemannian manifold  $\mathcal{M}$ , and they do  
not dependent on the geometry of  $\mathcal{M}$ .

**Proposition 4.** Let  $\mathcal{M}$  be a Riemannian manifold, and let  $f, g \in$   
 $\mathcal{F}(\mathcal{M}, \overline{\mathbb{R}})$  two real valued images  $\mathcal{M}$ . We have the following  
properties for the canonic Riemannian operators.

1. (Increasesness) If  $f(x) \leq g(x)$ ,  $\forall x \in \mathcal{M}$  then  $\delta_{\lambda}(f)(x) \leq$   
 $\delta_{\lambda}(g)(x)$  and  $\varepsilon_{\lambda}(f)(x) \leq \varepsilon_{\lambda}(g)(x)$ ,  $\forall x \in \mathcal{M}$  and  $\forall \lambda > 0$ .
2. (Extensivity and anti-extensivity)  $\delta_{\lambda}(f)(x) \geq f(x)$  and  
 $\varepsilon_{\lambda}(f)(x) \leq f(x)$ ,  $\forall x \in \mathcal{M}$  and  $\forall \lambda > 0$ .
3. (Ordering property) If  $0 < \lambda_1 < \lambda_2$  then  $\delta_{\lambda_2}(f)(x) \geq$   
 $\delta_{\lambda_1}(f)(x)$  and  $\varepsilon_{\lambda_2}(f)(x) \leq \varepsilon_{\lambda_1}(f)(x)$ .
4. (Invariance under isometry) If  $T : \mathcal{M} \rightarrow \mathcal{M}$  is an isometry  
of  $\mathcal{M}$  and if  $f$  is invariant under  $T$ , i.e.,  $f(Tz) = f(z)$   
for all  $z \in \mathcal{M}$ , then the Riemannian dilation and erosion  
are also invariant under  $T$ , i.e.,  $\delta_{\lambda}(f)(Tz) = \delta_{\lambda}(f)(z)$  and  
 $\varepsilon_{\lambda}(f)(Tz) = \varepsilon_{\lambda}(f)(z)$ ,  $\forall z \in \mathcal{M}$  and  $\forall \lambda > 0$ .
5. (Extrema preservation) We have  $\sup \delta_{\lambda}(f) = \sup f$  and  
 $\inf \varepsilon_{\lambda}(f) = \inf f$ , moreover if  $f$  is lower (resp. upper)  
semicontinuous then every minimizer (resp. maximizer) of  
 $\varepsilon_{\lambda}(f)$  (resp.  $\delta_{\lambda}(f)$ ) is a minimizer (resp. maximizer) of  $f$ ,  
and conversely.

#### 3.2. Flat isotropic Riemannian dilation and erosion

In order to obtain the counterpart of flat isotropic Euclidean  
dilation and erosion, we replace the quadratic structuring func-  
tion  $q_{\lambda}(x, y)$  by a flat structuring function given by the geodesic  
ball of radius  $r$  centered at  $x$ , i.e.,

$$B_r(x) = \{y : d_{\mathcal{M}}(x, y) \leq r\}, \quad r > 0. \quad (7)$$

The corresponding flat isotropic Riemannian dilation and  
erosion of size  $r$  are given by:

$$\delta_{B_r}(f)(x) = \sup \{f(y) : y \in \check{B}_r(x)\}, \quad (8)$$

$$\varepsilon_{B_r}(f)(x) = \inf \{f(y) : y \in B_r(x)\}. \quad (9)$$

where  $\check{B}_r(x)$  is the transposed shape of ball  $B_r(x)$ . Correspond-  
ing flat isotropic Riemannian opening and closing are obtained  
by composition of operators (8) and (9):

$$\gamma_{B_r}(f) = \delta_{B_r}(\varepsilon_{B_r}(f)); \quad \varphi_{B_r}(f) = \varepsilon_{B_r}(\delta_{B_r}(f)). \quad (10)$$

All the properties formulated for canonic operators hold for flat  
isotropic ones too. For practical applications, it should be noted  
that flat operators typically lead to stronger filtering effects than  
the quadratic ones.

172 3.3. Riemannian granulometries:

173 scale-space properties of  $\gamma_\lambda(f)$  and  $\varphi_\lambda(f)$

174 For the canonic Riemannian opening and closing, we have  
175 also the classical properties which are naturally proved as a con-  
176 sequence of the adjunction, see (Heijmans, 1994).

177 **Proposition 5.** Let  $\gamma_\lambda(f)$  and  $\varphi_\lambda(f)$  be respectively the canonic  
178 Riemannian opening and closing of an image  $f \in \mathcal{F}(\mathcal{M}, \mathbb{R})$ .

- 179 1.  $\gamma_\lambda(f)$  and  $\varphi_\lambda(f)$  are both increasing operators.  
2.  $\gamma_\lambda(f)$  is anti-extensive and  $\varphi_\lambda(f)$  extensive with the fol-  
lowing ordering relationships, i.e., for for  $0 < \lambda_1 \leq \lambda_2$ , we  
have:

$$\gamma_{\lambda_2}(f)(x) \leq \gamma_{\lambda_1}(f)(x) \leq f(x) \leq \varphi_{\lambda_1}(f)(x) \leq \varphi_{\lambda_2}(f)(x); \quad (11)$$

- 180 3. idempotency of both operators,  $\gamma_\lambda(\gamma_\lambda(f)) = \gamma_\lambda(f)$  and  
181  $\varphi_\lambda(\varphi_\lambda(f)) = \varphi_\lambda(f)$

182 Property 3 on idempotency together with the increaseness de-  
183 fines a family of so-called algebraic openings/closings (Serra,  
184 1988; Heijmans, 1994) larger than the one associated to the  
185 composition of dilation/erosion. Idempotent and increasing  
186 operators are also known as ethmomorphisms by Kiselman  
187 (2007). Anti-extensivity and extensivity involves that  $\gamma_\lambda$  is a  
188 anoiktomorphism and  $\varphi_\lambda$  a cleistomorphism. One of the most  
189 classical results in morphological operators provided us an ex-  
190 ample of algebraic opening: given a collection of openings  $\{\gamma_i\}$ ,  
191 increasing, idempotent and anti-extensive operators for all  $i$ , the  
192 supremum of them  $\sup_i \gamma_i$  is also an opening (Matheron, 1975).  
193 A dual result is obtained for the closing by changing the sup by  
194 the inf.

195 The class of openings (resp. closings) is neither closed under  
196 infimum (resp. opening) or a generic composition. There is  
197 however a semi-group property leading to a scale-space frame-  
198 work for opening/closing operators, known as granulometries.  
199 The notion of granulometry in Euclidean morphology is sum-  
200 marized in the following results (Matheron, 1975; Serra, 1988).

**Theorem 6 (Matheron (1975), Serra (1988)).** A parameter-  
ized family  $\{\gamma_\lambda\}_{\lambda>0}$  of flat openings from  $\mathcal{F}(E, \mathcal{T})$  into  $\mathcal{F}(E, \mathcal{T})$   
is a granulometry (or size ditribution) when

$$\gamma_{\lambda_1} \gamma_{\lambda_2} = \gamma_{\lambda_2} \gamma_{\lambda_1} = \gamma_{\sup(\lambda_1, \lambda_2)}; \quad \lambda_1, \lambda_2 > 0. \quad (12)$$

Condition (12) is equivalent to both

$$\gamma_{\lambda_1} \leq \gamma_{\lambda_2}; \quad \lambda_1 \geq \lambda_2 > 0; \quad (13)$$

$$\mathcal{B}_{\lambda_1} \subseteq \mathcal{B}_{\lambda_2}; \quad \lambda_1 \geq \lambda_2 > 0$$

201 where  $\mathcal{B}_\lambda$  is the invariance domain of the opening at scale  $\lambda$ ;  
202 i.e., the family of structuring elements  $B$ s such that  $B = \gamma_\lambda(B)$   
203 (Serra, 1988).

204 By duality, we introduce antysize distributions as the families of  
205 closings  $\{\varphi_\lambda\}_{\lambda>0}$ .

Axiom (12) shows how translation invariant flat openings are  
composed and highlights their semi-group structure. Equivalent

condition (13) emphasizes the monotonicity of the granulome-  
try with respect to  $\lambda$ : the opening becomes more and more ac-  
tive as  $\lambda$  increases. When dealing with Euclidean spaces, Math-  
eron (1975) introduced the notion of Euclidean granulometry  
as the size distribution being translationally invariant and com-  
patible with homothetics, i.e.,  $\gamma_\lambda(f(x)) = \lambda \gamma_1(f(\lambda^{-1}x))$ , where  
 $f \in \mathcal{F}(E, \mathcal{T})$  is an Euclidean grey-level images. More pre-  
cisely, a family of mappings  $\gamma_\lambda$  is an Euclidean granulometry if  
and only if there exist a class  $\mathcal{B}'$  such that

$$\gamma_\lambda(f) = \bigvee_{B \in \mathcal{B}', \mu \geq \lambda} \gamma_{\mu B}(f).$$

Then the domain of invariance  $\mathcal{B}_\lambda$  are equal to  $\lambda \mathcal{B}$ , where  $\mathcal{B}$  is  
the class closed under union, translation and homothetics  $\geq 1$ ,  
which is generated by  $\mathcal{B}'$ . If we reduce the class  $\mathcal{B}'$  to a single  
element  $B$ , the associated size distribution becomes

$$\gamma_\lambda(f) = \bigvee_{\mu \geq \lambda} \gamma_{\mu B}(f).$$

The following key result simplifies the situation. The size dis-  
tribution by a compact structuring element  $B$  is equivalent to  
 $\gamma_\lambda(f) = \gamma_{\lambda B}(f)$  if and only if  $B$  is convex. The extension  
of granulometric theory to non-flat structuring functions was  
deeply studied in (Kraus et al., 1993). In particular, it was  
proven that one can build grey-level Euclidean granulometries  
with one structuring function if and only if this function has a  
convex compact domain and is constant there (flat function).

We can naturally extend Matheron axiomatic to the general  
case of openings in Riemannian supported images. We start  
by giving a result which is valid for families of openings  $\{\psi_\lambda\}$   
(idempotent and anti-extensive operators) more general than the  
canonic Riemannian openings.

**Proposition 7.** Given the set of Riemannian openings  $\{\psi_\lambda\}_{\lambda>0}$   
indexed according to the positive parameter  $\lambda$ , but not nec-  
essary ordered between them, the corresponding Riemannian  
granulometry on image  $f \in \mathcal{F}(\mathcal{M}, \mathbb{R})$  is the family of multi-  
scale openings  $\{\Gamma_\lambda\}_{\lambda>0}$  generated as

$$\Gamma_\lambda(f) = \bigvee_{\mu \geq \lambda} \psi_\mu(f)$$

such that the granulometric semi-group law holds for any pair  
of scales:

$$\Gamma_{\lambda_1}(\Gamma_{\lambda_2}(f)) = \Gamma_{\lambda_2}(\Gamma_{\lambda_1}(f)) = \Gamma_{\sup(\lambda_1, \lambda_2)}(f). \quad (14)$$

In the particular case of canonic Riemannian openings,  $\{\gamma_\lambda\}_{\lambda>0}$ ,  
we always have  $\gamma_{\lambda_1} \leq \gamma_{\lambda_2}$  if  $\lambda_1 \geq \lambda_2 > 0$ . Hence,  $\Gamma_\lambda(f) = \gamma_\lambda$   
and consequently  $\{\gamma_\lambda\}_{\lambda>0}$  is a granulometry. This is also valid  
for flat isotropic Riemannian openings.

The Riemannian case closest to Matheron's Euclidean gran-  
ulometries corresponds to the flat isotropic Riemannian open-  
ings  $\gamma_{B_r}$  associated to a concave quadratic geodesic structuring  
function  $q_\lambda(x, y)$ . Or in other terms, the case of a Riemannian  
manifold  $\mathcal{M}$  where the Riemannian distance is always a convex  
function, since this fact involves that  $B_r(x)$  as defined in (7)

is a convex set for any  $r$  at any  $x \in \mathcal{M}$ . Obviously, the flat convex Riemannian granulometry  $\{\gamma_{B_r}\}_{r>0}$  is not translation invariant but we have that  $B_{r_1}(x) \subseteq B_{r_2}(x)$ , for  $r_2 \geq r_1$  and for any  $x \in \mathcal{M}$ , which involves a natural sieving selection of features in the neighborhood of any point  $x$ .

A Riemannian distance function which is convex is not only useful for scale-space properties. As discussed just below, one has powerful results of regularization too.

### 3.4. Concavity of $q_\lambda(x; y)$ and Lipschitz image regularization using $(\varepsilon_\lambda, \delta_\lambda)$

Lasry–Lions regularization (Lasry and Lions, 1986) is a theory of nonsmooth approximation for functions in Hilbert spaces using combinations of Euclidean dilation and erosion with quadratic structuring functions, which leads to the approximation of bounded lower or upper-semicontinuous functions with Lipschitz continuous derivatives which approximate  $f$ , without assuming convexity of  $f$ . The approach was generalized in (Attouch and Aze, 1993) to semicontinuous, non necessarily bounded, quadratically minorized/majorized functions defined on  $\mathbb{R}^n$ . More precisely, we have.

**Theorem 8 (Lasry and Lions(1986), Attouch and Aze(1993)).** For all  $0 < \mu < \lambda$ , let us define for a given image  $f$  the Lasry–Lions regularizers based on Euclidean dilation and erosion by a quadratic structuring function  $q_\lambda$  as:

$$(f_\lambda)^\mu(x) = ((f \ominus q_\lambda) \oplus q_\mu)(x),$$

$$(f^\lambda)_\mu(x) = ((f \oplus q_\lambda) \ominus q_\mu)(x).$$

- Let  $f$  be a bounded uniformly continuous scalar functions in  $\mathbb{R}^n$ . Then the functions  $(f_\lambda)^\mu$  and  $(f^\lambda)_\mu$  converge uniformly to  $f$  when  $\lambda, \mu \rightarrow 0$ , and belong to the class  $C_b^{1,1}(\mathbb{R}^n)$  (i.e., bounded continuously differentiable with a Lipschitz continuous gradient), namely  $|\nabla(f_\lambda)^\mu(x) - \nabla(f_\lambda)^\mu(y)| \leq M_{\lambda,\mu}\|x - y\|$  and  $|\nabla(f^\lambda)_\mu(x) - \nabla(f^\lambda)_\mu(y)| \leq M_{\lambda,\mu}\|x - y\|$ , where  $M_{\lambda,\mu} = (\mu^{-1}, (\lambda - \mu)^{-1})$ .
- Let  $f : E \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$  be a lower-semicontinuous function and  $g : E \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty\}$  an upper-semicontinuous. We assume the growing conditions  $f(x) \geq -\frac{c}{2}(1 + \|x\|^2)$ ,  $c \geq 0$  (quadratically minorized), then  $g(x) \leq \frac{c'}{2}(1 + \|x\|^2)$ ,  $c' \geq 0$  (quadratically majorized). Then for  $0 < \mu < \lambda < c^{-1}$  and  $0 < \mu < \lambda < c'^{-1}$  the regularizes  $(f_\lambda)^\mu$  and  $(g^\lambda)_\mu$  are  $C_b^{1,1}(\mathbb{R}^n)$  functions, whose gradient is Lipschitz continuous with constant  $\max(\mu^{-1}, (1 - \lambda c)^{-1}c)$ . In addition they converge point-wise respectively to  $f$  and  $g$  when  $\lambda, \mu \rightarrow 0$ .

Hence, we can replace the bounded and uniformly continuous assumptions by rather general growing conditions. The idea is that given a quadratically majorized function  $g$  of parameter  $c'$  the quadratic dilation  $f \oplus q_\lambda$  with  $\lambda < c'^{-1}$  produces a  $\lambda$ -weakly convex function. Then for any  $\mu < \lambda$  (strictly smaller than the dilation scale), the corresponding quadratic erosion  $(f \oplus q_\lambda) \ominus q_\mu$  produces a function belongs to the class of bounded  $C^1$ , with Lipschitz continuous gradient. Note that the key element

of this approximation is the transfer of the regularity of the quadratic kernel associated to its concavity and smoothness of  $q_\lambda$  to the function  $f$ .

Lasry–Lions regularization has been recently generalized to finite dimensional compact manifolds Bernard (2010); Bernard and Zavidovique (2013), and consequently these results can be used to show how Riemannian morphological operators are appropriate for image regularization. More precisely, let us focus on the case where  $\mathcal{M}$  is finite dimensional compact Cartan–Hadamard manifold, hence every two points can be connected by a minimizing geodesic. We remind that a Cartan–Hadamard manifold is a simply connected Riemannian manifold  $\mathcal{M}$  with sectional curvature  $K \leq 0$  (Lang, 1999). Let  $A$  be a closed convex subset of  $\mathcal{M}$ . Then the distance function to  $A$ ,  $x \mapsto d_{\mathcal{M}}(x, A)$ , where  $d_{\mathcal{M}}(x, A) = \inf \{d_{\mathcal{M}}(x, y) : y \in A\}$  is  $C^1$  smooth on  $\mathcal{M} \setminus A$  and, moreover, the square of the distance function  $x \mapsto d_{\mathcal{M}}(x, A)^2$  is  $C^1$  smooth and convex on all of  $\mathcal{M}$  (Azagra and Ferrera, 2006). Consequently, if  $\mathcal{M}$  is a Cartan–Hadamard manifold, the structuring function  $x \mapsto q(x, y)$ ,  $\forall y \in \mathcal{M}$ , is always a concave function; or equivalently,  $-q(x, y)$  is a convex function.

**Theorem 9.** Let  $\mathcal{M}$  be a compact finite dimensional Cartan–Hadamard manifold. Let  $\Omega \subset \mathcal{M}$  be a bounded set of  $\mathcal{M}$ . Given a image  $f \in \mathcal{F}(\Omega, \mathbb{R})$ , for all  $0 < \mu < \lambda$  let us define the Riemannian Lasry–Lions regularizers:

$$(f_\lambda)^\mu(x) = \sup_{z \in \mathcal{M}} \inf_{y \in \mathcal{M}} \left\{ f(y) + \frac{1}{2\lambda} d_{\mathcal{M}}(z, y)^2 - \frac{1}{2\mu} d_{\mathcal{M}}(z, x)^2 \right\}$$

$$(f^\lambda)_\mu(x) = \inf_{z \in \mathcal{M}} \sup_{y \in \mathcal{M}} \left\{ f(y) - \frac{1}{2\lambda} d_{\mathcal{M}}(z, y)^2 + \frac{1}{2\mu} d_{\mathcal{M}}(z, x)^2 \right\}$$

We have  $(f_\lambda)^\mu \leq f$  and  $(f^\lambda)_\mu \geq f$ .

- Let  $f$  be a bounded uniformly continuous image in  $\Omega$ . Then the images  $(f_\lambda)^\mu$  and  $(f^\lambda)_\mu$  belong to the class  $C_b^{1,1}(\Omega)$  and converges uniformly to  $f$  on  $\Omega$ .
- Assume that there exists  $c, c' > 0$ , such that we have the following growing conditions for semicontinuous functions  $\mathcal{F}(\Omega, \mathbb{R})$ :

$$f(x) \geq -\frac{c}{2}(1 + d(x, x_0)^2), \quad g(x) \leq \frac{c'}{2}(1 + d(x, x_0)^2), \quad x_0 \in \mathcal{M}.$$

Then, for all  $0 < \mu < \lambda < c^{-1}$  the pseudo-opened image  $(f_\lambda)^\mu$  and for all  $0 < \mu < \lambda < c'^{-1}$  the pseudo-closed image  $(g^\lambda)_\mu$  are of class  $C_b^{1,1}(\Omega)$ . In addition, they converge point-wise respectively to  $f$  and  $g$ .

We remark that this result is theoretically valid only for bounded images supported on bounded subsets on manifolds of nonpositive sectional curvature. However, in practice we observe that it works for bounded images on bounded surfaces of positive and negative curvature. By the way, one should note that our result conjectured in (Angulo and Velasco-Forero, 2013) was too general since the support space of the image should be a bounded set  $\Omega$ . As discussed in Bernard (2010) and Bernard and Zavidovique (2013), more general versions of

320 Lasry-Lions regularization can be obtained in Riemannian man-  
 321 ifolds. In particular the case of compact nonnegative curvature  
 322 manifolds is relevant for optimal transport problems (Villani,  
 323 2009).

#### 324 4. Generalized Riemannian morphological operators

325 We have discussed the canonic case on Riemannian math-  
 326 ematical morphology associated to the structuring function<sup>357</sup>  
 327  $q_\lambda(x, y)$ . Let consider now the most general family of Riemannian<sup>358</sup>  
 328 operators. We start by introducing the minimal properties<sup>359</sup>  
 329 that a Riemannian structuring function should verify.<sup>360</sup>

330 **Definition 10.** *Let  $\mathcal{M}$  be a Riemannian manifold. A mapping<sup>362</sup>  
 331  $\mathfrak{b} : \mathcal{M} \times \mathcal{M} \rightarrow \overline{\mathbb{R}}$  defined for any pair of points in  $\mathcal{M}$  is said an<sup>363</sup>  
 332 admissible Riemannian structuring function in  $\mathcal{M}$  if and only if<sup>364</sup>*

- 333 1.  $\mathfrak{b}(x, y) \leq 0, \forall x, y \in \mathcal{M}$  (non-positivity);<sup>365</sup>
- 334 2.  $\mathfrak{b}(x, x) = 0, \forall x \in \mathcal{M}$  (maximality at the diagonal).<sup>366</sup>

335 Now, we can introduce the pair of dilation and erosion for any<sup>367</sup>  
 336 image  $f$  according to  $\mathfrak{b}$ .<sup>368</sup>

337 **Definition 11.** *Given an admissible Riemannian structuring<sup>371</sup>  
 338 function  $\mathfrak{b}$  in a Riemannian manifold  $\mathcal{M}$ , the Riemannian di-<sup>372</sup>  
 339 lation and Riemannian erosion of an image  $f \in \mathcal{F}(\mathcal{M}, \overline{\mathbb{R}})$  by  $\mathfrak{b}$   
 340 are given respectively by*

$$\delta_{\mathfrak{b}}(f)(x) = \sup_{y \in \mathcal{M}} \{f(y) + \mathfrak{b}(x, y)\}, \quad (15)$$

$$\varepsilon_{\mathfrak{b}}(f)(x) = \inf_{y \in \mathcal{M}} \{f(y) - \mathfrak{b}(y, x)\}. \quad (16)$$

341 Note that this formulation has been considered recently in  
 342 the framework of adaptive morphology (Ćurić and Luengo-  
 343 Hendriks, 2013). Both are increasing operators which, by the<sup>373</sup>  
 344 maximality at the diagonal, preserves the extrema. By the non-<sup>374</sup>  
 345 positivity, Riemannian dilation is extensive and erosion is anti-<sup>375</sup>  
 346 extensive. In addition, we can easily check that the pair  $(\varepsilon_{\mathfrak{b}}, \delta_{\mathfrak{b}})$ <sup>376</sup>  
 347 forms an adjunction as in Proposition 3. Consequently, their<sup>377</sup>  
 348 composition leads to the *Riemannian opening and closing ac-*<sup>378</sup>  
 349 *cording to the admissible Riemannian structuring function  $\mathfrak{b}$* <sup>379</sup>  
 350 *given respectively by:*<sup>380</sup>

$$\gamma_{\mathfrak{b}}(f)(x) = \sup_{z \in \mathcal{M}} \inf_{y \in \mathcal{M}} \{f(y) - \mathfrak{b}(y, z) + \mathfrak{b}(z, x)\}, \quad (17)^{382}$$

$$\varphi_{\mathfrak{b}}(f)(x) = \inf_{z \in \mathcal{M}} \sup_{y \in \mathcal{M}} \{f(y) + \mathfrak{b}(z, y) - \mathfrak{b}(x, z)\}. \quad (18)$$

351 Remarkably, the symmetry of  $\mathfrak{b}$  is not a necessary condition for  
 352 the adjunction. Examples of such asymmetric structuring func-  
 353 tions have recently appeared in the context of stochastic mor-  
 354 phology (Angulo and Velasco-Forero, 2013), non-local mor-  
 355 phology (Velasco-Forero and Angulo, 2013) and saliency-based  
 356 adaptive morphology (Ćurić and Luengo-Hendriks, 2013).

In our framework, we propose a general form of any admissi-  
 ble Riemannian structuring function  $\mathfrak{b}(x, y), \forall x, y \in \mathcal{M}$ , which  
 should be decomposable into the sum of two terms:

$$\mathfrak{b}(x, y) = \alpha \mathfrak{b}^{sym}(x, y) + \beta \mathfrak{b}^{asym}(x, y), \quad \alpha, \beta \geq 0. \quad (19)$$

**Symmetric structuring function.** The symmetric term of  
 the structuring function will be a scaled p-norm shaped func-  
 tion depending exclusively on the Riemannian distance, i.e.,  
 $\mathfrak{b}^{sym}(x, y) = \mathfrak{b}^{sym}(y, x) = k_{\lambda, p}(d_{\mathcal{M}}(x, y))$  such that

$$k_{\lambda, p}(\eta) = -C_p \frac{\eta^{\frac{p}{p-1}}}{\lambda^{\frac{1}{p-1}}}; \quad \lambda > 0, \quad p > 1,$$

where the normalization factor is given by  $C_p = (p-1)p^{-\frac{p}{p-1}}$ .  
 We note that with the shape parameter  $p = 2$  we recover the  
 canonic quadratic structuring function. In fact, this general-  
 ization of the quadratic structuring is inspired from the solu-  
 tion of a generalized morphological PDE (Lions et al., 1987):  
 $u_t(t, x) + \|u_x(t, x)\|^p = 0, (t, x) \in (0, +\infty) \times E; u(0, x) = f(x),$   
 $x \in E$ , since the quadratic one is the solution of the class-  
 ical (Hamilton-Jacobi) morphological PDE (Bardi et al., 1984;  
 Crandall et al., 1992):  $u_t(t, x) + \|u_x(t, x)\|^2 = 0$ . Asymptotically,  
 one is dealing with almost flat shapes over  $\mathcal{M}$  as  $p \rightarrow 1$ ; as  
 $p > 2$  increases and  $p \rightarrow \infty$  the shape of  $k_{\lambda, p}(\eta)$  evolves from a  
 parabolic shape  $p = 2$ , i.e., term on  $d_{\mathcal{M}}(x, y)^2$ , to the limit case,  
 which is a conic shape, i.e., term on  $d_{\mathcal{M}}(x, y)$ .

We note that if  $\mathcal{M}$  is a Cartan-Hadamard manifold, the sym-  
 metric part  $\mathfrak{b}^{sym}(x, y)$  is a concave function for any  $\lambda > 0$  and  
 any  $p > 1$ .

**Asymmetric structuring function.** Relevant forms of the  
 asymmetric term is an open issue on Riemannian morphology,  
 which will probably allows to introduce more advanced mor-  
 phological operators. For instance, we can fix a reference point  
 $o \in \mathcal{M}$  and define, for  $x, y \in \mathcal{M}, y \neq o$ , the function

$$\mathfrak{b}_{\lambda, o}^{asym}(x, y) = -\frac{1}{2\lambda} \frac{d_{\mathcal{M}}(x, y)^2}{d_{\mathcal{M}}(y, o)^2}.$$

The assignment  $x \mapsto \mathfrak{b}_{\lambda, o}^{asym}(x, y)$  involves a shape strongly de-  
 formed near the reference point. One can also replace the refer-  
 ence point by a set  $O \subset \mathcal{M}$ , hence changing  $d_{\mathcal{M}}(y, o)$  by the  
 distance function  $d_{\mathcal{M}}(y, O)$ .

An alternative asymmetric function could be based on the  
 notion of Busemann function (Ballmann et al., 1985). Given  
 a point  $x \in \mathcal{M}$  and a ray  $\gamma$  starting at  $x$  in the direction of  
 the tangent vector  $\mathbf{v}$ , i.e., a unit-speed geodesic line  $\gamma : [0, \infty)$   
 $\rightarrow \mathcal{M}$  such that  $d_{\mathcal{M}}(\gamma(0), \gamma(t)) = t$  for all  $t \geq 0$ , one defines its  
 Busemann function  $b_{\gamma, \mathbf{v}}$  by the formula

$$\begin{aligned} b_{\gamma, \mathbf{v}}(y) &= \lim_{t \rightarrow \infty} [d_{\mathcal{M}}(x, \gamma_{x, \mathbf{v}}(t)) - d_{\mathcal{M}}(y, \gamma_{x, \mathbf{v}}(t))] \\ &= \lim_{t \rightarrow \infty} [t - d_{\mathcal{M}}(y, \gamma_{x, \mathbf{v}}(t))]. \end{aligned}$$

Since  $t - d_{\mathcal{M}}(y, \gamma_{x, \mathbf{v}}(t))$  is bounded above by  $d_{\mathcal{M}}(x, \gamma_{x, \mathbf{v}}(0))$  and  
 is monotone non-decreasing in  $t$ , the limit always exists. It fol-  
 lows that  $|b_{\gamma, \mathbf{v}}(y) - b_{\gamma, \mathbf{v}}(z)| \leq d_{\mathcal{M}}(y, z)$ , i.e., Busemann function  
 is Lipschitz with constant 1. If  $\mathcal{M}$  has non-negative sectional  
 curvature  $b_{\gamma, \mathbf{v}}(y)$  is convex. If  $\mathcal{M}$  is Cartan-Hadamard mani-  
 fold, it is concave. Consequently, we can define our asymmetric  
 structuring function as

$$\mathfrak{b}_{\lambda, \mathbf{v}}^{asym}(x, y) = \begin{cases} -(2\lambda)^{-1} b_{\gamma, \mathbf{v}}(y) & \text{if sect. curvature of } \mathcal{M} \geq 0 \\ (2\lambda)^{-1} b_{\gamma, \mathbf{v}}(y) & \text{if sect. curvature of } \mathcal{M} < 0 \end{cases}$$

383 From a practical viewpoint, asymmetric structuring functions<sup>403</sup>  
 384 obtained by Busemann functions allow to introduce a shape<sup>404</sup>  
 385 which depends on the distance between the point  $x$  and a kind<sup>405</sup>  
 386 of orthogonal projection of point  $y$  on the geodesic along the<sup>406</sup>  
 387 direction  $\mathbf{v}$ . Hence, it could be a way to introduce directional<sup>407</sup>  
 388 Riemannian operators.<sup>408</sup>

## 389 5. Parallel transport of a fixed external structuring func-<sup>411</sup> 390 tion

391 Previous Riemannian morphological operators are based on  
 392 geodesic structuring functions  $\mathfrak{b}(x; y)$  which are defined by the  
 393 geodesic distance function on  $\mathcal{M}$ . Let us consider now the case<sup>412</sup>  
 394 where a prior (semi-continuous) structuring function  $b$  external<sup>413</sup>  
 395 to  $\mathcal{M}$  is given and it should be adapted to each point  $x \in \mathcal{M}$ .<sup>414</sup>  
 396 Our approach is inspired from Roerdink (1994) formulation of<sup>415</sup>  
 397 dilation/erosion for binary images on smooth surfaces.

### 398 5.1. Manifold morphology

The idea behind the binary Riemannian morphology on smooth surfaces introduced in (Roerdink, 1994) is to replace the translation invariance by the parallel transport (the transformations are referred to as ‘‘covariant’’ operations). Let  $\mathcal{M}$  be a (geodesically complete) Riemannian manifold and  $\mathcal{P}(\mathcal{M})$  denotes the set of all subsets of  $\mathcal{M}$ . A binary image  $X$  on the manifold is just  $X \in \mathcal{P}(\mathcal{M})$ . Let  $A \subset \mathcal{M}$  be the basic structuring, a subset which is defined on the tangent space at a given point  $\omega$  of  $\mathcal{M}$  by  $\tilde{A} = \text{Log}_\omega(A) \subset T_\omega\mathcal{M}$ . Let  $\gamma = \gamma_{[p,q]}$  be a path from  $p$  to  $q$ , then the operator

$$\tau_\gamma(A) = \text{Exp}_q P_\gamma \text{Log}_p(A) = B,$$

transports the subset  $A$  of  $p$  to the set  $B$  of  $q$ . As the image of the set  $X$  under parallel translation from  $p$  to  $q$  will depend in general on which path is taken; the solution proposed in (Roerdink, 1994), denoted by  $\delta_A^{\text{Roerdink}}$ , is to consider all possible paths from  $p$  to  $q$ . The mapping  $\delta_A^{\text{Roerdink}} : \mathcal{P}(\mathcal{M}) \rightarrow \mathcal{P}(\mathcal{M})$  given by

$$\delta_A^{\text{Roerdink}}(X) = \bigcup_{x \in \mathcal{M}} \bigcup_{\gamma} \tau_\gamma(A) = \bigcup_{x \in \mathcal{M}} \bigcup_{\gamma} \text{Exp}_x P_{\gamma_{[\omega,x]}} \text{Log}_\omega(A), \quad (20)$$

is a dilation of image  $X$  according to the structuring element  $A$ . Using the symmetry group morphology (Roerdink, 2000), this operator can be rewritten as

$$\delta_A^{\text{Roerdink}}(X) = \bigcup_{x \in \mathcal{M}} \text{Exp}_x P_{\gamma_{[\omega,x]}} \text{Log}_\omega(\tilde{A}),$$

399 where  $\tilde{A} = \bigcup_{s \in \Sigma} sA$ , with  $\Sigma$  being the holonomy group around  
 400 the normal at  $\omega$ . For instance, if  $\tilde{A} = \text{Log}_\omega(A)$  is a line segment  
 401 of length  $r$  starting at  $\omega$  then  $\tilde{A}$  is a disk of radius  $r$  centered at  
 402  $\omega$ .

### 5.2. $b_\omega$ -transported Riemannian dilation and erosion

Coming back to our framework of real-valued images on a geodesically complete Riemannian manifold  $\mathcal{M}$ . From our viewpoint, it seems more appropriate to fix the reference structuring element as a Boolean set  $S$  on the tangent space at the reference point  $\omega \in \mathcal{M}$ , i.e.,  $S_\omega \subset T_\omega\mathcal{M}$ . More precisely, let  $S_\omega$  be a compact set which contains the origin of  $T_\omega\mathcal{M}$ . We can now formulate the  $S_\omega$ -transported flat Riemannian dilation and erosion as

$$\check{\delta}_{S_\omega}(f)(x) = \sup \{f(y) : y \in \text{Exp}_x P_{\gamma_{[\omega,x]}^{\text{geo}}} \check{S}_\omega\}, \quad (21)$$

$$\check{\epsilon}_{S_\omega}(f)(x) = \inf \{f(y) : y \in \text{Exp}_x P_{\gamma_{[\omega,x]}^{\text{geo}}} S_\omega\}. \quad (22)$$

Thus, in comparison to dilation (20), we prefer to consider in our case that the parallel transport from  $\omega$  to  $x$  is done exclusively along the geodesic path  $\gamma_{[\omega,x]}^{\text{geo}}$  between  $\omega$  and  $x$ , i.e., if  $S_\omega$  is a line in  $\omega$  then it will be also at  $x$  a line, but rotated.

This idea leads to a natural extension to the case where the fixed datum is an upper-semicontinuous structuring function  $b_\omega(\mathbf{v})$ , defined in the Euclidean tangent space at  $\omega$ , i.e.,  $b_\omega : T_\omega\mathcal{M} \rightarrow [-\infty, 0]$ . Let consider now the upper level sets (or cross-section) of  $b_\omega$  obtained by thresholding at a value  $l$ :

$$X_l(b_\omega) = \{\mathbf{v} \in T_\omega\mathcal{M} : b_\omega(\mathbf{v}) \geq l\}, \quad \forall l \in [-\infty, 0]. \quad (23)$$

The set of upper level sets constitutes a family of decreasing closed sets:  $l \geq m \Rightarrow X_l \subseteq X_m$  and  $X_l = \bigcap \{X_m, m < l\}$ . Any function  $b_\omega(\mathbf{v})$  can be now viewed as a unique stack of its cross-sections, which leads to the following reconstruction property:

$$b_\omega(\mathbf{v}) = \sup \{l \in [-\infty, 0] : \mathbf{v} \in X_l(b_\omega)\}, \quad \forall \mathbf{v} \in T_\omega\mathcal{M}. \quad (24)$$

Using this representation, the corresponding Riemannian structuring function at  $\omega$  is given by  $\mathfrak{b}_\omega(\omega, y) = \sup \{l \in [-\infty, 0] : z \in \text{Exp}_\omega X_l(b_\omega)\}$ . In the case of a different point  $x \in \mathcal{M}$ , the cross-section should be transported to the tangent space of  $x$  before mapping back to  $\mathcal{M}$ , i.e.,

$$\mathfrak{b}_\omega(x, y) = \sup \{l \in [-\infty, 0] : z \in \text{Exp}_x P_{\gamma_{[\omega,x]}^{\text{geo}}} X_l(b_\omega)\}.$$

416 Finally, the  $b_\omega$ -transported Riemannian dilation and erosion of  
 417 image  $f$  are given respectively by

$$\check{\delta}_{b_\omega}(f)(x) = \sup_{y \in \mathcal{M}} \{f(y) + \mathfrak{b}_\omega(x, y)\}, \quad (25)$$

$$\check{\epsilon}_{b_\omega}(f)(x) = \inf_{y \in \mathcal{M}} \{f(y) - \mathfrak{b}_\omega(y, x)\}. \quad (26)$$

Obviously, the case of a concave structuring function  $b_\omega$  is particularly well defined since in such a case, its cross-sections are convex sets. In addition, if  $\mathcal{M}$  is a Cartan–Hadamard manifold, the corresponding Riemannian structuring function  $b_\omega(x, y)$  is also a concave function.

A typical useful case consists in taking at reference  $\omega$  the structuring function:

$$b_\omega(\mathbf{v}) = -\frac{\mathbf{v}^T Q \mathbf{v}}{2}$$



423 where  $Q$  is a  $d \times d$  symmetric positive definite matrix,  $d$  being  
 424 the dimension of manifold  $\mathcal{M}$ . It corresponds just to a general-  
 425 ized quadratic function such that the eigenvectors of  $Q$  define  
 426 the principal directions of the concentric ellipsoids and the  
 427 eigenvalues their eccentricity. Therefore, we can introduce by  
 428 means of  $Q$  an anisotropic/directional shape on  $b_\omega(x, y)$ . We  
 429 can easily check that  $Q = \frac{1}{\lambda}I$ ,  $I$  being the identity matrix of di-  
 430 mension  $d$ , corresponds just to the canonic Riemannian dilation  
 431 and erosion (1) and (2).

432 Without an explicit expression of the exponential map, we  
 433 cannot compute straightforwardly the  $b_\omega$ -transported Riemann-  
 434 nian dilation and erosion on a Riemannian manifold  $\mathcal{M}$ . This is  
 435 for instance the situation when  $f$  is an image on a 3D smooth  
 436 surface. Hence, in the case of applications to valued surfaces,  
 437 manifold learning techniques as LOGMAP (Brun et al., 2005)  
 438 can be used to numerically obtain the transported cross-sections  
 439 on  $\mathcal{M}$ .

## 440 6. Connections with classical Euclidean morphology

### 441 6.1. Spatially-invariant operators

442 First of all, it is obvious that the Riemannian dilation/erosion  
 443 naturally extends the quadratic Euclidean dilation/erosion for  
 444 images  $\mathcal{F}(\mathbb{R}^d, \mathbb{R})$  by considering that the intrinsic distance is  
 445 the Euclidean one (or the discrete one for  $\mathbb{Z}^d$ ), i.e.,  $d_{\mathcal{M}}(x, y) =$   
 446  $\|x - y\| = d_{space}(x, y)$ .

447 By the way, we note also that definition of the Riemannian  
 448 flat dilation and erosion of size  $r$  given in (8) and (9) are com-  
 449 patible with the formulation of the classical geodesic dilation  
 450 and erosion (Lantuejoul and Beucher, 1981) of size  $r$  of im-  
 451 age  $f$  (marker) constrained by the image  $g$  (reference or mask),  
 452  $\delta_{g,\lambda}(f)$  and  $\varepsilon_{g,\lambda}(f)$ , which underly the operators by recon-  
 453 struction (Soille, 1999), where the upper-level sets of the reference  
 454 image  $g$  are considered as the manifold  $\mathcal{M}$  where the geodesic  
 455 distance is defined.

### 456 6.2. Adaptive (spatially-variant) operators

457 From (Kimmel et al., 1997), the idea of embedding a 2D  
 458 grey-level image  $f \in \mathcal{F}(\mathbb{R}^2, \mathbb{R})$ ,  $x = (x_1, x_2)$ , into a surface  
 459 embedded in  $\mathbb{R}^3$ , i.e.,

$$f(x) \mapsto \xi_x = (x_1, x_2, \alpha f(x_1, x_2)), \alpha > 0,$$

457 where  $\alpha$  is a scaling parameter useful for controlling intensity  
 458 distances, has become popular in differential geometry inspired  
 459 image processing. This embedded Riemannian manifold  $\mathcal{M} =$   
 460  $\mathbb{R}^2 \times \mathbb{R}$  has a product metric of type  $ds_{\mathcal{M}}^2 = ds_{space}^2 + \alpha ds_f^2$ , where  
 461  $ds_{space}^2 = dx_1^2 + dx_2^2$  and  $ds_f^2 = df^2$ . The geodesic distance  
 462 between two points  $\xi_x, \xi_y \in \mathcal{M}$  is the length of the shortest path  
 463 between the points, i.e.,  $d_{\mathcal{M}}(\xi_x, \xi_y) = \min_{\gamma \in \Gamma(\xi_x, \xi_y)} \int_{\gamma} ds_{\mathcal{M}}$ .

464 As shown in (Welk et al., 2011), this is essentially the frame-  
 465 work behind the morphological amoebas (Lerallut et al., 2007),  
 466 which are flat spatially adaptive structuring functions centered  
 467 in a point  $x$ ,  $A_\lambda(x)$ , computed by thresholding the geodesic dis-  
 468 tance at radius  $\lambda > 0$ , i.e.,  $A_\lambda(x) = \{y \in E : d_{\mathcal{M}}(\xi_x, \xi_y) < \lambda\}$ . In

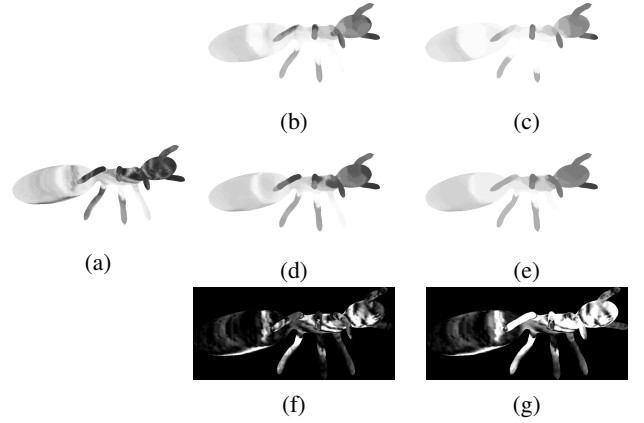


Figure 2: Morphological processing of real valued 3D surface: (a) original image on a surface  $S \subset \mathbb{R}^3$ ,  $f(x) \in \mathcal{F}(S, \mathbb{R}_+)$ ; (b) and (c) Riemannian dilation  $\delta_\lambda(f)(x)$  with respectively  $\lambda = 4$  and  $\lambda = 8$ ; (d) and (e) Riemannian closing  $\varphi_\lambda(f)(x)$  with respectively  $\lambda = 4$  and  $\lambda = 8$ ; (f) and (g) residue between the original surface and the Riemannian closings  $\varphi_\lambda(f)(x) - f(x)$ ,  $\lambda = 4$  and  $\lambda = 8$ .

441 the discrete setting, the geodesic distance is given by

$$d_{\mathcal{M}}(\xi_x, \xi_y) = \min_{\{\xi^1 = \xi_x, \xi^2, \dots, \xi^N = \xi_y\}} \sum_{i=1}^N \alpha |f(x^i) - f(x^{i+1})| + \sqrt{(x_1^i - x_1^{i+1})^2 + (x_2^i - x_2^{i+1})^2}. \quad (27)$$

442 We should remark that for  $x \rightarrow y$  and assuming a smooth man-  
 443 ifold, the geodesic distance is asymptotically equivalent to the  
 444 corresponding distance in the Euclidean product space, i.e.,

$$d_{\mathcal{M}}(\xi_x, \xi_y)^2 \approx d_{space}(x, y)^2 + \alpha^2 |f(x) - f(y)|^2, \quad (28)$$

445 which is the distance appearing in the bilateral structuring func-  
 446 tions (Angulo, 2013). We can also see that the salience maps  
 447 behind the salience adaptive structuring elements (Ćurčić et al., 2012)  
 448 can be approached in a Riemannian formulation by  
 449 choosing the appropriate metric.

## 456 7. Various useful case studies

### 457 7.1. Hyperbolic embedding of an Euclidean positive image into Poincaré half-space $\mathcal{H}^3$

458 Shortest path distance (27) is not invariant to scaling of  
 459 image intensity, i.e.,  $f \mapsto f' = \beta f$ ,  $\beta > 0$  involves that  
 460  $|f'(x^i) - f'(x^{i+1})| = \beta |f(x^i) - f(x^{i+1})|$  and hence the shape of  
 461 the corresponding Riemannian structuring function for  $f$  and  $f'$   
 462 will be different. This lack of contrast invariance can be easily  
 463 solved by using a logarithmic metric in the intensities. Hence, if  
 464 we assume positive intensities,  $f(x) > 0$ , for all  $x \in \mathcal{M}$ , we can  
 465 consider the distance  $d_{\mathcal{M}}(\xi_x, \xi_y) = \min_{\gamma \in \Gamma(\xi_x, \xi_y)} \sum_{i=1}^N d_{space}(x^i, x^{i+1}) +$   
 466  $\alpha |\log f(x^i) - \log f(x^{i+1})|$ . This metric can be connected to the  
 467 logarithmic image processing (LIP) model (Joulin and Pinoli,  
 468 1988). This geometry can be also justified from a human per-  
 469 ception viewpoint. The classical Weber-Fechner law states that  
 470 human sensation is proportional to the logarithm of the stim-  
 471 ulus intensity. In the case of vision, the eye senses brightness

492 approximately according to the Weber-Fechner law over a moderate range.  
493

Following the same assumption of positive intensities, we can also consider that a 2D image can be embedded into the hyperbolic space  $\mathcal{H}^3$  (Cannon et al., 1997). More particularly the (Poincaré) upper half-space model is the domain  $\mathcal{H}^3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\}$  with the Riemannian metric  $ds_{\mathcal{H}^3}^2 = \frac{dx_1^2 + dx_2^2 + dx_3^2}{x_3^2}$ . This space has constant negative sectional curvature. If we consider the image embedding  $f(x) \mapsto \xi_x = (x_1, x_2, f(x_1, x_2)) \in \mathcal{H}^3$ , the Riemannian distance needed for morphological operators will be given by

$$d_M(\xi_x, \xi_y) = \min_{\gamma_{\xi_x, \xi_y}} \sum_{i=1}^N \cosh^{-1} \left( 1 + \frac{(x_1^i - x_1^{i+1})^2 + (x_2^i - x_2^{i+1})^2 + (f(x^i) - f(x^{i+1}))^2}{2f(x^i)f(x^{i+1})} \right). \quad (29)$$

494 The geometry of this space is extremely rich in particular concerning the invariance and isometric symmetry. Hence, distance (29) is for instance invariant to translations  $\xi \mapsto \xi + \alpha$ ,  $\alpha \in \mathbb{R}$ , scaling  $\xi \mapsto \beta\xi$ ,  $\beta > 0$ . A specific theory on granulometric scale-space properties in this embedding can be intended.  
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## 500 7.2. Embedding an Euclidean image into the structure tensor manifold

501 Besides the space×intensity embeddings discussed just above, we can consider other more alternative non-Euclidean geometric embedding of scalar images, using for instance the local structure.  
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More precisely, given a 2D Euclidean image  $f(x) = f(x_1, x_2) \in \mathcal{F}(\mathbb{R}^2, \mathbb{R})$ , the structure tensor representing the local orientation and edge information (Förstner and Gülch, 1987) is obtained by Gaussian smoothing of the dyadic product  $\nabla f \nabla f^T$ :

$$S(f)(x) = G_\sigma * (\nabla f(x_1, x_2) \nabla f(x_1, x_2)^T) = \begin{pmatrix} s_{x_1 x_1}(x_1, x_2) & s_{x_1 x_2}(x_1, x_2) \\ s_{x_1 x_2}(x_1, x_2) & s_{x_2 x_2}(x_1, x_2) \end{pmatrix} \quad 528$$

506 where  $\nabla f(x_1, x_2) = \left( \frac{\partial f(x_1, x_2)}{\partial x_1}, \frac{\partial f(x_1, x_2)}{\partial x_2} \right)^T$  is the 2D spatial intensity gradient and  $G_\sigma$  stands for a Gaussian smoothing with  
507 a standard deviation of  $\sigma$ . From a mathematical viewpoint,  
508  $S(f)(x) : E \rightarrow \text{SPD}(2)$  is an image where at each pixel  
509 we have a symmetric positive (semi-)definite matrix  $2 \times 2$ .  
510 The differential geometry in the manifold  $\text{SPD}(n)$  is very  
511 well-known (Bhatia, 2007). Namely, the metric is given  
512 by  $ds_{\text{SPD}(n)}^2 = \text{tr}(M^{-1} dM M^{-1} dM)$  and the Riemannian dis-  
513 tance is defined as  $d_{\text{SPD}(n)}(M_1, M_2) = \|\log(M_1^{-1/2} M_2 M_1^{-1/2})\|_F$ ,  
514  $\forall M_1, M_2 \in \text{SPD}(n)$ . Let consider now the embedding  $f(x) \mapsto$   
515  $\xi_x = (x_1, x_2, \alpha S(f)(x_1, x_2))$ ,  $\alpha > 0$ , in the product manifold  
516  $\mathcal{M} = \mathbb{R}^2 \times \text{SPD}(2)$ , which has the product metric  $ds_{\mathcal{M}}^2 =$   
517  $ds_{\text{space}}^2 + \alpha ds_{\text{SPD}(2)}^2$ . It is a (complete, not compact, nega-  
518 tive sectional curved) Riemannian manifold of geodesic dis-  
519 tance given by  $d_M(\xi_x, \xi_y) = \min_{\gamma_{\xi_x, \xi_y}} \sum_{i=1}^N d_{\text{space}}(x^i, x^{i+1}) + \alpha$   
520  $d_{\text{SPD}(n)}(S(f)(x^i), S(f)(x^{i+1}))$ , which is asymptotically equal to  
521  $d_M(\xi_x, \xi_y)^2 \approx d_{\text{space}}(x, y)^2 + \alpha d_{\text{SPD}(2)}(S(f)(x), S(f)(y))^2$ .  
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By means of this embedding, we can compute anisotropic  
524 morphological operators following the flow coherence of im-  
525 age structures. This embedding is related to previous adaptive  
526 approaches such as (Verdú et al., 2011) and (Landström and  
527 Thurley, 2013).  
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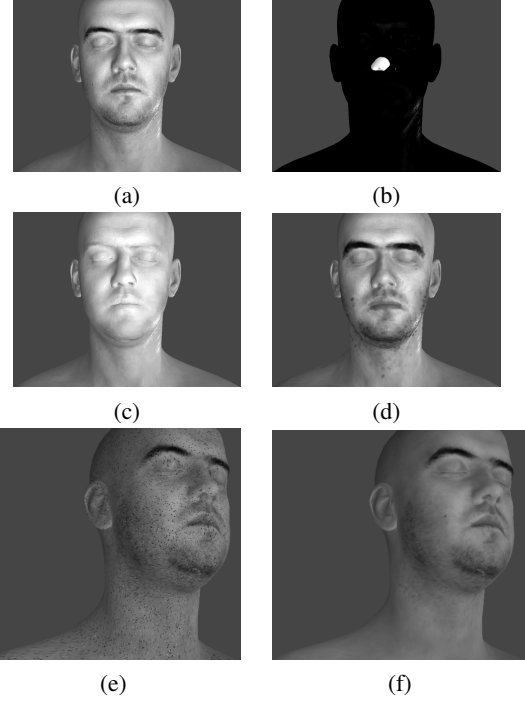


Figure 3: Morphological processing of real valued 3D surface of a face: (a) original image on a surface  $S \subset \mathbb{R}^3$ ,  $f(x) \in \mathcal{F}(S, \mathbb{R}_+)$ ; (b) example of geodesic ball  $B_r(x)$  at a given point  $x \in S$ ; (d) and (e) Riemannian dilation  $\delta_\lambda(f)(x)$  and Riemannian erosion  $\varepsilon_\lambda(f)(x)$  with  $\lambda = 0.5$ ; (e) nonsmooth version of surface (added impulse noise); (f) filtered surface obtained by Lasry–Lions regularizers.

## 528 7.3. Morphological processing of real valued 3D surfaces

In Fig. 2(a) is given an example of real-valued 3D surface, i.e., the image to be processed is  $f : S \subset \mathbb{R}^3 \rightarrow \mathbb{R}$ . In practice, the 3D surface is represented by a mesh (i.e., triangulated manifold with a discrete structure composed of vertices, edges and faces). In our example, the grey-level intensities are supported on the vertices. In the case of a discrete approximation of a manifold based on mesh representation, the geodesic distance  $d_S(x, y)$  can be calculated by the Floyd–Warshall algorithm for finding shortest path in the weighted graph of vertices of the mesh. Efficient algorithms are based on Fast Marching generalized to arbitrary triangulations Kimmel and Sethian (1998). Fig. 2 depicts examples of Riemannian dilation  $\delta_\lambda(f)$  and Riemannian closing  $\varphi_\lambda(f)$ , for two different scales ( $\lambda = 4$  and  $\lambda = 8$ ) and the corresponding dual top-hats.

Another example of real valued surface is given in Fig. 3. It corresponds to the 3D acquisition of a face. We observe how the canonic Riemannian dilation and erosion are able to locally process the face details taking into the geometry of the surface. In Fig. 3 is also given an example of image filtering using the composition our Lasry–Lions regularizers (15) (with  $\lambda = 4$  and  $\mu = 2$ ), where the original surface is a nonsmooth version obtained by adding impulse noise.

## 8. Conclusions

We have introduced in this paper a general theory for the formulation of mathematical morphology operators for images valued on Riemannian manifolds. We have defined the main operators and studied their fundamental properties. We have considered two main families of operators. On the one hand, morphological operators based on an admissible Riemannian structuring function which is adaptively obtained for each point  $x$  according to the geometry of the manifold. On the other hand, morphological operators founded on an external Euclidean structuring function which is parallel transported to the tangent space at each point  $x$  and then mapped to the manifold. We have also discussed some original Riemannian embeddings of Euclidean images onto Cartan–Hadamard manifolds. This is the case of the Poincaré half-space  $\mathcal{H}^3$  as well as the structure tensor manifold. Riemannian structuring functions defined on Cartan–Hadamard manifolds are particular rich in terms of scale-space properties as well as in Lipschitz regularization.

**Acknowledgment.** The authors would like to thank the anonymous reviewer who pointed out the problem of the result on Lasry-Lions regularization for the case of unbounded support space or unbounded functions.

**Remark on related work.** In the last stages of writing this paper, we learned of the work (Azagra and Ferrera, 2014) where it is provided a complete analysis of the generalization of Lasry-Lions regularization for bounded functions in manifolds of bounded sectional curvature.

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