



HAL
open science

Generalised Morphological Image Diffusion

Jesus Angulo

► **To cite this version:**

| Jesus Angulo. Generalised Morphological Image Diffusion. 2011. hal-00789162v1

HAL Id: hal-00789162

<https://minesparis-psl.hal.science/hal-00789162v1>

Preprint submitted on 16 Feb 2013 (v1), last revised 17 Jan 2016 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Generalised Morphological Image Diffusion

Jesús Angulo

CMM-Centre de Morphologie Mathématique,

Mathématiques et Systèmes, MINES ParisTech;

35, rue Saint-Honoré, 77305 Fontainebleau cedex - France

`jesus.angulo@mines-paristech.fr`

June 2011 *

Abstract

Relationships between linear and morphological scale-spaces have been considered by various previous works. The aim of this paper is to study how to generalise the diffusion-based approaches in order to introduce nonlinear filters whose effects mimic morphological dilation and erosion. A methodology based on the counter-harmonic mean is adopted here. Details of numerical implementation are discussed and results are provided to illustrate the behaviour of various studied cases: isotropic, nonlinear and coherence-enhancing diffusion. We also rediscover the classical link between Gaussian scale-space and dilation/erosion scale-spaces based on quadratic structuring functions.

Keywords: counter-harmonic mean, mathematical morphology, quadratic structuring function, image diffusion, nonlinear scale-space theory

*Submitted to CVIU. The first draft of this study was started in June 2010.

1 Introduction

Two fundamental paradigms of image filtering appear distinguished nowadays in the state-of-the-art. On the one hand, differential methods inspired from the (parabolic) heat equation, including isotropic diffusion [22], nonlinear diffusion [27, 11], anisotropic diffusion [35], etc. The main properties of these techniques are the appropriateness to deal with the notion of scale-space of image structures and the ability to process symmetrically the bright/dark image structures. Practical algorithms involve (local-adaptive) kernel convolution as well as PDE-formulation and subsequent numerical solutions. The interested reader should refer to basic references [13] and [34]. On the other hand, mathematical morphology operators [30, 32] which are formulated in terms of geometric notions as well as in terms of complete lattice theory. Morphological filters entail mainly the computation of supremum and infimum values in neighbourhoods (or structuring elements) which correspond respectively to the dilation and the erosion, the two basic operators. Morphological operators present also good scale-space properties [19, 4] but, by the natural duality of complete lattices, most operators appear by pairs and one acts on bright structures and the other one on dark structures. This latter property of asymmetry is in fact an advantage which allows defining evolved operators by product of a pair of dual ones. For instance, the opening (resp. closing) is obtained by the product of an erosion (resp. dilation) followed by a dilation (resp. erosion), then the product of openings and closings leads to the alternate filters and other families of morphological filters [30, 32]. Diffusion involves blurring image structures whereas morphological dilation and erosion involve enhancement of image structure transitions. In fact, morphological operators are related to geometric optics models and in particular to the (hyperbolic) eikonal

equation. Hence, there exists also a well motivated formulation of morphological operators using PDEs [1, 3, 8, 24]. This differential or continuous-scale morphology can be solved using numerical algorithms for curve evolution [29]. Thus multiscale flat dilation/erosion by disks as structuring elements (resp. unflat dilation/erosion by parabolic structuring functions) can be modelled in a continuous framework. Morphological operators using geometric structuring elements are today one of the most successful areas in image processing. However, it is obvious that the soundness and maturity of numerical methods to implement the different versions of image diffusion constitute an advantage against continuous morphology implementation, which requires more specific numerical schemas to achieve robust results [5, 7].

The aim of this paper is to study how to generalise the diffusion-based approaches in order to introduce nonlinear filters whose effects mimic morphological dilation and erosion. Or using other words, our goal is to propose a new approach of generalised morphological image diffusion based on standard numerical implementations.

Related work. Theoretical investigation of the relation between the scale-space concepts of linear and morphological scale spaces has been carried out by various studies. A parallelism between the role which plays the Fourier transform in the convolution with Gaussian kernels and the dilation using quadratic structuring function has been established using the notion of slope transform. More generally, the slope transform and its applications to morphological systems was developed independently and simultaneously by [12] and [23]. In [10] it was shown that the slope transform in the $(\max, +)$ -algebra corresponds to the logarithmic multivariate Laplace transform in the $(+, \cdot)$ -algebra; and that the Cramer transform as the Legendre-Fenchel transform of the logarithmic Laplace transform. It was studied in [14]

the structural similarities between linear and morphological processes in order to construct a one-parameter semi-linear process that incorporates Gaussian scale-space, and both types of morphological scale-spaces by quadratic structuring elements as limiting processes of a one-parameter transformation of grey-level values. More recently, it was proposed in [37] a morphological scale-space by deforming the algebraic operations related to Minkowski (or L^p) norms and generalised means, where the Gaussian scale-space is a limit case. We adopted here a different methodology in order to link diffusion-based image filtering and morphological image filtering. The starting point of our approach is the notion of counter-harmonic mean [9]. In fact, the idea of using the counter-harmonic mean for constructing robust morphological-like operators, without the notions of supremum and infimum, was proposed in [33]. Our purpose in this paper is to go further and to exploit the counter-harmonic mean to propose a more general framework which can be exploited for the various algorithms of image diffusion.

Paper organisation. This paper is an extended and improved version of a conference contribution [2]. In particular, in the present manuscript, a complete proof of the different results and properties is given, as well as a more illustrative examples are included.

The outline of the paper is as follows. In the next section we review the notion of counter-harmonic mean. The appropriateness of counter-harmonic mean to approximate flat dilation/erosion is considered in Section 3. Section 4 introduces the novel counter-harmonic Gaussian scale-space, or counter-harmonic isotropic diffusion. The limit relationships with parabolic dilation/erosion as well as other theoretical properties are also considered in Section 4. Section 5 extends these investigations to the nonlinear diffusion, in particular to the

Perona and Malik model and to the Weickert model of coherence-enhanced diffusion. The paper is concluded with a summary and perspectives in Section 6.

2 Counter-Harmonic Mean (CHM)

Let us start by presenting the basic notion of this paper.

Definition 1 Let $\mathbf{a} = (a_1, a_2, \dots, a_n)$ and $\mathbf{w} = (w_1, w_2, \dots, w_n)$ be real n -tuples, i.e., $\mathbf{a}, \mathbf{w} \in \mathbb{R}^n$. If $r \in \overline{\mathbb{R}}$ then the r -th counter-harmonic mean (CHM) of \mathbf{a} with weight \mathbf{w} is given by [9]

$$\mathfrak{K}^{[r]}(\mathbf{a}; \mathbf{w}) = \begin{cases} \frac{\sum_{i=1}^n w_i a_i^r}{\sum_{i=1}^n w_i a_i^{r-1}} & \text{if } r \in \mathbb{R} \\ \max(a_i) & \text{if } r = +\infty \\ \min(a_i) & \text{if } r = -\infty \end{cases} \quad (1)$$

It will be denoted $\mathfrak{K}^{[r]}(\mathbf{a})$ the equal weight case. We notice that $\mathfrak{K}^{[1]}(\mathbf{a}; \mathbf{w})$ is the weighted arithmetic mean and $\mathfrak{K}^{[0]}(\mathbf{a}; \mathbf{w})$ is the weighted harmonic mean.

The following two properties are useful for the rest of the paper.

Proposition 2 If $1 \leq r \leq +\infty$ then $\mathfrak{K}^{[r]}(\mathbf{a}; \mathbf{w}) \geq \mathfrak{M}^{[r]}(\mathbf{a}; \mathbf{w})$; and if $-\infty \leq r \leq 1$ then the following stronger results holds: $\mathfrak{K}^{[r]}(\mathbf{a}; \mathbf{w}) \leq \mathfrak{M}^{[r-1]}(\mathbf{a}; \mathbf{w})$; where $\mathfrak{M}^{[r]}(\mathbf{a}; \mathbf{w}) = \left(\frac{1}{W} \sum_{i=1}^n w_i a_i^r\right)^{1/r}$ is the r -th power-mean, or Minkowski weighted mean of order r , defined for $r \in \mathbb{R}^*$. Inequalities are strict unless $r = 1, +\infty, -\infty$ or \mathbf{a} is constant.

Proposition 3 If \mathbf{a} and \mathbf{w} are n -tuples and if $-\infty \leq r \leq s \leq +\infty$ then $\mathfrak{K}^{[r]}(\mathbf{a}; \mathbf{w}) \leq \mathfrak{K}^{[s]}(\mathbf{a}; \mathbf{w})$, with equality if and only if \mathbf{a} is constant.

Proofs of the results given above as well as other properties can be found in [9].

3 Robust Pseudo-Morphological Operators using CHM

The CHM has been considered in the state-of-the-art of image processing as an appropriate filter to deal with salt and pepper noise [15]. More precisely, let $v = f(x, y)$ be a grey-level image: $f : \Omega \rightarrow \mathcal{V}$. Typically, for digital 2D images, $(x, y) \in \Omega$ where $\Omega \subset \mathbb{Z}^2$ is the discrete support of the image. The pixel values are $v \in \mathcal{V} \subset \mathbb{Z}$ or \mathbb{R} , but for the sake of simplicity of our study, we consider that $\mathcal{V} = [0, 1]$.

Definition 4 *The CHM filter is obtained as*

$$\kappa_B^P(f)(x, y) = \frac{\sum_{(s,t) \in B(x,y)} f(s, t)^{P+1}}{\sum_{(s,t) \in B(x,y)} f(s, t)^P} = \mathfrak{R}^{[P+1]}(\{f(s, t)\}_{(s,t) \in B(x,y)}) \quad (2)$$

where $B(x, y)$ is the window of the filter, centered at point (x, y) , i.e., the structuring element in the case of morphological operators.

This filter is well suited for reducing the effect of pepper noise for $P > 0$ and of salt noise for $P < 0$. In the pioneering paper [33], starting from the natural observation that morphological dilation and erosion are the limit cases of the CHM, i.e.,

$$\lim_{P \rightarrow +\infty} \kappa_B^P(f)(x, y) = \max_{(s,t) \in B(x,y)} (f(s, t)) = \delta_B(f)(x, y) \quad (3)$$

and

$$\lim_{P \rightarrow -\infty} \kappa_B^P(f)(x, y) = \min_{(s,t) \in B(x,y)} (f(s, t)) = \varepsilon_B(f)(x, y); \quad (4)$$

it was proposed to use the CHM to calculate robust nonlinear operators which approach the morphological ones but without using max and min operators. In addition, these operators

are more robust to outliers (i.e., to noise) and consequently they can be considered as an alternative to rank-based filters in the implementation of pseudo-morphological operators.

3.1 Comparison with Minkowski Power Means

It is easy to see that for $P \gg 0$ ($P \ll 0$) the pixels with largest (smallest) values in the local neighbourhood B will dominate the result of the weighted sum. Of course, in practice, the range of P is limited due to the precision in the computation of the floating point operations. Proposition 2 of the previous Section justifies theoretically the suitability of CHM with respect to the alternative approach by high-order Minkowski mean, as considered by Welk [37]. Let us illustrate empirically how both means converge to the supremum (resp. infimum) when positive P increases (negative P decreases). Fig. 1 depicts convergence with respect to the value of P for the erosion (in blue) and dilation (in red), using Minkowski mean in (a) and using CHM in (b). The convergence is measured as the average difference value between the CHM for each P and the exact dilation/erosion obtained by max and min. The curves correspond to the image results provided on the website ¹, which includes other comparative examples of the methods discussed in the present paper. The practical advantage of CHM to approach morphological operators is obvious: even for $P = 100$ (or $P = -100$) the dilation (resp. erosion) is not reached for Minkowski mean whereas the error in the results for CHM is already negligible for $P = 20$ (resp. $P = -20$). We notice also in the empirical curves, as expected from Proposition 2, that the convergence to the erosion with $P \ll 0$ is faster than

¹<http://cmm.enscm.fr/~angulo/research/>

to the dilation with equivalent $P \gg 0$, i.e., for $P > 0$

$$|\kappa_B^P(f)(x, y) - \delta_B(f)(x, y)| \geq |\kappa_B^{-P}(f)(x, y) - \varepsilon_B(f)(x, y)|, \quad \forall (x, y) \in \Omega$$

3.2 Properties

The asymmetry involves that $\kappa_B^P(f)$ and $\kappa_B^{-P}(f)$ are not dual operators with respect to the complement, i.e., for $P > 0$

$$\kappa_B^P(f) \neq \mathbb{C}\kappa_B^{-P}(\mathbb{C}f)$$

with $\mathbb{C}f = 1 - f$. In Fig. 1(c) are also given rates of convergence of CHM to the opening and closing using the product of $\kappa_B^P(f)$ and $\kappa_B^{-P}(f)$: results are extremely good already for values $P = 10$. As it was already pointed out in [33], the fundamental drawback of $\kappa_B^P(f)$ (resp. $\kappa_B^{-P}(f)$) is the fact that $f(x, y) \not\leq \kappa_B^P(f)(x, y)$ with $P > 0$ (resp. $f(x, y) \not\geq \kappa_B^{-P}(f)(x, y)$ with $P < 0$). Or in other words, the extensivity (resp. anti-extensivity) for $P > 0$ (resp. $P < 0$) is not guaranteed. However, according to proposition 3, the following ordering relationship holds for $P > 0$:

$$\kappa_B^{-P}(f)(x, y) \leq \kappa_B^P(f)(x, y).$$

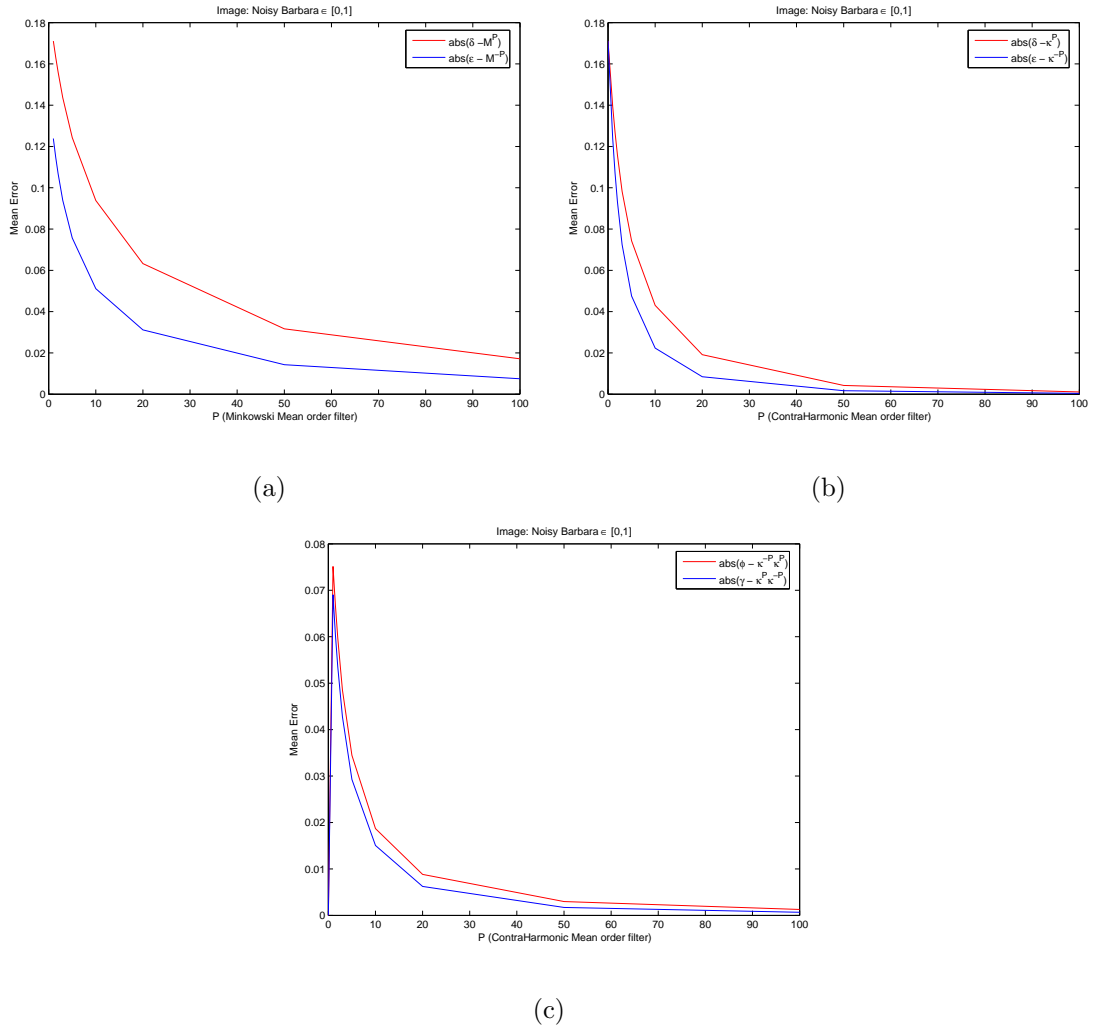


Figure 1: Convergence with respect to P of nonlinear power means-based operators to morphological operators: (a) pseudo-dilation and pseudo-erosion using Minkowski mean, (b) pseudo-dilation and pseudo-erosion using Counter-Harmonic mean, (c) pseudo-opening and pseudo-closing using Counter-Harmonic mean.

4 Counter-Harmonic Gaussian Scale-Space

Canonic multiscale image analysis involves obtaining the multiscale linear convolutions of the original image:

$$\psi(f)(x, y; t) = (f * K_\sigma)(x, y) = \int_{\Omega} f(u, v) K_\sigma(x - u, y - v) dudv, \quad (5)$$

where K_σ is the two-dimensional Gaussian function $K_\sigma(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(\frac{-(x^2+y^2)}{2\sigma^2}\right)$, whose variance (or width) σ^2 is proportional to the scale t ; i.e., $\sigma^2 = 2t$. Larger values of t lead to simpler image representations.

We can define according to our paradigm the following generalised morphological scale-space.

Definition 5 *The counter-harmonic Gaussian scale-space of order P is defined as*

$$\eta(f)(x, y; t; P) = \frac{(f^{P+1} * K_{\sqrt{2t}})(x, y)}{(f^P * K_{\sqrt{2t}})(x, y)} = \frac{\int_E f(u, v)^{P+1} K_{\sqrt{2t}}(x - u, y - v) dudv}{\int_E f(u, v)^P K_{\sqrt{2t}}(x - u, y - v) dudv}. \quad (6)$$

By choosing $P > 0$ (resp. $P < 0$), $\eta(f)(x, y; t; P)$ leads to a scale-space of pseudo-dilations (resp. pseudo-erosions), whose filtering effects for a given scale t depend on the “nonlinearity order” of P , which skew the Gaussian weighted values towards the supremum or infimum value.

Fig. 2 depicts a comparative example of image filtering using the CHM Gaussian scale-space for a fixed t and variable P . The terminology of “pseudo-dilation”, or more generally “pseudo-morphological operators” is probably inappropriate: so many operators have been published as pseudo-morphological ones, without relation with the present approach. However, due to the fact that the nonlinear filter $\eta(f)(x, y; t; P)$ for $P > 0$ (resp. $P < 0$) is not

extensive (resp. anti-extensive) and does not commute with the supremum (resp. with the infimum), it cannot be considered *stricto sensu* as a dilation (resp. erosion). We notice also that, for a fixed value of the nonlinear parameter P , the associated CHM Gaussian scale-space for a variable scale t leads to families of multi-scale pseudo-dilations/-erosions.

4.1 Limit statements

We know that $\eta(f)(x, y; t; P = +\infty) = \delta_B(f)(x, y)$ and $\eta(f)(x, y; t; P = -\infty) = \varepsilon_B(f)(x, y)$, i.e., flat dilation and flat erosion, where B is the square support of the kernel $\sigma = \sqrt{2t}$. Let us consider in detail the limit cases for $P \uparrow$ and $P \downarrow$.

Proposition 6 *For a given scale parameter t , the limits of $\eta(f)(x, y; t; P)$ with respect to P exist and are given by*

$$\lim_{P \rightarrow +\infty} \eta(f)(x, y; t; P) = \sup_{(u,v) \in \Omega} \left(f(x-u, y-v) - \frac{(u^2 + v^2)}{2P(2t)} \right), \quad (7)$$

$$\lim_{P \rightarrow -\infty} \eta(f)(x, y; t; P) = \inf_{(u,v) \in \Omega} \left(f(x-u, y-v) + \frac{(u^2 + v^2)}{2P(2t)} \right), \quad (8)$$

which can be interpreted respectively as the dilation and the erosion of $f(x, y)$ with a quadratic structuring function

$$b_{\sqrt{2t}}(x, y; P) = \frac{(x^2 + y^2)}{2P(2t)},$$

i.e., the numerical dilation and erosion [30, 32] are defined by

$$(f \oplus b_{\sqrt{2t}})(x, y) = \sup_{(u,v) \in \Omega} \left(f(x-u, y-v) - b_{\sqrt{2t}}(u, v) \right),$$

$$(f \ominus b_{\sqrt{2t}})(x, y) = \inf_{(u,v) \in \Omega} \left(f(x-u, y-v) + b_{\sqrt{2t}}(u, v) \right).$$

Note that in these limiting cases the CHM framework involves a “normalization” by P of the original Gaussian kernel scale parameter during unlinearization, i.e., the nonlinear asymptotic scale parameter is $\tilde{\sigma} = \sqrt{P}\sigma = \sqrt{P(2t)}$. This result is perfectly coherent with those obtained from totally different paradigms [37, 14]. Also, we notice again that for $P = +\infty$ the structuring function becomes flat and hence we obtain the flat dilation.

Proof. We only give the proof for the case $P \rightarrow +\infty$ since the case $P \rightarrow -\infty$ is exactly similar. By rewriting $f^P = \exp(P \log(f))$, taking first order Taylor expansion $\log(f) \approx f - 1$ and first order Taylor expansion of exponential function such that

$$\frac{N}{D} = \exp\left(\log\left(\frac{N}{D}\right)\right) \approx 1 + \log(N) - \log(D),$$

we have:

$$\begin{aligned} \lim_{P \rightarrow +\infty} \eta(f)(x, y; t; P) &= 1 + \log \int_{\Omega} \exp\left((P+1)\left[f(x-u, y-v) - \frac{(u^2+v^2)}{2(P+1)(2t)} - 1\right]\right) dudv \\ &\quad - \log \int_{\Omega} \exp\left(P\left[f(x-u, y-v) - \frac{(u^2+v^2)}{2P(2t)} - 1\right]\right) dudv, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} &1 + (P+1) \log \left(\int_{\Omega} \left(\exp\left(f(x-u, y-v) - \frac{(u^2+v^2)}{2(P+1)(2t)} - 1\right)\right)^{(P+1)} dudv \right)^{\frac{1}{(P+1)}} \\ &- P \log \left(\int_{\Omega} \left(\exp\left(f(x-u, y-v) - \frac{(u^2+v^2)}{2P(2t)} - 1\right)\right)^P dudv \right)^{\frac{1}{P}}, \end{aligned}$$

Using now the standard result

$$\lim_{P \rightarrow +\infty} \left\{ \int_{\Omega} g^P(x) dx \right\}^{1/P} = \sup_{x \in \Omega} g(x),$$

which holds for positive and bounded function g with support space Ω , and and considering continuity and monotonicity of the logarithm, we obtain:

$$\begin{aligned} \lim_{P \rightarrow +\infty} \eta(f)(x, y; t; P) &= 1 + (P + 1) \sup_{(u, v) \in \Omega} \left(f(x - u, y - v) - \frac{(u^2 + v^2)}{2(P + 1)(2t)} - 1 \right) - \\ &P \sup_{(u, v) \in \Omega} \left(f(x - u, y - v) - \frac{(u^2 + v^2)}{2P(2t)} - 1 \right). \end{aligned}$$

By considering that both supremum operations gives closer values, i.e.,

$$\sup_{(u, v) \in \Omega} \left(f(x - u, y - v) - \frac{(u^2 + v^2)}{2P(2t)} - 1 \right) \approx \sup_{(u, v) \in \Omega} \left(f(x - u, y - v) - \frac{(u^2 + v^2)}{2(P + 1)(2t)} - 1 \right),$$

we finally obtain the corresponding result. ■

4.2 Properties of CHM Gaussian Scale-Space

We discuss now in more detail several features of the double scale-space $\eta(f)(x, y; t; P)$.

4.2.1 Continuity

The generic kernel function $K_{\sqrt{2t}}(x, y)$ is a continuous function in \mathbb{R}^2 .

Proposition 7 *The scale-space image $\eta(f)(x, y; t; P)$ is continuous for all $(x, y) \in \Omega$, for any $t \geq 0$ and for any $P \in \overline{\mathbb{R}}$.*

Proof. Consider $K_{\sqrt{2(t+\Delta t)}}(x + \Delta x, y + \Delta y) = K_{\sqrt{2t}}(x, y) + E(x, y)$ where assuming the continuity of $K(x, y)$, which implies the continuity of $K_{\sqrt{2t}}(x, y)$, we have

$$\begin{aligned} \lim_{\substack{|\Delta x| \rightarrow 0 \\ |\Delta y| \rightarrow 0 \\ \Delta t \rightarrow 0}} E(x, y) &= 0 \quad \text{for all } (x, y) \in \Omega \end{aligned} \tag{9}$$

For $t > 0$, we have

$$\eta(f)(x + \Delta x, y + \Delta y; t + \Delta t; P) = \frac{\int_E f(x - u + \Delta x, y - v + \Delta y)^{P+1} K_{\sqrt{2(t+\Delta t)}}(u, v) du dv}{\int_E f(x - u + \Delta x, y - v + \Delta y)^P K_{\sqrt{2(t+\Delta t)}}(u, v) du dv}$$

Setting $w = u - \Delta x$ and $z = v - \Delta y$ we have

$$\begin{aligned} \eta(f)(x + \Delta x, y + \Delta y; t + \Delta t; P) &= \frac{\int_E f(x - w, y - z)^{P+1} K_{\sqrt{2(t+\Delta t)}}(w + \Delta x, z + \Delta y) dw dz}{\int_E f(x - w, y - z)^P K_{\sqrt{2(t+\Delta t)}}(w + \Delta x, z + \Delta y) dw dz} \\ &= \frac{\int_E f(x - w, y - z)^{P+1} \left(K_{\sqrt{2t}}(w, z) + E(w, z) \right) dw dz}{\int_E f(x - w, y - z)^P \left(K_{\sqrt{2t}}(w, z) + E(w, z) \right) dw dz} \end{aligned}$$

So, in the limit as $|\Delta x| \rightarrow 0$, $|\Delta y| \rightarrow 0$, $\Delta t \rightarrow 0$ and assuming that the expression (9) holds, hence

$$\eta(f)(x + \Delta x, y + \Delta y; t + \Delta t; P) \rightarrow \eta(f)(x, y; t; P)$$

establishing the continuity. ■

4.2.2 Ordering

We have the following property of ordering w.r.t. scale parameter t .

Proposition 8 *There exists a value of P such that for any $Q \geq P > 0$, we have that for any pair of ordered scales $0 < t_1 < t_2$ involves $\eta(f)(x, y; t_1; Q) \leq \eta(f)(x, y; t_2; Q)$. Similarly, there is a P such that for $R \leq P < 0$, if $0 < t_1 < t_2$ then $\eta(f)(x, y; t_1; R) \geq \eta(f)(x, y; t_2; R)$.*

Proof. If we consider that for a large enough $P > 0$, with $Q \geq P$ we have

$$\eta(f)(x, y; t_1; Q) = \sup_{(u,v) \in \Omega} \left(f(x - u, y - v) - \frac{(u^2 + v^2)}{4Qt_1} \right)$$

and we have also for $0 < t_1 < t_2$: $\frac{(u^2+v^2)}{4Qt_2} < \frac{(u^2+v^2)}{4Qt_1}$, $\forall (u, v) \in D$ where $D \subset \Omega$. Then we obtain

$$\sup_{(u,v) \in \Omega} \left(f(x-u, y-v) - \frac{(u^2+v^2)}{4Qt_1} \right) \leq \sup_{(u,v) \in \Omega} \left(f(x-u, y-v) - \frac{(u^2+v^2)}{4Qt_2} \right)$$

thus proving the proposition. Similarly, the proof can be obtained for $P < 0$ ■

Hence, the ordering properties associated to the spatial scale t appears only in the pseudo-morphological behaviour of the CHM for large enough $|P|$. The fundamental ordering relationship w.r.t. P is directly inherited from Proposition 3 of the CHM.

Proposition 9 *For any pair of nonlinearity orders $-\infty \leq R \leq S \leq +\infty$, we have*

$$\eta(f)(x, y; t; R) \leq \eta(f)(x, y; t; S)$$

for any $t \geq 0$, with equality for $t = 0$.

4.2.3 Dimensionality

Dimensionality is a fundamental concept in measurements. This principle was extended in [28] to image processing, in particular, in the framework of mathematical morphology, as a way to take into account the homogeneity of physical properties computed from images. Later, it was studied in [18] the case of nonflat scale-space mathematical morphology using elliptic poweroid structuring functions. Making a measurement on an image consists in applying a functional on the image, where a “functional” is a global parameter associated with a function. More precisely, a functional W on a function f is said to be dimensional if and only if exists constants k_1, k_2 such that for all $\lambda_1, \lambda_2 > 0$

$$W(f') = \lambda_1^{k_1} \lambda_2^{k_2} W(f)$$

where $f'(x, y) = \lambda_1 f(\lambda_2 x, \lambda_2 y)$ and where k_1 is the intensity dimension and k_2 the space dimension. Hence the parameter λ_1 accounts for the affinity along the gray tone axis and λ_2 for the homothety of the support space of image f . The relation $W(f')$ restricts the way in which affinities and homotheties of an image affect dimensional measurements on it and results in a decoupling between affinity and homothety measurements. We notice that if $k_1 = 0$ the functional W is invariant under intensity affinities and if $k_2 = 0$ involves invariance under homotheties. Let us consider under which conditions a dimensional functional of the CHM Gaussian scale-space is also a dimensional functional of the underlying image.

Let us define the transformed scale-space: $\lambda_1 \eta(f)(\lambda_2 x, \lambda_2 y; \lambda_3 t; \lambda_4 P)$, for any $\lambda_1, \lambda_2, \lambda_3, \lambda_4 > 0$.

Proposition 10 *A functional W of the nonlinear scale-space $\eta(f)(x, y; t; P)$ is dimensional if and only if $\lambda_2^2 = \lambda_3 \lambda_4$ and if $|P|$ is large enough or if $\lambda_2^2 = \lambda_3$ and $P = 0$. In such a case, there exist constants k_1 and k_2 and we have*

$$W(\eta(f')(x, y; t; P)) = \lambda_1^{k_1} \lambda_2^{k_2} W(\eta(f)(x, y; t; P))$$

Proof. We start by writing

$$\lambda_1 \eta(f)(\lambda_2 x, \lambda_2 y; \lambda_3 t; \lambda_4 P) = \lambda_1 \frac{\int_E f(\lambda_2(x-u), \lambda_2(y-v))^{\lambda_4 P+1} K_{\sqrt{2\lambda_3 t}}(\lambda_2 u, \lambda_2 v) dudv}{\int_E f(\lambda_2(x-u), \lambda_2(y-v))^{\lambda_4 P} K_{\sqrt{2\lambda_3 t}}(\lambda_2 u, \lambda_2 v) dudv}.$$

On the one hand, we have $K_{\sqrt{2\lambda_3 t}}(\lambda_2 x, \lambda_2 y) = \frac{1}{4\pi\lambda_3 t} \exp\left(\frac{-(x^2+y^2)\lambda_2^2}{4t\lambda_3}\right)$, on the other hand, we define $f'(x, y) = \lambda_1 f(\lambda_2(x, y))$. In addition, $\lambda_1^2 f^{\lambda_4 P+1}(\lambda_2(x, y)) = \lambda_1 f^{\lambda_4 P}(\lambda_2(x, y)) \lambda_1 f(\lambda_2(x, y)) = (f')^{\lambda_4 P}(x, y) f'(x, y) = (f')^{\lambda_4 P+1}(x, y)$. Thus

$$\lambda_1 \eta(f)(\lambda_2 x, \lambda_2 y; \lambda_3 t; \lambda_4 P) = \frac{\int_E f'(x-u, y-v)^{\lambda_4 P+1} e^{-\frac{(u^2+v^2)\lambda_2^2}{4t\lambda_3}} dudv}{\int_E f'(x-u, y-v)^{\lambda_4 P} e^{-\frac{(u^2+v^2)\lambda_2^2}{4t\lambda_3}} dudv}$$

Consequently if we fix $\lambda_2^2 = \lambda_3$, for the case $P = 0$ we have

$$\lambda_1 \eta(f)(\lambda_2 x, \lambda_2 y; \lambda_3 t; 0) = \eta(f')(x, y; t; 0)$$

and for the case $P \rightarrow +\infty$ and fixing $\lambda_2^2 = \lambda_3 \lambda_4$ we have

$$\lambda_1 \eta(f)(\lambda_2 x, \lambda_2 y; \lambda_3 t; \lambda_4 P) = \eta(f')(x, y; t; P)$$

■

In conclusion, a functional W will be dimensional only when a modification of homothety is compensated by an affinity on the spatial scale, which involves also to increase the non-linearity order. This point can be also studied under the viewpoint of the Heijmans theory of [16, 17].

4.2.4 Invariance

By its linearity, Gaussian scale-space is invariant under linear transformations $f \mapsto \hat{f} = \alpha f + \beta \Rightarrow (\hat{f} * K_{\sqrt{2t}})(x, y) = \alpha (f * K_{\sqrt{2t}})(x, y) + \beta$. Morphological dilation/erosion are invariant only under grey-value shifts $f \mapsto \hat{f} = f + \beta \Rightarrow (\hat{f} \oplus b_{\sqrt{2t}})(x, y) = (f \oplus b_{\sqrt{2t}})(x, y) + \beta$. We notice that the flat dilation/erosion are invariant under anamorphosis [30, 25], i.e., any strictly increasing mapping is an anamorphosis. Let us look now at invariance properties of the CHM Gaussian scale-space.

Proposition 11 *The CHM Gaussian scale-space $\eta(f)(x, y; t; P)$ is invariant under scalar multiplication of grey-values for any P , $-\infty \leq P \leq +\infty$, i.e., $\eta(\alpha f)(x, y; t; P) = \alpha \eta(f)(x, y; t; P)$.*

In the limit cases $P = 0$ (equivalent to Gaussian scale-space) and $P \rightarrow \pm\infty$, the CHM

Gaussian scale-space is invariant under linear transformations: $\eta(\alpha f + \beta)(x, y; t; P = 0) = \alpha\eta(f)(x, y; t; P = 0) + \beta$ and $\lim_{P \rightarrow \pm\infty} \eta(\alpha f + \beta)(x, y; t; P) = \alpha \lim_{P \rightarrow \pm\infty} \eta(f)(x, y; t; P) + \beta$.

Proof. The proofs are direct from the expressions of the limit cases. ■

Hence the most interesting cases of the CHM Gaussian scale-space are invariant to grey-level addition and multiplication.

4.2.5 Asymptotic separability and semi-group law

In the classical state-of-the-art on linear scale-spaces, the fundamental semi-group property [21], or recursivity principle [26], states as follows: $\forall t_1, t_2 \geq 0$, $\left((f * K_{\sqrt{2t_1}}) * K_{\sqrt{2t_2}}\right) = \left(f * K_{\sqrt{2(t_1+t_2)}}\right)$. This is due to the fact that the Gaussian convolution kernels are the only separable and rotationally invariant functions that preserve the shape under Fourier transform. That allows also a separability in the implementation of the 2D Gaussian kernel convolution at scale t , using two successive 1D Gaussian kernel convolutions (at directions x and y) at scale t . From the works by Van den Boomgaard [4], it is also well known that in mathematical morphology the parabolic structuring function (in fact, any quadratic structuring function) are the equivalent class of functions which are dimensionally separable and closed with respect to the dilation/erosion, i.e., $\forall t_1, t_2 \geq 0$, $\left((f \oplus b_{\sqrt{2t_1}}) \oplus b_{\sqrt{2t_2}}\right) = \left(f \oplus b_{\sqrt{2(t_1+t_2)}}\right)$, and similarly for the erosion. This result is usually proved in the slope transform domain [12, 23].

The filter family $\eta(f)(x, y; t; P)$ for a given $P \in \overline{\mathbb{R}}$ has no semi-group structure and is, therefore, not a scale-space in strict sense. However, using the limit expressions it is easy to

proof that for the asymptotic cases $P = 0$ and $P \rightarrow \pm\infty$, the following semi-group holds:

$$\eta(\eta(f)(x, y; t_1; P))(x, y; t_2; P) = \eta(f)(x, y; t_1 + t_2; P); \quad \forall t_1, t_2 \geq 0$$

This result constitutes an advantage for an efficient numerical computation of an acceptable approximation to the limit cases of 2D CHM Gaussian scale-space using 1D separable Gaussian kernel for the convolutions of f^{P+1} and of f^P . Nevertheless, since computational efficiency is often important, the result of the operator $\eta(f)(x, y; t; P)$ for any P can be computed using low-order recursive filters for the numerator and denominator 2D Gaussian convolutions of the CHM filter. For example, the approach introduced in [38] uses a third-order recursive filter with one real pole and a pair of complex poles, applied forward and backward to make a sixth-order symmetric approximation to the Gaussian with low computational complexity for any scale t .

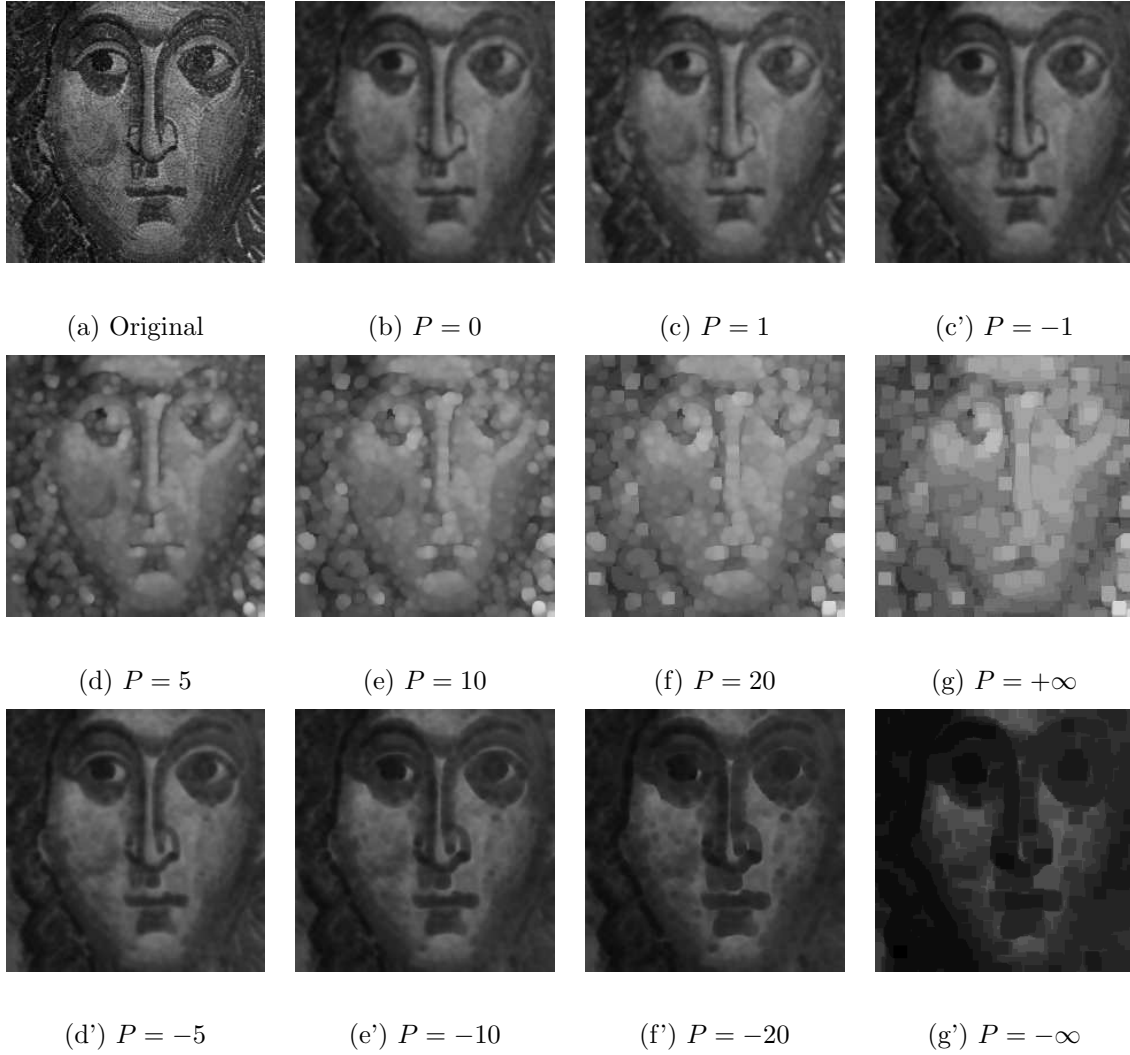


Figure 2: Generalised morphological Gaussian convolution (isotropic linear diffusion) $\eta(f)(x, y; t; P)$ at scale $t = 5$: First row, original image, standard Gaussian filtered image ($P = 0$) and cases $P = 1$ and $P = -1$; middle row, counter-harmonic Gaussian pseudo-dilations ($P > 0$); bottom row, counter-harmonic Gaussian pseudo-erosions ($P < 0$).

4.3 Interpretation as CHM Linear Diffusion and iterative framework

A filtered image $u(x, y; t)$ of $f(x, y)$ is calculated by solving the diffusion equation with the original image as initial state, and reflecting boundary conditions:

$$\begin{aligned} \partial_t u &= \operatorname{div}(c \nabla u) = c \Delta u = c \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \\ u(x, y; 0) &= f(x, y) \\ \partial_{\mathbf{n}} u|_{\partial \Omega} &= 0 \end{aligned} \quad (10)$$

where c is the conductivity and \mathbf{n} denotes the normal to the image boundary $\partial \Omega$. The popularity of the Gaussian scale-space is due to its linearity and the fact that the multiscale function $\psi(f)(x, y; t)$ can be generated from the isotropic heat diffusion, i.e.,

$$u(x, y; t) = (f * K_{\sqrt{2t}})(x, y) = \psi(x, y; t), \quad t > 0.$$

4.3.1 Numerical solution

The PDE can also be solved using finite differences in an explicit schema. Pixel i represents the location (x_i, y_i) . Let h_l denote the grid size in the l direction (working on square grid, we assume 1 for horizontal and vertical directions and $\sqrt{2}$ for 45° and -45° directions); and τ denote the time step size (to guarantee stability, the step size must satisfy [35]: $\tau = 1/(\sum_{l=1}^m 2/h_l^2)$), considering discrete times $t_k = k\tau$ (with k positive integer). By u_i^k we denote approximation to $u(x_i, y_i; t_k)$. The simplest discretization can be written in a compact way as [36]:

$$u_i^{k+1} = u_i^k + \tau \left(\sum_{l=1}^m \sum_{j \in \mathcal{N}_l(i)} \frac{c_j^k + c_i^k}{2h_l^2} (u_j^k - u_i^k) \right), \quad (11)$$

where $\mathcal{N}_l(i)$ consists of the two neighbours of pixel i along the l discretized direction. The conduction coefficients c_i^k are considered here constant in time and space.

Thus, the pseudo-morphological isotropic diffusion of order P can be rewritten as

$$\eta(f)(x, y; t; P) = \frac{[u(x, y; t)]^{P+1}}{[u(x, y; t)]^P}, \quad (12)$$

where $[u(x, y; t)]^P$ is the solution of diffusion equation 11 with the initial condition

$$u(x, y; 0) = f(x, y)^P.$$

4.3.2 Coupled schema

Furthermore, starting from the PDE numerical solution, it is also possible to define an iterative CHM-based isotropic diffusion, $\hat{\eta}(f)(x, y; t; P)$, by a coupled evolution of both power terms, according to the following discretization:

$$u_i^{k+1} = \frac{(u_i^k)^{P+1} + \tau \left(\sum_{l=1}^m \sum_{j \in \mathcal{N}_l(i)} \frac{c_j^k + c_i^k}{2h_l^2} ((u_j^k)^{P+1} - (u_i^k)^{P+1}) \right)}{(u_i^k)^P + \tau \left(\sum_{l=1}^m \sum_{j \in \mathcal{N}_l(i)} \frac{c_j^k + c_i^k}{2h_l^2} ((u_j^k)^P - (u_i^k)^P) \right)}. \quad (13)$$

As we can expect, the results of decoupled $\eta(f)(x, y; t; P)$ and coupled $\hat{\eta}(f)(x, y; t; P)$ diffusions are different. In Fig. 3 is given a comparison of CHM linear diffusion using the decoupled iterative scheme (equivalent to the generalised Gaussian convolution) vs. the coupled iterative scheme, at spatial scale $t = 5$ and order $P = 10$. We observe from this experiment of pseudo-dilation, and other similar results provided in the above mentioned website, that the effects of $\hat{\eta}(f)(x, y; t; P)$ are more “morphological” for equivalent order P , with sharper transitions between structures. We can conclude that the iteration n times of a unitary (i.e., $\sigma = \sqrt{2}$) generalised morphological Gaussian convolution introduces a nonlinearity degree superior than the corresponding generalised morphological Gaussian convolution at scale

$\sigma = \sqrt{2n}$. Another example is depicted in Fig. 6(e) and (f), which just corresponds to a comparison of the pseudo-opening obtained by product of pseudo-erosion and pseudo-dilation $\hat{\eta}(\hat{\eta}(f)(x, y; t; -P))(x, y; t; P)$ and the counterpart decoupled diffusion operators.

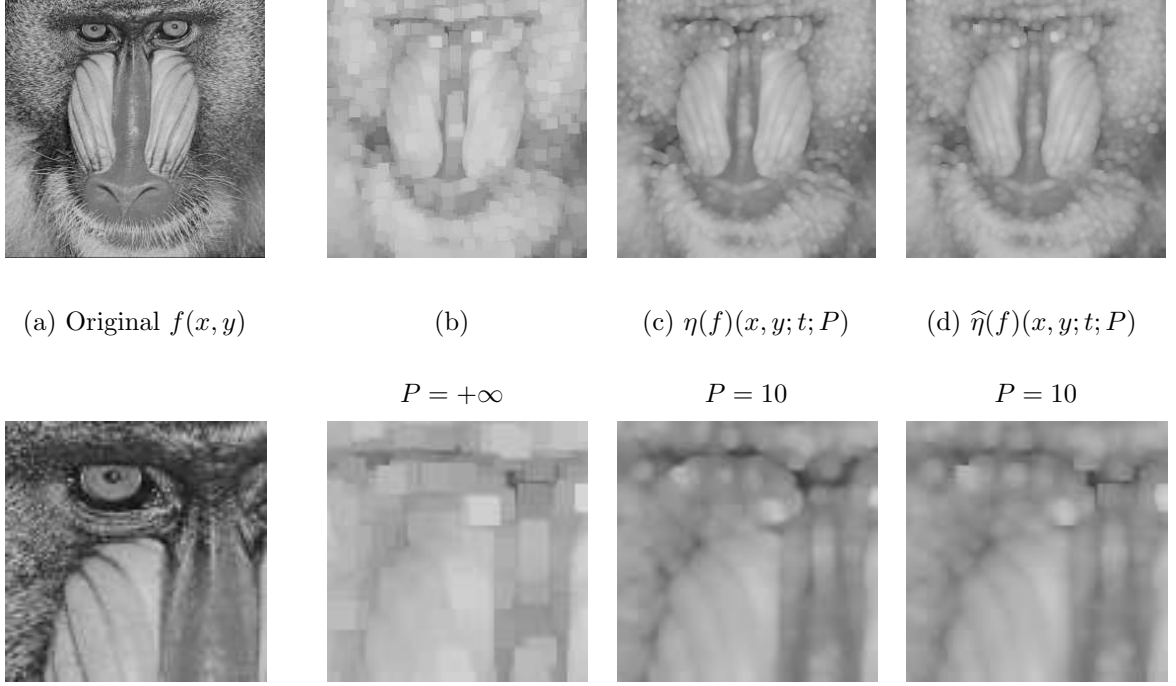


Figure 3: Comparison of CHM linear diffusion using the decoupled iterative scheme (equivalent to the generalised Gaussian convolution) vs. the coupled iterative scheme, at spatial scale $t = 5$ and order $P = 10$: (a) original image $f(x, y)$, (b) standard dilation, (c) decoupled $\eta(f)(x, y; t; P)$ and (d) coupled $\hat{\eta}(f)(x, y; t; P)$. At the bottom are zoom-in frames of a square section cropped from the corresponding images.

5 Counter-Harmonic Nonlinear Diffusion

Ideas introduced above are extended in this section to two well-known cases of evolved image diffusion.

5.1 Nonlinear diffusion

The big disadvantage of isotropic diffusion is the fact that linear smoothers do not only smooth the noise but also blur and shift structure edges, which are important image features in some applications, i.e., segmentation and object recognition. The pioneering idea introduced by Perona and Malik [27] to reduce these problems is to locally adapt the diffusivity, i.e., the value of the conductivity c , to the gradient of the image at each iteration. More precisely, this nonlinear diffusion involves replacing the diffusion equation 10 by the following model:

$$\partial_t u = \operatorname{div} (g (\|\nabla u\|^2) \nabla u) \tag{14}$$

$$u(x, y; 0) = f(x, y)$$

$$\partial_{\mathbf{n}} u|_{\partial\Omega} = 0$$

In this model the diffusivity has to be such that $g (\|\nabla u\|^2) \rightarrow 0$ when $\|\nabla u\|^2 \rightarrow +\infty$ and $g (\|\nabla u\|^2) \rightarrow 1$ when $\|\nabla u\|^2 \rightarrow 0$. One of the diffusivities Perona and Malik proposed is the function

$$g(s^2) = 1 / (1 + (s^2/\lambda^2)), \quad \lambda > 0,$$

where λ is a threshold parameter that separates forward and backward diffusion. This model accomplishes the aim of blurring small fluctuations (noise) while enhancing edges (by preventing excessive diffusion). To avoid some numerical and theoretical drawbacks of this model,

it was proposed in [11], a new version of Perona and Malik theory, based on replacing diffusivity $g(\|\nabla u\|^2)$ by a regularized version $g(\|\nabla u_\sigma\|^2)$ with $\nabla u_\sigma = \nabla(K_\sigma * u)$ where K_σ is a Gaussian kernel. This latter model is just used in our framework and the numerical solution can be obtained using the explicit scheme 11, where the local conductivity is here the approximation to the regularized term $g(\|\nabla u_\sigma\|^2)$, i.e.,

$$c_i^k = \frac{1}{1 + \frac{\|\nabla u_\sigma(x_i, y_i; t_k)\|^2}{\lambda^2}}, \quad (15)$$

with the gradient computed by central differences.

Similarly to the isotropic case, we can now define the pseudo-morphological counter-harmonic nonlinear diffusion of order P as

$$\xi(f)(x, y; t; P) = \frac{[u(x, y; t)]_{nl}^{P+1}}{[u(x, y; t)]_{nl}^P}, \quad (16)$$

where $[u(x, y; t)]_{nl}^Q$ is the solution of regularized Perona and Malik diffusion equation with the initial condition $u(x, y; 0) = f(x, y)^Q$. Fig. 4 provides the results of pseudo-dilation and pseudo-erosion ($P = 10$ and $P = -10$) using the regularized Perona and Malik model on a noisy image. Comparing with respect to the standard filtering ($P = 0$), the good properties of denoising without blurring are conserved but in addition, an effect of dilation/erosion is obtained. This kind of pseudo-morphology is useful for instance to compute morphological gradient, i.e.,

$$\xi(f)(x, y; t; P) - \xi(f)(x, y; t; -P),$$

of noisy images or to construct filters as morphological openings, i.e.,

$$\xi(\xi(f)(x, y; t; -P))(x, y; t; P).$$

See in Fig. 6(c) and (g) the comparison of the standard Perona and Malik diffusion and the corresponding pseudo-opening diffusion, which removes some small bright structures. It is also possible to define the coupled case by computing the counter-harmonic but this discussion is out of the paper scope.

A critical point here is the choice of parameter λ , which is an ill-posed problem. Sophisticated approaches in the state-of-the-art are based on noise estimation. We have determined empirically that the value

$$\lambda = \frac{1}{m} \left(\sum_{l=1}^m \text{Mean}(\nabla_l u(x, y; t)) \right), \quad (17)$$

where m is the number of discretized directions, and which of course is different for P and $P + 1$, leads to stable results.

5.2 Coherence-enhanced diffusion

Coherence-enhanced diffusion, or tensor-driven diffusion, was introduced by Weickert [35] in order to achieve an anisotropic diffusion filtering based on directionality information. The idea is to adapt the diffusion process to local image structure using the following nonlinear diffusion equation:

$$\partial_t u = \text{div}(\mathbf{D}\nabla u). \quad (18)$$

where the conductivity function becomes a symmetric positive definite diffusion tensor \mathbf{D} , which is a function adapted to the structure tensor

$$\mathbf{J}_\rho(\nabla u_\sigma) = K_\rho * (\nabla u_\sigma \otimes \nabla u_\sigma).$$

The eigenvectors of \mathbf{J}_ρ are orthonormal and the eigenvalues are positive. The corresponding eigenvalues (let us call them $\mu_1 \geq \mu_2$) describe the local structure. Flat areas give $\mu_1 \approx \mu_2$, straight edges give $\mu_1 \gg \mu_2 = 0$ and corners give $\mu_1 \geq \mu_2 \gg 0$. In order to control the diffusion, \mathbf{J}_ρ is not used directly, but tensor \mathbf{D} has the same eigenvectors as \mathbf{J}_ρ , but different eigenvalues, thus controlling the diffusion in both directions. The eigenvalues are

$$\begin{cases} \lambda_1 &= \alpha \\ \lambda_2 &= \alpha + (1 - \alpha) \exp(-C/\kappa) \end{cases}$$

where κ is the orientation coherence and $C > 0$ serves as a threshold parameter. Parameter $\alpha > 0$ is quite important and serves as a regularization parameter which keeps \mathbf{D} uniformly positive definite. For this diffusion, we have used in our tests a numerical implementation using the additive operator splitting (AOS) scheme [35], which is particularly efficient and has the advantage of being rotationally invariant compared to their multiplicative counterparts.

As previously, the pseudo-morphological counter-harmonic coherence-enhanced diffusion of order P is defined as

$$\chi(f)(x, y; t; P) = \frac{[u(x, y; t)]_{ce}^{P+1}}{[u(x, y; t)]_{ce}^P}, \quad (19)$$

where $[u(x, y; t)]_{ce}^Q$ is the solution of Weickert diffusion equation 18 with the initial condition $u(x, y; 0) = f(x, y)^Q$. As in the previous case, we have to adapt the regularization parameter α to the dynamics of the power images $f(x, y)^P$ in order to have numerical stable results.

Empirically, we have observed that

$$\alpha = \begin{cases} 0.01|P| & \text{if } P \neq 0 \\ 0.005 & \text{if } P = 0 \end{cases} \quad (20)$$

lead to satisfactory results. Moreover, we have also observed in experimental tests that numerical solution of the CHM coherence diffusion given in expression (19) can introduce some instabilities and errors for $P > 0$, whereas for $P < 0$ the results are numerically regular. To avoid this drawback, we propose to compute, by duality, the coherence-enhanced pseudo-dilation $\chi(f)(x, y; t; P)$, with $P > 0$, by means of the operation $1 - \chi(1 - f)(x, y; t; -P)$. In any case, a deeper study on the numerical implementation of counter-harmonic Weickert diffusion, in particular in the case of coupled diffusion, is required.

Examples of pseudo-morphological anisotropic diffusion are given in Fig. 5 and Fig. 6(d)-(h). Dark regions are anisotropically pronounced in pseudo-erosion schemes ($P < 0$) whereas bright regions are anisotropically emphasized in pseudo-dilation as well as in their products, respectively the pseudo-openings and pseudo-closings.

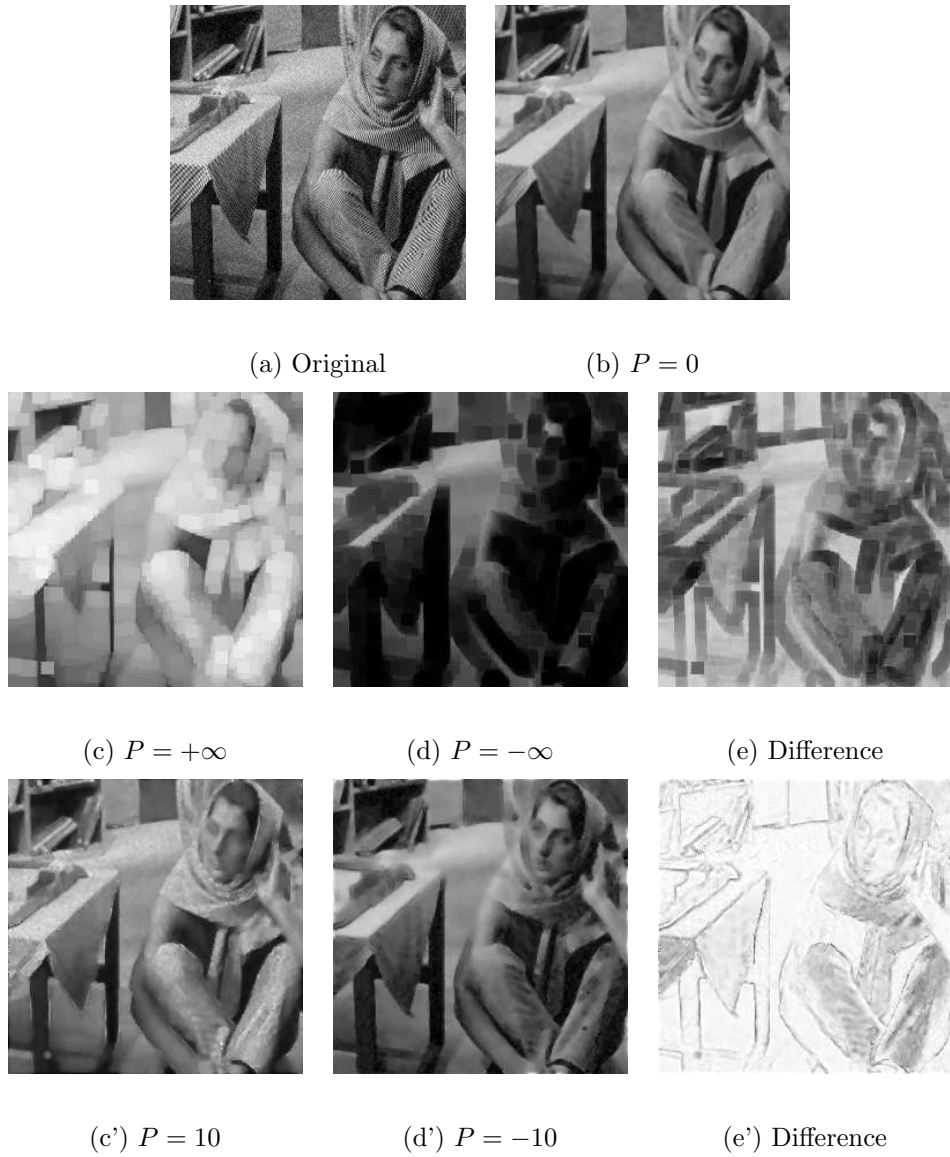


Figure 4: Pseudo-morphological nonlinear diffusion $\xi(x, y; t)$ at scale $t = 5$ (regularization parameter $\sigma = 0.5$): (a) original image, (b) standard nonlinear diffusion, (c)/(d) standard dilation/erosion, (e) image difference between dil./ero. (gradient), (c')/(d') pseudo-dilation/erosion by counter-harmonic nonlinear diffusion, (e') image difference between pseudo-dil./ero.

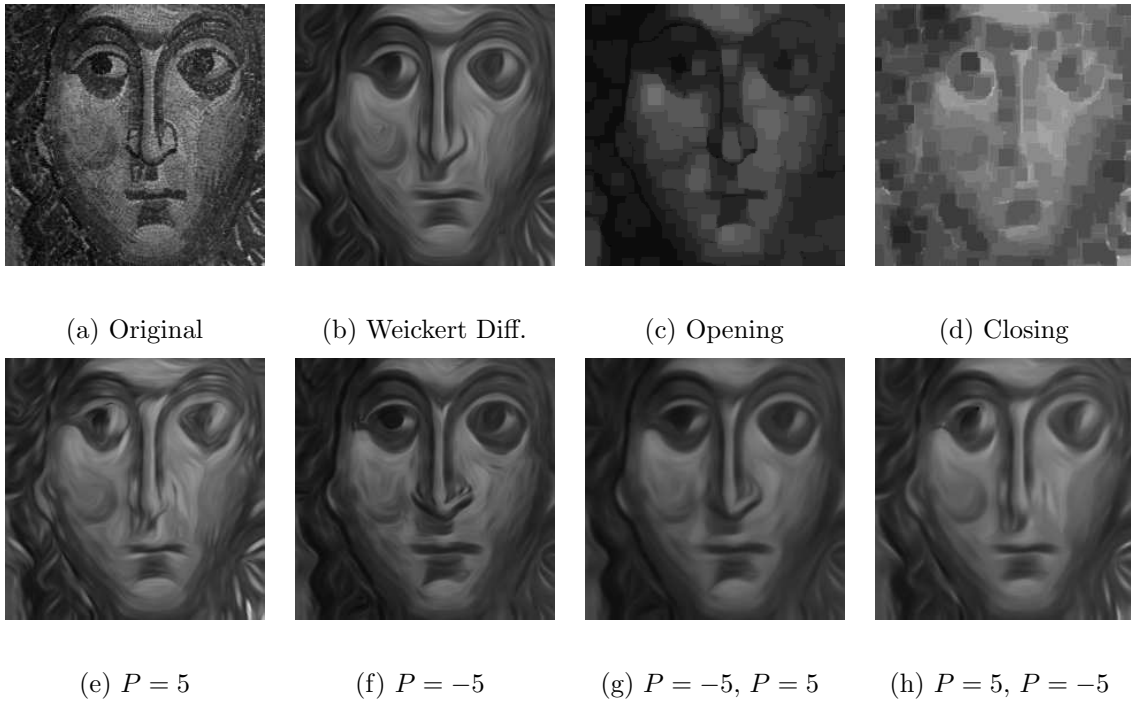


Figure 5: Pseudo-morphological anisotropic diffusion $\chi(x, y; t; P)$ at scale $t = 10$ (with regularization parameters: local scale $\sigma = 1.5$, integration scale $\rho = 6$): (a) original image, (b) standard anisotropic diffusion, (c)-(d) opening and closing of size equivalent to $t = 5$, (e)/(f) pseudo-dilation/erosion by counter-harmonic anisotropic diffusion, (g)/(h) pseudo-opening/closing by counter-harmonic anisotropic diffusion.

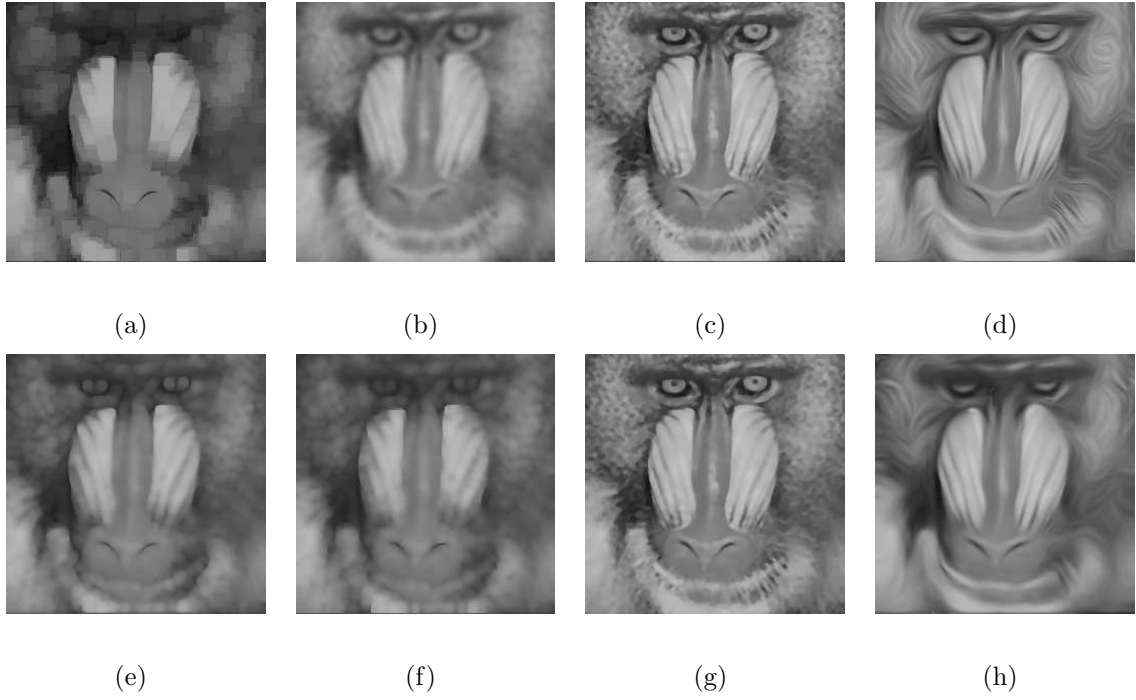


Figure 6: Comparison of standard diffusions and pseudo-morphological openings (the original image is given in Fig. 3: (a) standard opening using a square 11×11 , (b) isotropic diffusion $t = 5$, (c) Perona and Malik diffusion $t = 5$, (d) Weickert diffusion $t = 10$, (e) pseudo-opening using isotropic diffusion $t = 5$, $P = 10$, $P = -10$, (f) pseudo-opening using coupled isotropic diffusion $t = 5$, $P = 10$, $P = -10$, (g) pseudo-opening using Perona and Malik diffusion $t = 5$, $P = 5$, $P = -5$, (h) pseudo-opening using Weickert diffusion $t = 10$, $P = 5$, $P = -5$).

6 Conclusions and perspectives

We have introduced a mathematical framework of pseudo-morphological image diffusion using the notion of counter-harmonic mean. In particular, we have proposed the counter-harmonic Gaussian scale-space, and we have studied in detail its properties. In the case of 2D images, the nonlinear smoothed signals $\eta(f)(x, y; t; P)$ taken as a whole can be considered as a continuous function on \mathbb{R}^{2+2} , that is, a double pseudo-scale-space with respect to t and with respect to P . This family of filters expands and includes the Gaussian scale-space and the morphological erosion/dilation scale-spaces. We have also proposed the corresponding generalization of Perona and Malik diffusion model as well as of Weickert model of coherence-enhanced diffusion.

As the dilation/erosion operators depend on the extreme values of the (additively weighted signal) in the neighbourhood of a point, impulse noise and other kind of high-frequency noise cause spurious results of the standard morphological processing of noisy image. Linear diffusion filtering is able to deal with the noise, however the averaging process blurs and limits also the effect of the filtering. The fundamental property of the present pseudo-morphological image diffusion is its ability to robustly filter out the noisy images in a nonlinear way, which can selectively bias the result toward high (low) values such as the dilation (erosion) filter.

We have given a first analysis about the limit statements, but a deeper study is still required about the behaviour of the counter-harmonic linear diffusion with respect to value of P . In particular, we believe that the relationship between our numerical framework and

an hybrid evolutionary PDE such as

$$\partial_t u = \alpha \Delta u \pm \beta \|\nabla u\|^2,$$

with variable coefficients α and β which determines the part of linear *versus* morphological smoothing w.r.t. P , should be explored. This hybrid equation was already mentioned in [8], considered in [14] and it is related also to Kuwahara-Nagao PDE introduced in [6] but, to our knowledge, it remains an open problem. Additionally, it should be considered how the present nonlinearization of image diffusion can be turned to the Laplace-Beltrami framework to describe diffusion process [31, 20].

A deeper study on the numerical implementation of counter-harmonic Perona-Malik and Weickert diffusions, and in particular in the case of coupled diffusion schemas, should be achieved. We have pointed out that, for data defined in the interval $[0, 1]$, the behaviour of filters with $P < 0$ is generally more stable than the case $P > 0$ and this is probably related with one of the properties of Proposition 2. However a better understating of these effects is still to be investigated.

References

- [1] L. Alvarez, F. Guichard, P-L. Lions, J-M. Morel. Axioms and fundamental equations of image processing. *Arch. for Rational Mechanics*, 123(3): 199–257, 1993.
- [2] J. Angulo. Pseudo-Morphological Image Diffusion using the Counter-Harmonic Paradigm. In *Proc.of Acivs'2010 (2010 Advanced Concepts for Intelligent Vision Systems)*, LNCS Vol. 6474, Part I, 426–437, Springer, Sydney, Australia, December 2010.

- [3] A.B. Arehart, L. Vincent, B.B. Kimia. Mathematical morphology: The Hamilton-Jacobi connection. In *Proc. of IEEE 4th Inter. Conf. on Computer Vision (ICCV'93)*, 215–219, 1993.
- [4] R. van den Boomgaard, L. Dorst. The morphological equivalent of Gaussian scale-space. In *Proc. of Gaussian Scale-Space Theory*, 203–220, Kluwer, 1997.
- [5] R. van den Boomgaard. Numerical solution schemes for continuous-scale morphology. In *Proc. of Scale-Space Theories in Computer Vision*, LNCS Vol. 1682, 199–210, Springer, Berlin, 1999.
- [6] R. van den Boomgaard. Decomposition of the Kuwahara-Nagao operator in terms of a linear smoothing and a morphological sharpening. In *Proc. of ISMM'2002*, 283–292, CSIRO Publishing, 2002.
- [7] M. Breuß, J. Weickert. Highly accurate PDE-based morphology for general structuring elements. In *Proc. of Scale Space and Variational Methods in Computer Vision*, LNCS Vol. 5567, 758–769, Springer, Berlin, 2009.
- [8] R.W. Brockett, P. Maragos. Evolution equations for continuous-scale morphology. *IEEE Trans. on Signal Processing*, 42(12):3377–3386, 1994.
- [9] P.S. Bullen. *Handbook of Means and Their Inequalities*, 2nd edition, Springer, 1987.
- [10] B. Burgeth, J. Weickert. An Explanation for the Logarithmic Connection between Linear and Morphological System Theory. *International Journal of Computer Vision*, 64(2–3):157–169, 2005.

- [11] F. Catte, P.-L. Lions, J.-M. Morel, T. Coll. Image selective smoothing and edge detection by nonlinear diffusion. *SIAM Journal on Numerical Analysis*, 29(1):182–193: 1992.
- [12] L. Dorst, R. van den Boomgaard. Morphological Signal Processing and the Slope Transform. *Signal Processing*, 38:79–98, 1994.
- [13] L. Florack. *Image Structure*, Kluwer Academic Publishers, 1997.
- [14] L. Florack, R. Maas, W. Niessen. Pseudo-Linear Scale-Space Theory, *International Journal of Computer Vision*, 31(2-3): 1–13, 1999.
- [15] R.C.Gonzalez, R.E. Woods. *Digital Image Processing*, Addison-Wesley, Boston, MA, USA, 1992.
- [16] H.J.A.M. Heijmans, R. van den Boomgaard. Algebraic Framework for linear and morphological scale-spaces. *Journal of Visual Communication and Image Representation*, 13(1-2): 269–301, 2002.
- [17] H.J.A.M. Heijmans. Morphological scale-spaces, scale invariance, and Lie groups. In *Proc. of International Symposium on Mathematical Morphology (ISMM 2002)*, CSIRO Ed., Sydney, 2002.
- [18] P.T. Jackway. On Dimensionality in Multiscale Morphological Scale-Space with Elliptic Poweroid Structuring Functions. *Journal of Visual Communication and Image Representation*, 6(2), 189–195, 1995.
- [19] P.T. Jackway, M. Deriche. Scale-Space Properties of the Multiscale Morphological Dilation-Erosion. *IEEE Trans. Pattern Anal. Mach. Intell.*, 18(1): 38–51, 1996.

- [20] R. Kimmel, R. Malladi, N.A. Sochen. Image Processing via the Beltrami Operator. In *Proc. of Third Asian Conference on Computer Vision (ACCV'98)*, LNCS Vol. 1351, Vol. I, 574–581, 1998.
- [21] T. Lindeberg. Scale-Space for Discrete Signals. *IEEE Trans. Pattern Anal. Mach. Intell.*, 12(3):234–254, 1990.
- [22] T. Lindeberg. *Scale-Space Theory in Computer Vision*, Kluwer Academic Publishers, 1994.
- [23] P. Maragos. Slope Transforms: Theory and Application to Nonlinear Signal Processing. *IEEE Trans. on Signal Processing*, 43(4): 864–877, 1995.
- [24] P. Maragos. Differential morphology and image processing. *IEEE Transactions on Image Processing*, 5(1): 922-937, 1996.
- [25] G. Matheron. Les treillis compacts. *Technical Report - Paris School of Mines, N-23/90/G* (available online), November 1990.
- [26] E.J. Pauwels, L.J. Van Gool, P. Fiddelaers, T. Moons. An Extended Class of Scale-Invariant and Recursive Scale Space Filters. *IEEE Trans. Pattern Anal. Mach. Intell.*, 17(7): 691–701, 1995.
- [27] P. Perona, J. Malik. Scale-Space and Edge Detection Using Anisotropic Diffusion. *IEEE Trans. Pattern Anal. Mach. Intell.*, 12(7): 629–639, 1990.
- [28] J.F. Rivest, J. Serra, P. Soille. Dimensionality in Image Analysis. *Journal of Visual Communication and Image Representation*, 3(2): 137–146, 1992.

- [29] G. Sapiro, R. Kimmel, D. Shaked, B.B. Kimia, A.M. Bruckstein. Implementing continuous-scale morphology via curve evolution. *Pattern recognition*, 26(9): 1363–1372, 1993.
- [30] J. Serra. *Image Analysis and Mathematical Morphology. Vol I, and Image Analysis and Mathematical Morphology. Vol II: Theoretical Advances*, London: Academic Press, 1982,1988.
- [31] N.A. Sochen, R. Kimmel, R. Malladi. A general framework for low level vision. *IEEE Transactions on Image Processing*, 7(3): 310–318, 1998.
- [32] P. Soille. *Morphological Image Analysis*, Springer-Verlag, Berlin, 1999.
- [33] L.J. van Vliet. Robust Local Max-Min Filters by Normalized Power-Weighted Filtering. In *Proc. of IEEE 17th International Conference of the Pattern Recognition (ICPR'04)*, Vol. 1, 696–699, 2004.
- [34] J. Weickert. *Anisotropic Diffusion in Image Processing*, ECMI Series, Teubner-Verlag, Stuttgart, Germany, 1998.
- [35] J. Weickert. Coherence-Enhancing Diffusion Filtering. *Int. J. Comput. Vision*, 31(2-3): 111–127, 1999.
- [36] J. Weickert. Efficient image segmentation using partial differential equations and morphology. *Pattern Recognition*, 31(9): 1813–1824, 2001.
- [37] M. Welk. Families of generalised morphological scale spaces. In *Proc. of 4th Inter. Conf. of Scale-Space Methods in Computer Vision*, LNCS Vol. 2695, 770–784, Springer, 2003.

- [38] I.T. Young, L.J. van Vliet. Recursive implementation of the Gaussian filter. *Signal Processing*, 44: 139–151, 1995.