Introduction to some basic random morphological models

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Abstract

The Boolean RF are a generalization of the Boolean RACS. Their construction based on the combination of a sequence of primary RF by the operation $\vee$ (supremum) or $\wedge$ (infimum), and their main properties (among which the supremum or infimum infinite divisibility) are given in the case of scalar RF built on a Poisson point process.

1 Introduction

This text reviews a family of random functions (RF) which is an extension of the binary Boolean model, and is of wide use for applications, the Boolean RF.

This family owns the interesting property of supremum (or infimum, according to the chosen type of construction) infinite divisibility. These models are particularly interesting for applications in physics, such as in fracture statistics [6, 9, 10]. The basic idea of the Boolean RF (BRF) was born about the modelling of rough surfaces by D. Jeulin (1979), by a generalization of the Boolean model of G. Matheron. The first presentations and applications are given in [3, 25]. In [4] an anisotropic version is developed. Out of the field of materials, other examples of applications are given for biomedical images [19, 21, 22], for Scanning Electron Microscope images [28], and for solving problems of exploitation of oceanographic reserves [1]. The first theoretical studies of the BRF are given in [3, 25, 4, 26]. In [27, 28], J. Serra introduces a general BRF model, connected to a non stationary Poisson point process in R^{n+1}. In [20, 21], F. Preteux and M. Schmitt proved some characteristic properties of the BRF, useful for the identification of a model from images. Finally, a generalization of the BRF at two levels was proposed and developed by D. Jeulin [7, 6]: introduction of Boolean varieties RF (including the Poisson point process as a particular case), and of the multivariate case.

In what follows, a reminder on random closed sets and on semi-continuous RF is given. Then we review the properties of the Boolean RF model.

2 Reminder on random closed sets and on semi-continuous random functions

The heterogeneity of materials can be handled through a probabilistic approach, which enables us to generate models and simulation of the microstructures. Two-phase media can be modelled by realizations of random closed sets. More general microstructures involve the use of random functions.

2.1 Random closed sets

When considering two-phase materials (for instance a set of particles A embedded in a matrix A^c), we use a model of random closed set (RACS) A [16, 18, 25, 8], fully characterized from a probabilistic point of view by its Choquet capacity $T(K)$ defined on the compact sets K, from (1) below, where P denotes a probability:

$$T(K) = P(K \cap A \neq \emptyset) = 1 - Q(K) = 1 - P(K \subset A^c). \qquad (1)$$

In the Euclidean space R^n, the Choquet capacity is related to the dilation operation of Mathematical Morphology $A \oplus \check{K}$. We have:

$$T(K_x) = P\{K_x \cap A \neq \emptyset\} = P\{x \in A \oplus \check{K}\}$$

In practice, $T(K)$ can be estimated by area fraction measurements on 2D images, or from volume fraction estimation on 3D images (from true microstructures, or from simulations), after a morphological dilation of the set A by the set K [16, 18, 25, 8], or calculated for a given theoretical model. Equation (1) is used for the identification of a model (estimation of its parameters, and test of its validity). Particular cases of morphological properties deduced from (1) are the volume fraction V_v, the covariance (a useful tool to detect the presence of scales or anisotropies), the distribution of distances of a point in A^c to the boundary of A. The access to 3D images of microstructures by means of X-ray microtomography [23] makes it possible to use 3D compact sets K (like balls $B(r)$ with various radii r) to characterize the random set.

2.2 Upper semi-continuous random functions

We consider semi-continuous (upper, lower) random functions, for which the changes of supports by $\vee$ or by $\wedge$ provide random variables [17]:

$$Z_\vee(K) = \vee_{x \in K}\{Z(x)\}$$

$$Z_\wedge(K) = \wedge_{x \in K}\{Z(x)\}$$

Random Functions (RF) and Random Sets are related by means of their subgraph and of their overgraph [2].

Definition 1 *The subgraph* Γ^{φ} *of the function* φ *is made of the pairs* $\{x, z\}$, $x \in E$, $z \in \overline{R}$, *with* $z \leq \varphi(x)$. *The overgraph* Γ_{φ} *is made of the pairs* $\{x, z\}$, $x \in E$, $z \in \overline{R}$, *with* $z \geq \varphi(x)$.

We have the following result connecting semi-continuous functions and closed sets [2]:

Proposition 2 *The function* φ *is lsc* $\Leftrightarrow$ *its overgraph* Γ_{φ} *is a closed set in* $E \times \overline{R}$; φ *is usc* $\Leftrightarrow$ *its subgraph* Γ^{φ} *is a closed set in* $E \times \overline{R}$.

Theorem 3 *A random function* $Z(x)$ *defined in* R^n, *upper semi continuous (usc), is characterized by its Choquet capacity* $T(g)$ *defined over lower semi continuous functions (lsc)* g *with a compact support* K

$$\begin{aligned} T(g) &= P\{x \in D_Z(g)\} = 1 - Q(g) \\ D_Z(g)^c &= \{x, Z(x+y) < g(y), \forall y \in K\} \end{aligned} \tag{2}$$

Particular cases of the Choquet capacity are obtained from Equation (2), depending on the choice of the test function g.

- When $g(x_i) = z_i$ for x_i $(i = 1, 2, ..., n)$, else $g(x) = +\infty$:

  $$T(g) = 1 - P\{Z(x_1) < z_1, ..., Z(x_n) < z_n\}$$

  $1 - T(g)$ gives the spatial law. In what follows we note $A_Z(z)$ the random closed set obtained by thresholding the RF $Z(x)$ at level z:

  $$A_Z(z) = \{x, \ Z(x) \geq z\}$$

- For the function $g(x) = z$ if $x \in K$, and $g(x) = +\infty$ if $x \notin K$,

  $$D_Z(g)^c = \{x, Z(x+y) < z, \forall y \in K\} = \left[A_{Z_{\vee}(K)}(z)\right]^c$$

  We have:

  $$Z_{\vee}(K)(x) < z \Leftrightarrow K_x \subset (A_Z(z))^c \Leftrightarrow x \in A_Z(z)^c \ominus \check{K}$$

  and

  $$Z_{\vee}(K)(x) \geq z \Leftrightarrow x \in (A_Z(z)^c \ominus \check{K})^c = A_Z(z) \oplus \check{K}$$

  and therefore

  $$A_{Z_{\vee}(K)}(z) = D_Z(g) = A_Z(z) \oplus \check{K}$$

  For this type of test function g, we have:

  $$T(g) = P\{x \in D_Z(g)\} = 1 - Q(g) = 1 - P\{Z_{\vee}(K) < z\}$$

  The Choquet capacity gives the probability distribution of the RF $Z(x)$ after a change of support by $\vee$ over the compact set K.

Let $Z_1(x)$ and $Z_2(x)$ be two usc RF and $Z(x) = Z_1(x) \vee Z_2(x)$. We have

$$
\begin{aligned}
(D_{Z_1 \vee Z_2}(g))^c &= D_Z(g)^c = \{x, Z_1(x+y) \vee Z_2(x+y) < g(y), \forall y \in K\} \\
&= \{x, Z_1(x+y) < g(y) \text{ and } Z_2(x+y) < g(y)\ , \forall y \in K\} \\
&= D_{Z_1}(g)^c \cap D_{Z_2}(g)^c
\end{aligned}
$$

Therefore

$$D_{Z_1 \vee Z_2}(g) = D_{Z_1}(g) \cup D_{Z_2}(g) \tag{3}$$

and

$$
\begin{aligned}
T_{Z_1 \vee Z_2}(g) &= P\{x \in D_Z(g)\} = 1 - Q(g) = P\{x \in D_{Z_1}(g) \cup D_{Z_2}(g)\} &(4)\\
\text{and } Q(g) &= P\{x \in D_{Z_1}(g)^c \cap D_{Z_2}(g)^c\} &(5)
\end{aligned}
$$

Furthermore we have:

$$
\begin{aligned}
A_Z^c(z) &= \{x, Z(x) < z\} = \{x, Z_1(x) \vee Z_2(x) < z\} \\
&= \{x, Z_1(x) < z \text{ and } Z_2(x) < z\} = A_{Z_1}^c(z) \cap A_{Z_2}^c(z)
\end{aligned}
$$

$$A_{Z_1 \vee Z_2}(z) = A_{Z_1}(z) \cup A_{Z_2}(z) \tag{6}$$

If the two RF $Z_1(x)$ and $Z_2(x)$ are independent, the two random sets $D_{Z_1}(g)$ and $D_{Z_2}(g)$ are independent, and the relation 5 writes:

$$
\begin{aligned}
Q(g) &= P\{x \in D_Z(g)^c\} = P\{x \in D_{Z_1}(g)^c \cap D_{Z_2}(g)^c\} \\
&= P\{x \in D_{Z_1}(g)^c\} P\{x \in D_{Z_2}(g)^c\} = Q_1(g) Q_2(g)
\end{aligned}
$$

More generally, if $Z(x) = \vee_{i=1}^{i=n} Y_i(x)$ and if the $Y_i(x)$ are independent realizations of the same RF $Y(x)$ with $P\{x \in D_Y(g)^c\} = Q_Y(g)$, we get:

$$Q_Z(g) = P\{x \in D_Z(g)^c\} = P\{x \in \cap_{i=1}^{i=n} D_{Yi}(g)^c\} = Q_Y(g)^n \tag{7}$$

2.3 Principle of random sets and of random function modeling

The main steps to follow when designing a random model of structure are as follows:

1. Choice of basic assumptions
2. Computation or estimation of the Choquet's capacity functionnal $T(K)$

The functional $T(K), T(g)$ is obtained as a function of

1. the assumptions
2. the parameters of the model
3. the compact K or the function g.

For a given model, the functionnal T is obtained by theoretical calculation or by estimation, either on simulations, or on real structures. This gives access to a possible estimation of the parameters from the "experimental" T, and to tests of the validity of assumption for model identification.

3 The Boolean random functions

In what follows, we review the main properties of the BRF built on the Poisson point process.

3.1 Construction of the BRF

We are concerned in this section by Boolean RF with support in the Euclidean space R^n, and note $\mu_n(dx)$ and $\theta(dt)$ the Lebesgue measure in R^n and a σ finite measure on R (such that $\int_B \theta(dt)$ remains finite for every bounded Borel set B in R). We consider:

- i) a Poisson point process $\mathcal{P}$, with the intensity measure $\mu_n(dx) \otimes \theta(dt)$ in $R^n \times R$;
- ii) a family of independent lower semi continuous primary RF $Z'_t(x)$, with a subgraph $\Gamma^{Z'_t} = A'(t)$ having almost surely compact sections $A_{Z'_t}(z)$.

Definition 4 *The Boolean random function (BRF) with the primary function $Z'_t(x)$ and with the intensity $\mu_n(dx) \otimes \theta(dt)$ is the RF $Z(x)$ obtained by*

$$Z(x) = \vee_{(t_k,x_k)\in\mathcal{P}}\{Z'_{t_k}(x - x_k)\} \quad (8)$$

We can notice the following points.

- i) This definition, given in [6], is more general than the one proposed by J. Serra in [27, 28]; it covers the previous definitions:
  - the Boolean islands, for which the measure $\theta(dt)$ is the Dirac distribution concentrated in a point t of R: $\theta(dt) = \theta\delta_0(t)$ [3, 25, 4];

– the "generalized" BRF, where we have $Z'_t(x) = Y'_t(x) + t$, where $Y'_t(x)$ is a family of primary RF. The addition of t comes from a definition of the BRF as a non stationary Boolean RACS in R^{n+1} with Poisson germs in R^{n+1} and with primary random sets $A'(t)$ defined at the origin $(0, 0)$ of the coordinates of R^{n+1}. To introduce BRF on more general lattices, where the addition is not necessarily defined, this construction process cannot be used.

- ii) The parameter t, that can be assimilated to z in the definition [27, 28], as for examples given in section 3.8, can also be interpreted as a time, leading to the notion of sequential RF. In these conditions, for the time interval $(t, t + dt)$ is defined an infinitesimal BRF.

- iii) It is possible to parametrize the primary functions by $t \in R^k$, with a $\sigma-$finite measure $\theta(dt)$ on R^{n-k}.This enables us to introduce a primary function depending on several indexes. Instead of R^k, an abstract space E and a measure θ defined on E can be chosen. Similarly, the Lebesgue measure on R^n, $\mu_n(dx)$, can be replaced by a $\sigma-$finite measure $\theta(dx)$ on R^n, dropping the stationarity in R^n. This process can be used to build multiscale RF, as illustrated in section 4.

- iv) From the definition (8), the "floor" value of $Z(x)$ is $-\infty$. This value can be bounded (z_0) by use of primary functions such that $A_{Z'_t}(z_0) = R^n$, or by taking $Y(x) = z_0 \vee Z(x)$.

- v) From lower semi-continuous primary functions $Z'(t)$ (with overgraph $\Gamma_{Z'_t}$), it is possible to build a $\wedge$ BRF [6], by replacing in Eq. (8) the operation $\vee$ by $\wedge$, and starting from a $+\infty$ ceiling value of $Z(x)$. It is equivalent to build a $\vee$ BRF Y from the primary RF $Y'_t(x) = -Z'_t(x)$ and to consider as a $\wedge$ BRF $Z(x) = -Y(x)$. For this reason, we limit this presentation mainly to the $\vee$ BRF given by Eq. (8).

- vi) From the point of view of subgraphs (closed in R^{n+1} for lower semi-continuous functions), the relation (8) involves:

$$\Gamma^Z = \cup_{(t_k, x_k) \in \mathcal{P}} A'(t_k)_{x_k} \tag{9}$$

By definition, Γ^Z is a Boolean RACS in R^n with primary grain $A'(t)$.

For illustration, simulations of BRF are shown on figure 1.

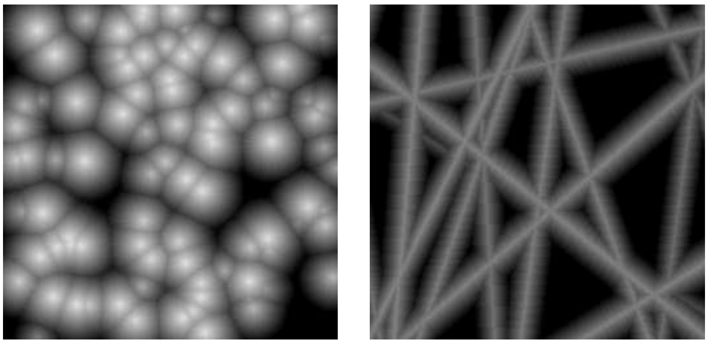

Figure 1: Examples of realizations of a Boolean random function with cone primary functions (left), and built from Poisson lines (right)

3.2 Boolean Random function and Boolean model of random sets

Using the definition 8 and the property 6, we have for the BRF $Z(x)$:

$$A_Z(z) = \cup_{(t_k,x_k)\in\mathcal{P}} A_{Z'_{t_k}}(z)_{x_k}$$

As a consequence,

Proposition 5 *Every random closed set $A_Z(z)$ obtained by thresholding a BRF $Z(x)$ at level z is a Boolean random set with primary grain $A_{Z'_t}(z)$.*

This property will be useful for the identification of a model of BRF, since available tools for the Boolean model can be used for this purpose.

3.3 Choquet capacity of the BRF

As mentioned in theorem 3, we can characterize a BRF by means of the functional $T(g)$ defined on lower semi-continuous functions g with a compact support K :

$$T(g) = P\{x \in D_Z(g)\}; D_Z(g)^c = \{x, Z(y) < g(y-x), \forall y \in K\}$$

Since $D_{Z_1 \vee Z_2}(g) = D_{Z_1}(g) \cup D_{Z_2}(g)$, we get for a BRF $Z(x)$:

$$D_Z(g) = \cup_{(t_k,x_k)\in\mathcal{P}} D_{Z'_{t_k}}(g)_{x_k} \tag{10}$$

and $D_Z(g)$ is a Boolean RACS with the primary grain $D_{Z'_t}(g)$. Since $D_Z(g)$ corresponds to the event $A^c(Z) = \{\exists y \in R^n, Z(y) \geq g(x-y)\}$, the two following theorems result.

Theorem 6 *Consider a BRF $Z(x)$ and a lower semi continuous function g translated in x. The number of primary functions Z'_t for which the event $A^c(Z'_t)$ is satisfied, follows a Poisson distribution with parameter $\int_R \overline{\mu}_n(D_{Z'_t}(g))\, \theta(dt)$.*

Theorem 7 *The Choquet capacity of the BRF $Z(x)$ is given by:*

$$1 - T(g) = Q(g) = \exp\left(-\int_R \overline{\mu}_n(D_{Z'_t}(g))\, \theta(dt)\right) \tag{11}$$

For the Boolean islands model $\theta(dt) = \theta\delta_0(t)$ and $Z'_0 = Z'$

$$1 - T(g) = Q(g) = \exp\left(-\theta\overline{\mu}_n(D_{Z'}(g))\right) \tag{12}$$

As particular functions g, let us examine the following cases:

- i) If $g(x_i) = z_i$ for points x_i $(i = 1, 2, ..., n)$, and else $g(x) = +\infty$, we obtain the spatial law of the BRF:

  $$\begin{aligned} 1 - T(g) &= P\{Z(x_1) < z_1, ..., Z(x_n) < z_n\} \\ &= \exp\left(-\int_R \overline{\mu}_n(A_{Z'_t}(z_1)_{x_1} \cup ... \cup A_{Z'_t}(z_n)_{x_n})\, \theta(dt)\right) \end{aligned} \tag{13}$$

  For a single point x, is obtained the cumulative distribution function $F(z)$

  $$F(z) = P\{Z(x) < z\} = \exp\left(-\int_R \overline{\mu}_n(A_{Z'_t}(z))\, \theta(dt)\right) \tag{14}$$

  For two points x and $x + h$, Eq. (13) gives the bivariate distribution $F(h, z_1, z_2)$ as a function of the cross geometrical covariogram $K(h, z_1, z_2, t)$ between the two sets $A_{Z'_t}(z_1)$ and $A_{Z'_t}(z_2)$:

  $$\begin{aligned} F(h, z_1, z_2) &= P\{Z(x) < z_1, Z(x + h) < z_2\} \\ &= \exp\left(-\int_R \overline{\mu}_n(A_{Z'_t}(z_1) \cup A_{Z'_t}(z_2)_{-h})\, \theta(dt)\right) \\ &= F(z_1)F(z_2)\exp\left(\int_R \overline{\mu}_n(A_{Z'_t}(z_1) \cap A_{Z'_t}(z_2)_{-h})\, \theta(dt)\right) \\ &= F(z_1)F(z_2)\exp\left(\int_R K(h, z_1, z_2, t)\, \theta(dt)\right) \end{aligned} \tag{15}$$

  From Eq. (15), it is clear that for the BRF we always have $F(h, z_1, z_2) \geq F(z_1)F(z_2)$, so that no negative correlation can occur.

- ii) If $g(x) = z$ for $x \in K$ and else $g(x) = +\infty$, K being a compact set, Eq. (11) enables us to calculate the distribution of $Z(x)$ after a change of support by the operator $\vee$ taken over the compact set K $(Z_\vee(x) = \vee_{x \in K}\{Z(x)\})$; we have in that case $D_{Z'_t}(g) = A_{Z'_t}(z) \oplus \check{K}$ and

  $$P\{Z_\vee(K) < z\} = \exp\left(-\int_R \overline{\mu}_n(A_{Z'_t}(z) \oplus \check{K})\, \theta(dt)\right) \tag{16}$$

  From the definition (8) and from Eq. (10), the following result is obtained:

Proposition 8 *The RF $Z_\vee(x)$ is a BRF with the primary function $Z'_\vee(x)$.*

All previous results can be specialized to the Boolean island version of the model, when $\theta(dt) = \theta\delta_0(t)$ and $Z'_0 = Z'$. Starting form the Choquet capacity 12, the spatial law becomes:

$$\begin{gathered} 1 - T(g) = P\{Z(x_1) < z_1, ..., Z(x_n) < z_n\} \\ = \exp\left(-\theta\overline{\mu}_n(A_{Z'}(z_1)_{x_1} \cup ... \cup A_{Z'}(z_n)_{x_n})\; \theta(dt)\right) \end{gathered}$$

The bivariate distribution is given by

$$F(h, z_1, z_2) = F(z_1)F(z_2)\exp\left(\theta K(h, z_1, z_2)\right)$$

and the change of support by the operator $\vee$ follows

$$P\{Z_\vee(K) < z\} = \exp\left(-\theta\overline{\mu}_n(A_{Z'}(z) \oplus \check{K})\right)$$

3.4 Supremum stability and infinite divisibility

Let $Z_1(x)$ and $Z_2(x)$ be two independent BRF with the primary functions Z'_{1t} and Z'_{2t}, and the intensities $\theta_1(t)$ and $\theta_2(t)$. From Eq. (9) is obtained:

$$\Gamma^Z = \Gamma^{Z_1} \cup \Gamma^{Z_2} = \cup_{(t_k,x_k)\in\mathcal{P}_1} A'_1(t_k)_{x_k} \;\; \cup_{(t_k,x_k)\in\mathcal{P}_2} A'_2(t_k)_{x_k}$$

and therefore Γ^Z is a Boolean model in R^{n+1}; as a consequence, $Z(x)$ is a BRF with intensity $\theta(t) = \theta_1(t) + \theta_2(t)$ and with a mixture of primary functions.

Proposition 9 *Every supremum of a family of independent BRF $Z_i(x)$ is a BRF with intensity $\theta(t) = \sum_i \theta_i(t)$. The BRF is stable with respect to the supremum.*

The supremum stability property 9 of the BRF is shared with more recent RF models, namely so-called max-stable processes.

As a consequence of the infinite divisibility of the Boolean model for $\cup$, we get:

Theorem 10 *Every BRF $Z(x)$ is infinite divisible for $\vee$: $\forall n, Z(x) \equiv \vee_{k=1}^{k=n} Z_k(x)$ where the Z_k are independent BRF with the same law.*

This results immediately from the expression of the Choquet capacity of the BRF 11 and from the Choquet capacity of $\vee_{k=1}^{k=n} Z_k(x)$ derived from equation 7: for any integer n, we have

$$1 - T(g) = Q(g) = \exp\left(-\int_R \overline{\mu}_n(D_{Z'_t}(g))\; \frac{\theta(dt)}{n}\right)^n$$

3.5 Characteristics of the primary functions

Some characteristics of the pair (intensity, primary function) can be determined from information on the BRF $Z(x)$. These characteristics are directly deduced from the Choquet capacity (11) and from the derived properties (13-15).

3.5.1 Transformation by anamorphosis

Let φ an anamorphosis transformation (namely a monotonous non-decreasing transformation applying R into R). Let $Y = \varphi(Z)$.

Proposition 11 *Every anamorphosis of a BRF, $\varphi(Z)$, is a BRF obtained with the same intensity $\theta(t)$ and with the primary function $Y' = \varphi(Z')$.*

Proof. We have

$$\begin{aligned} A_{\varphi(Z)}(z) = \{x; \varphi(Z(x)) \geq z\} = \{x; Z(x) \geq \varphi^{-1}(z)\} \\ = A_Z(\varphi^{-1}(z)) = \cup_{(t_k, x_k)\in\mathcal{P}} A_{Z'_{t_k}}(\varphi^{-1}(z))_{x_k} \\ = \cup_{(t_k, x_k)\in\mathcal{P}} A_{Y'_{t_k}}(z)_{x_k} = A_Y(z) \end{aligned}$$

■

This result enables us to restrict our study to strictly positive BRF, since it is always possible to transform any function Z into a positive function $Y = \varphi(Z)$ (consider for instance the anamophosis obtained by an exponential transformation).

3.5.2 Moments of $Z'_\vee(K)$ and mathematical expectation of the anamorphosed of $Z'_\vee(K)$

We consider now positive BRF.

Proposition 12 *We have:*

$$\begin{aligned} M(i, K) &= -\int_R z^{i-1} \log\left(P\{Z_\vee(K) < z\}\right)\, dz \\ &= \frac{1}{i}\int_{R^+} E\left[\int_{R^n} (Z'_{t\vee}(K)(x))^i\, dx\right]\, \theta(dt) \end{aligned} \tag{17}$$

Let $\Phi(z)$ a strictly positive function with $\Phi(z) = \int_0^z \varphi(u)\, du$. We have:

$$\begin{aligned} &-\int_R \varphi(z) \log\left(P\{Z_\vee(K) < z\}\right)\, dz \\ &= \int_{R^+} E\left[\int_{R^n} \Phi(Z'_{t\vee}(K)(x))\, dx\right]\, \theta(dt) \end{aligned} \tag{18}$$

Proof. We note $1_{Z'_t \geq z}(x)$ the indicator function of the set $A_{Z'_t}(z)$ at point x ($1_{Z'_t \geq z}(x) = 1$ if $Z'_t(x) \geq z$ and else $1_{Z'_t \geq z}(x) = 0$). For a given realization of the primary function we have:

$$\mu_n(A_{Z'_t}(z)) = \int_{R^n} 1_{Z'_t \geq z}(x)\, dx$$

and

$$\int_{R^+} z^{i-1} 1_{Z'_t \geq z}(x)\, dz = \int_0^{Z'_t(x)} z^{i-1} dz = \frac{(Z'_t(x))^i}{i}$$

By integration in R^n we obtain

$$\int_{R^n} \frac{(Z'_t(x))^i}{i}\, dx = \int_{R^+} z^{i-1} \mu_n(A_{Z'_t}(z))\, dz$$

and by taking the mathematical expectation

$$E\left\{ \int_{R^n} \frac{(Z'_t(x))^i}{i}\, dx \right\} = \int_{R^+} z^{i-1} \overline{\mu}_n(A_{Z'_t}(z))\, dz$$

The moment $M(i)$ is deduced by integration of the last expression over the measure $\theta(dt)$, and similarly for the moment $M(i, K)$ after replacing Z and Z' by $Z_\vee(K)$ and by $Z'_\vee(K)$. Similarly, we have

$$\int_{R^+} \varphi(z) 1_{Z'_t \geq z}(x)\, dz = \int_0^{Z'_t(x)} \varphi(z)\, dz = \Phi(Z'_t(x))$$

and by integration in R^n

$$\int_{R^+} \varphi(z) \mu_n(A_{Z'_t}(z))\, dz = \int_{R^n} \Phi(Z'_t(x))\ \ dx$$

After taking the mathematical expectation and after integration over $\theta(dt)$ the expression (18) is immediate for Z and for $Z_\vee(K)$. ■

3.5.3 Geometrical covariogram of the primary function

Starting from the bivariate distribution given in Eq. (15), for $z = z_1 = z_2$ we obtain for a positive RF Z

$$\begin{aligned} &\int_0^\infty \log\left(P\{Z(x) < z, Z(x+h) < z\}/(F(z))^2\right) \\ &= \int_R \int_0^\infty K(h, z, z, t)\theta(dt)\, dz = \int_R K(h, t)\theta(dt) \end{aligned} \tag{19}$$

with the notation $K(h, t) = \mu_{n+1}(A'_+(t) \cap A'_+(t)_{-h})$ for the geometrical covariogram in R^{n+1} of the positive part of the subgraph of Z'_t, $A'_+(t)$. The Eq. (19) may be useful for the identification of primary functions from $K(h, t)$, often simpler for calculations than the bivariate distribution deduced from the cross geometrical covariogram $K(h, z_1, z_2, t)$.

3.6 Some stereological aspects of the BRF

As for the Boolean model, a BRF defined in R^n generates by section in R^k ($k < n$) BRF with induced intensity and primary functions. This is a property connected to the Poisson point process. For some families of primary functions (for instance when the positive part of the subgraph is made in R^{n+1} of spheres, similar cylinders, or similar parallelepipeds,...), it is possible to estimate the properties of the primary functions (up to the intensity), from the sole bivariate distribution known on profiles, through the function $\int_R K(h,t)\theta(dt)$. As far as these primary functions are well suited to real data, it can be relatively easy to implement them in applications.

3.7 BRF and counting

In this section, we consider digital images with support in R^2, modelled by Boolean island BRF.

As in [27, 28], we assume that the integral $V = \int_0^\infty \overline{\mu}_2(A_{Z'}(z))\ dz$ is known from a preliminary study. When considering a topographical surface in R^3, V is the volume covered by the primary function Z'. We wish to estimate θ for images considered as realizations of BRF with intensity θ. From the distribution function $F(z)$, we get:

$$-\int_0^\infty \log\left(F(z)\right) dz = \theta V \tag{20}$$

This counting algorithm is very convenient, since it does not require any segmentation or any choice of a threshold. Well suited to Boolean textures, it is weakly sensitive to noise, but it is sensitive to illumination conditions (through V), which should remain strictly constant between a standard experiment (to estimate V) and an image acquisition for counting.

3.8 Identification of a BRF model

To identify a BRF from data, both the family of primary function $Z'_t(x)$ and the measure $\theta(dt)$ must be known. However, the Choquet capacity (11), experimentally estimated from realizations of BRF, depends on the product of two factors: the intensity and a measure on the primary function. It is therefore not possible to know these two terms separately from their product, so that we have to face an indetermination. To raise it, we rely on the following results proved by M. Schmitt and F. Preteux [19, 20, 24]. We note (Z'_t, θ) the BRF defined by a choice of the primary function Z'_t and of the intensity $\theta(dt)$.

Proposition 13 ***Characterization of a BRF.*** *Consider a BRF* (Z'_t, θ)*; i) If* $\theta(R) = \theta < \infty$*, the BRF admits a unique representation as a Boolean* *island $(Z', \theta\delta)$, where Z' is centered on the projection on the plane $z = 0$ of the center m of the sphere in R^{n+1} circumscribing the maxima of the primary function; ii)If $\theta(R) = +\infty$, the BRF can be uniquely represented by (Z'_t, θ), where the Z'_t are centered in m and where $Z(x) = \vee_{(t_k, x_k) \in \mathcal{P}} \{Z'_{t_k}(x - x_k) + t_k\}$.*

From experimental data, we always access to a bounded range of variation for $Z(x)$. We can therefore mostly consider Boolean islands. It will be the same situation for simulations. However, at the level of a theoretical model, it is often interesting to consider the case ii) with $\theta(R) = +\infty$. For instance we can use the following BRF:

- The **Weibull model** is obtained by implantation of primary functions $Z'_t(x)$ with a point support ($Z'_t(x) = t\delta(x)$, $\delta(x)$ being the Dirac distribution in R^n) and $\theta(dt) = \theta m(z_0 - t)^{m-1}$ for $t \leq z_0 \leq 0$. With this definition, the BRF differs from $-\infty$ on points of a Poisson process. It cannot be characterized by its spatial law, which is equal to zero. Use must be made of the Choquet capacity (11) for functions g having a support with non zero measure in R^n. For instance, the distribution function of $Z_\vee(K)$ is derived from Eq. (16):

$$P\{Z_\vee(K) < z\} = \exp - \int_{-\infty}^{z_0} \theta m(z_0 - t)^{m-1} \overline{\mu}_n(A_{Z'_t}(z) \oplus \check{K})\, dt$$

with $A_{Z'_t}(z) \oplus \check{K} = \check{K}$ for $t \geq z$, else $= \emptyset$. It comes:

$$P\{Z_\vee(K) < z\} = \exp - \theta(z_0 - z)^m \mu_n(K) \tag{21}$$

In fracture statistics, the variable of interest is $Z > 0$ (the fracture stress), and use is made of the BRF $Y(x) = -Z(x)$, which can be directly obtained with the intensity $\theta(dt) = \theta m(t - z_0)^{m-1} dt$ $(t \geq z_0)$, by means of the operator $\wedge$ instead of $\vee$, and starting from the value $+\infty$ outside of the Poisson point process in R^n. For $z \geq z_0$

$$P\{Z_\wedge(K) \geq z\} = \exp - (\theta(z - z_0)^m \mu_n(K)) \tag{22}$$

- The **Pareto** model is obtained with the same construction as the Weibull model, with the intensity $\theta(dt) = \dfrac{-\theta dt}{t}$ for $t \leq z_0 \leq 0$ and else $\theta(dt) = 0$. We have

$$\begin{aligned} P\{Z_\vee(K) < z\} &= \exp \int_{-\infty}^{z_0} \theta \overline{\mu}_n(A_{Z'_t}(z) \oplus \check{K}) \frac{dt}{t} \\ &= \exp \theta \mu_n(K) \int_{-z}^{-z_0} \frac{dt}{t} = \left(\tfrac{z_0}{z}\right)^{\theta \mu_n(K)} \end{aligned} \tag{23}$$

Using the operator $\wedge$ instead of $\vee$, and starting from the value $+\infty$ outside of the Poisson point process in R^n. For $z \geq z_0$

$$P\{Z_\wedge(K) \geq z\} = \left(\frac{z_0}{z}\right)^{\theta\mu_n(K)} \tag{24}$$

3.9 Test of the BRF

Tests proposed for testing the BRF model are derived from tests proposed for the Boolean model. In a first step, it is possible to work on sets obtained by applying thresholds on $Z(x)$ at different levels z_i, which are Boolean models with intensity $\int_R (1 - G_t(z_i))\theta(dt)$, where $G_t(z)$ is the distribution function of the maximum of $Z'_t(x)$. Other tests can be directly applied to the function $Z(x)$. They involve the following criteria: convexity of the sections $A_{Z'_t}(z)$, change of support on convex sets, and infinite divisibility for $\vee$.

3.9.1 Convexity of $A_{Z'_t}(z)$

This is the most often used test used in applications until now. It is based on an additional assumption, the convexity of the sections of the primary function, $A_{Z'_t}(z)$. This is not satisfied in the general case. The test makes use of the Steiner formula to the distribution (16) $P\{Z_\vee(K) < z\}$ when K is a compact convex set. In these conditions, $\log(P\{Z_\vee(\lambda K) < z\})$ and similarly $\int_R \log(P\{Z_\vee(\lambda K) < z\})\ dz$ are polynomials of degree k in λ for $K \subset R^k$.

It is easy to implement these tests, since they only require the estimation of the distribution functions after change of support by the operator $\vee$ on convex sets with increasing sizes λK. The first test, based on a threshold z, is the same as for the Boolean random set model. The second test may be the source of numerical difficulties, since we may obtain $P\{Z_\vee(\lambda K) < z\} \simeq 0$ for weak values of z. In that case, we have to set a lower value z_0 for the numerical integration of the integral.

Examples of applications of these tests are given in [3, 4, 19, 21, 28, 1, 4]. In [4], the test was satisfactory for change of support on segments with increasing lengths; primary functions of different shapes were used for the simulation of the rough surface of steel plates: cylinders, paraboloids, cones.

3.9.2 Change of support on convex sets

Again we consider $Z_\vee(\lambda K)$ with K convex, and λ is chosen in such a way that $\mu_n(\lambda K) \gg \overline{\mu}_n(A_{Z'_t}(z))$. We do not need to make any assumption about the convexity of $A_{Z'_t}(z)$ for the proposed asymtotic tests [5].

Let z such that $\vee_t\{G_t(z)\} < 1$ and two convex sets $K_1 \subset R^{n_1}$, $K_2 \subset R^{n_2}$

with $n_1 \leq n$, $n_2 \leq n$. We have:

$$H(\lambda_1, \lambda_2) = \frac{\log(P\{Z_\vee(\lambda_1 K_1) < z\})}{\log(P\{Z_\vee(\lambda_2 K_2) < z\})} = \frac{\int_R \overline{\mu}_n(A_{Z'_t}(z) \oplus \lambda_1 \check{K}_1)\theta(dt)}{\int_R \overline{\mu}_n(A_{Z'_t}(z) \oplus \lambda_2 \check{K}_2)\theta(dt)} \tag{25}$$

For $\lambda_1 \to +\infty$ and for $\lambda_2 \to +\infty$, Eq. (25) becomes

$$H(\lambda_1, \lambda_2) = \frac{\lambda_1^{n_1}}{\lambda_2^{n_2}} \frac{\mu_{n_1}(K_1)}{\mu_{n_2}(K_2)} \tag{26}$$

For instance in R^3 :

- If K_1 is the cube with edge 1 and if K_2 is the square with edge 1

$$H(\lambda_1, \lambda_2) = \frac{\lambda_1^3}{\lambda_2^2} \text{ and } H(\lambda, \lambda) = \lambda$$

- If K_1 is the cube with edge 1 and if K_2 is the segment with length 1

$$H(\lambda_1, \lambda_2) = \frac{\lambda_1^3}{\lambda_2} \text{ and } H(\lambda, \lambda) = \lambda^2$$

- If K_1 is the square with edge 1 and if K_2 is the segment with length 1

$$H(\lambda_1, \lambda_2) = \frac{\lambda_1^2}{\lambda_2} \text{ and } H(\lambda, \lambda) = \lambda$$

In practice, it is also possible to set λ_1 and λ_2 constant and to vary z. The two curves $\log(P\{Z_\vee(\lambda_1 K_1) < z\})$ and $\log(P\{Z_\vee(\lambda_2 K_2) < z\})$ must be proportional, with a slope equal to $H(\lambda_1, \lambda_2)$.

These tests, not based on the assumption of convexity of the sections $A_{Z'_t}(z)$, can be implemented after a first change of support over a non convex set K (for instance $\{x, x+h\}$, or K made of any number of points), provided that λ stays larger than the range of $Y(x)$, deduced from $Z(x)$ by this first transformation.

3.10 BRF and random tesselations

Boolean random functions can be used for the generation of models of random tesselations [15]. A large class of random tesselation models combines a point process and a distance to the points. For instance, attaching to every Poisson point x_k a primary random function $Z'_k(x)$ defined according to the Euclidean distance, the standard Voronoi model can be deduced from a $\wedge$ BRF with primary function made of an increasing paraboloid of revolution:

$$Z(x) = \wedge_k Z'_k(x - x_k) \tag{27}$$

Sections of primary functions at level z are balls defined by the corresponding metric. Define

$$B'_k(z) = \{x, Z'_k(x) < z\}$$

From equation (27) we have

$$B(z) = \{x, Z(x) < z\} = \cup_{x_k} B'_k(z)_{x_k} \tag{28}$$

By construction equation (28) $B(z)$ is a Boolean random set with convex primary grains $B'_k(z)$. Consider a compact set K and the infimum $Z_\wedge(K) = \wedge_{y \in K}\{Z(y)\}$. We have for the stationary case

$$P\{Z_\wedge(K) \geq z\} = \exp -\{\theta E(\mu_n(B'_k(z) \oplus \check{K}))\} \tag{29}$$

More general random tesselations can be generated by the same process, starting from Boolean random functions with any primary random function $Z'(x)$. We consider that the realization k of $Z'(x)$ owns simply connected compact sections $B'_k(z)$, such that $B'_k(z_1) \subset B'_k(z_2)$ for $z_2 > z_1$. We consider primary random functions reaching their minimum $Z'(0)$ for $x = 0$. We associate to $Z'_k(x)$ the floor set A'_k defined by

$$A'_k = \{x, Z'_k(x) = Z'_k(0)\} \tag{30}$$

If we have $A' = \{O\}$, we can define the class C_k of the random tesselation, generated by the germ x_k and the primary random function $Z'(x)$ by:

$$C_k = \{x \in R^n, Z'_k(x - x_k) < Z'_l(x - x_l), x_k \in \mathcal{P}, x_l \in \mathcal{P}, l \neq k\} \tag{31}$$

For the simulation of random tesselations, we just need to simulate realizations of the Boolean random function with primary functions Z'_k corresponding to the model. The boundaries of the tesselation are provided by the crest lines of the random functions, obtained by the watershed of the random function using as markers apparent markers defined below. By construction of the Boolean random functions, the location of crest lines, and therefore the boundaries of the classes of the resulting tesselation are invariant by a non decreasing transformation Φ (anamorphosis) of the values of $Z'_k(x)$ (for instance using $Z'^p_k(x)$ instead of $Z'_k(x)$), that is compatible with the order relationship, namely such that $z_1 < z_2$ implies $\Phi(z_1) < \Phi(z_2)$.

An alternative extraction of classes is given by their labels C_k. Starting from the simulation, and from the germs x_k, we generate in each point x a set of labels $L(x)$:

$$L(x) = \{k, Z(x) = Z'_k(x - x_k)\} \tag{32}$$

Points x with the single label k generate the interior of cell C_k. Points with two labels k and l are on the boundaries between cells C_k and C_l. In R^3, points with three labels are on the edges of the tesselation, and points with four labels are its vertices. More details about the properties of such random tesselations are given in [15].

4 Multiscale Boolean random functions

In many practical situations, there is a non-homogeneous dispersion of objects in a matrix, and possibly arrangement of aggregates at different scales [11, 13, 14]. A convenient way to account for theses situations is to replace the Poisson point process by a Cox process, generating multi scale Cox Boolean Random Function.

In a first step, we can replace in the construction of the BRF the intensity measure $\mu_n(dx) \otimes \theta(dt)$ in $R^n \times R$ by the intensity $\theta(dx, dt)$, dropping the stationarity of the Poisson point process. In a second step, we use for $\theta(dx, dt)$ a realization of a positive random function, generating a Cox point process. The Choquet capacity becomes:

$$T(g) = 1 - E_\theta\{\exp(-E_{Z'_t}\{\int_R \theta(D_{Z'_t}(g), dt)\}\} = 1 - \varphi_g(1)$$

with $\varphi_g(\lambda)$ the Laplace transform of the positive random variable

$$E_{Z'_t}\{\int_R \theta(D_{Z'_t}(g), dt)\}$$

$E_{Z'_t}$being the expectation with respect to the random function Z'_t.

A typical example is given by a constant intensity θ inside a first random set A_1 (such as a stationary Boolean model of spheres with a large radius R). We keep the points of a Poisson point process contained in A_1, as germs for centers of primary RF. We have $\theta(dx) = \theta 1_{A_1}(x)dx$, where $1_{A_1}(x)$ is the indicator function of the set A_1. Then

$$T(g) = 1 - \varphi_g(\theta)$$

where $\varphi_g(\lambda)$ is the Laplace transform of the positive random variable

$$E_{Z'_t}\{\int_R \overline{\mu}_n(D_{Z'_t}(g) \cap A_1)\ \theta(dt)$$

For a deterministic primary function grain Z'_t, we have to use the change of support of the random set A_1 over the compact set $D_{Z'_t}(g)$), which is easily estimated from simulations. In [13, 14] use is made of of the Beta distribution for the Cox-Boolean model.

For the test function $g(x) = z$ if $x \in K$, and $g(x) = +\infty$ if $x \notin K$, we obtain the distribution of the supremum of $Z(x)$ over the compact set K, $Z_\vee(K)$

$$1 - T(K, z) = \varphi_K(z, 1)$$

where $\varphi_K(z, \lambda)$ is the Laplace transform of the positive random variable $E_{Z'_t}\{\int_R \overline{\mu}_n(A_{Z'_t}(z) \oplus \check{K} \cap A_1)\ \theta(dt)$.

An alternative way to generate multiscale BRF is to use a hierarchical model built from a random tesselation of space given in section (5.2).

5 Exercises

5.1 BRF with cylinder primary random functions

We consider primary RF defined in two steps: start with a compact random set A'_0. To every realization of A'_0 is attributed an independent realization of random variable Z' with distribution function $G(z) = P\{Z' < z\}$. A Boolean islands RF $Z(x)$ with intensity θ is built from this cylinder primary function. Give i) the univariate and bivariate distribution functions of $Z(x)$. Give ii) the distribution function of $Z_\vee(K)$ for this model.

Answer:

i) Using 12, we have $F(z) = P\{Z(x) < z\} = \exp\left(-\theta\overline{\mu}_n(A'_0)(1 - G(z))\right)$. Similarly, using the notation $r(h) = \frac{\overline{\mu}_n(A'_0 \cap A'_{0-h})}{\overline{\mu}_n(A'_0)}$ we get

$$\begin{aligned} F(h, z_1, z_2) &= P\{Z(x) < z_1, Z(x+h) < z_2\} \\ &= F(z_1)F(z_2)F(z_1 \wedge z_2)^{-r(h)} \end{aligned}$$

ii) We have

$$\begin{aligned} P\{Z_\vee(K) &< z\} = \exp\left(-\theta\overline{\mu}_n(A'_0 \oplus \check{K})(1 - G(z))\right) \\ &= F(z)^{\frac{\overline{\mu}_n(A'_0 \oplus \check{K})}{\overline{\mu}_n(A'_0)}} \end{aligned}$$

5.2 A hierarchical BRF model

A RF Z is built in two steps: a random stationary tessellation π of the space R^n delimits classes C_i. In every class C_i is considered a realization Z_i of a stationary Boolean island with primary RF $Z'(x)$ and with the random intensity Y_i. For two classes C_i and C_j the realizations of Z_i, Z_j, Y_i, Y_j are independent. Give for the RF Z, as a function of the statistical properties of π and of the Laplace transform Φ of the positive RV Y, the expressions of: i) the probability law $F(z) = P\{Z(x) < z\}$; ii) the bivariate distribution $F(h, z_1, z_2)$.

Answer:

i) We have $F(z) = P\{x \in A_Z(z)\}$. The restriction of $A_Z(z)$ to every class C_i is a Boolean random set with primary grain $A_{Z'}(z)$ and with intensity Y_i. Given $Y_i = y$, we have

$$F(z) = \exp(-y\overline{\mu}_n(A_{Z'}(z))$$

Taking the mathematical expectation we respect to the random variable Y, we get:

$$F(z) = \Phi(\overline{\mu}_n(A_{Z'}(z)))$$

ii) When $x \in C_i$ and $x + h \in C_i$, we use the bivariate distribution of the BRF with the intensity Y_i. When $x \in C_i$ and $x + h \in C_j$ (with $i \neq j$), the RV $Z(x)$ and $Z(x + h)$ are independent, with univariate distribution functions $F(z_1) = \exp(-y_1\overline{\mu}_n(A_{Z'}(z_1))$ and $F(z_2) = \exp(-y_2\overline{\mu}_n(A_{Z'}(z_2))$. We note

$$r(h) = \frac{\overline{\mu}_n(C \cap C_{-h})}{\overline{\mu}_n(C)}$$

the probability of $x \in C_i$ and $x + h \in C_i$. After deconditioning over π and over Y, we obtain:

$$\begin{aligned} F(h, z_1, z_2) &= r(h)\Phi(\overline{\mu}_n(A_{Z'}(z_1) \cup A_{Z'}(z_2)_h)) \\ &\quad +(1 - r(h))\Phi(\overline{\mu}_n(A_{Z'}(z_1)))\Phi(\overline{\mu}_n(A_{Z'}(z_2))) \end{aligned}$$

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