On stability of discrete-time quantum filters
Pierre Rouchon

To cite this version:
Pierre Rouchon. On stability of discrete-time quantum filters. 2010. hal-00492009

HAL Id: hal-00492009
https://minesparis-psl.hal.science/hal-00492009
Preprint submitted on 14 Jun 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
On stability of discrete-time quantum filters

Pierre Rouchon∗

May 18, 2010

Fidelity is known to increase through a Kraus map: the fidelity between two density matrices is less than the fidelity between their images via a Kraus map. We prove here that, in average, the square of the fidelity is also increasing for a quantum filter: the square of the fidelity between the density matrix of the underlying Markov chain and the density matrix of its associated quantum filter is a super-martingale. Thus discrete-time quantum filters are stable processes and tend to forget their initial conditions.

1 Kraus maps and quantum Markov chains

Take the Hilbert space $S = \mathbb{C}^n$ of dimension $n > 0$ and consider a quantum channel described by the Kraus map (see [3], chapter 4)

$$
\mathcal{K}(\rho) = \sum_{\mu=1}^{m} M_\mu \rho M_\mu^\dagger
$$

(1)

where

- $\rho$ is the density matrix describing the input quantum state, $\mathcal{K}(\rho)$ being then the output quantum state; $\rho \in \mathbb{C}^{n \times n}$ is a density matrix, i.e., an Hermitian matrix semi-positive definite and of trace one;
- for each $\mu \in \{1, \ldots, m\}$, $M_\mu \in \mathbb{C}^{n \times n}/\{0\}$, and $\sum_\mu M_\mu^\dagger M_\mu = I$.

To this quantum channel is associated the following discrete-time Markov chain:

$$
\rho_{k+1} = \mathcal{M}_{\mu_k}(\rho_k)
$$

(2)

where

- $\rho_k$ is the quantum state at sampling time $t_k$ and $k$ the sampling index ($t_k < t_{k+1}$).
- $\mu_k \in \{1, \ldots, m\}$ is a random variable; $\mu_k = \mu$ with probability $p_\mu(\rho_k) = \text{Tr} \left( M_\mu \rho_k M_\mu^\dagger \right)$.

∗Mines ParisTech: pierre.rouchon@mines-paristech.fr
\[ \mathcal{M}_\mu (\rho) = \frac{1}{\text{Tr}(M_\mu \rho M_\mu^\dagger)} M_\mu \rho M_\mu^\dagger = \frac{1}{p_\mu (\rho)} M_\mu \rho M_\mu^\dagger. \]

Kraus maps tend also to increase fidelity, where, for any Hermitian matrix \( A \) with spectrum \( \{ \lambda_i \}_{i \in \{1, \ldots, n\}} \), \( \text{Tr}(|A|) = \sum_{i=1}^{n} |\lambda_i| \). The Kraus map tends also to increase fidelity \( F \) (see [3], theorem 9.6, page 414): for all density matrices \( \rho, \sigma \), one has

\[ \text{Tr} (|\mathcal{K}(\sigma) - \mathcal{K}(\rho)|) \leq \text{Tr} (|\sigma - \rho|) \]

where, for any Hermitian matrix \( A \) with spectrum \( \{ \lambda_i \}_{i \in \{1, \ldots, n\}} \), \( \text{Tr}(|A|) = \sum_{i=1}^{n} |\lambda_i| \). The Kraus map tends also to increase fidelity \( F \) (see [3], theorem 9.6, page 414): for all density matrices \( \rho, \sigma \), one has

\[ \text{Tr} \left( \sqrt{\mathcal{K}(\sigma) \mathcal{K}(\rho)} \sqrt{\mathcal{K}(\sigma)} \right) = F(\mathcal{K}(\sigma), \mathcal{K}(\rho)) \geq F(\sigma, \rho) = \text{Tr} \left( \sqrt{\sigma \sigma} \right) \quad (3) \]

where, for any Hermitian semi-positive matrix \( A = U\Lambda U^\dagger \), \( U \) unitary matrix and \( \Lambda = \text{diag}\{ \lambda_i \}_{i \in \{1, \ldots, n\}} \), \( \sqrt{\Lambda} = U \sqrt{\Lambda} U^\dagger \) with \( \sqrt{\Lambda} = \text{diag}\{ \sqrt{\lambda_i} \}_{i \in \{1, \ldots, n\}} \).

The conditional expectation of \( \rho_{k+1} \) knowing \( \rho_k \) is given by the Kraus map:

\[ \mathbb{E} (\rho_{k+1}/\rho_k) = \mathcal{K}(\rho_k). \]

This result from the trivial identity \( \sum_{\mu=1}^{m} \text{Tr} \left( M_\mu \rho M_\mu^\dagger \right) \frac{M_\mu \rho M_\mu^\dagger}{\text{Tr}(M_\mu \rho M_\mu^\dagger)} = \mathcal{K}(\rho) \). In section 2, we show during the proof of theorem (1) the following inequality

\[ \sum_{\mu=1}^{m} \text{Tr} \left( M_\mu \rho M_\mu^\dagger \right) F^2 \left( \frac{M_\mu \sigma M_\mu^\dagger}{\text{Tr}(M_\mu \sigma M_\mu^\dagger)}, \frac{M_\mu \rho M_\mu^\dagger}{\text{Tr}(M_\mu \rho M_\mu^\dagger)} \right) \geq F^2(\sigma, \rho) \quad (4) \]

for any density matrices \( \rho \) and \( \sigma \). The left-hand side is related to a conditional expectation. Inequality (4), attached to the probabilistic mapping (2), can be seen as the stochastic counter-part of inequality (3) attached to the deterministic mapping (1). When for some \( \mu \), \( \text{Tr} (M_\mu \sigma M_\mu^\dagger) = 0 \) with \( \text{Tr} (M_\mu \rho M_\mu^\dagger) > 0 \), one term in the sum at the left-hand side of (4) is not defined. This is not problematic, since in this case, if we replace \( \frac{M_\mu \sigma M_\mu^\dagger}{\text{Tr}(M_\mu \sigma M_\mu^\dagger)} \) by \( \frac{M_\mu \xi M_\mu^\dagger}{\text{Tr}(M_\mu \xi M_\mu^\dagger)} \) where \( \xi \) is any density matrix such that \( \text{Tr} (M_\mu \xi M_\mu^\dagger) > 0 \), this term is then well defined (in a multi-valued way) and inequality (4) remains satisfied for any such \( \xi \).

During the proof of theorem (3), we extend this inequality to any partition of \( \{1, \ldots, m\} \) into \( p \geq 1 \) sub-sets \( \mathcal{P}_\nu \):

\[ \sum_{\nu=1}^{p} \text{Tr} \left( \sum_{\mu \in \mathcal{P}_\nu} M_\mu \rho M_\mu^\dagger \right) F^2 \left( \frac{\sum_{\mu \in \mathcal{P}_\nu} M_\mu \sigma M_\mu^\dagger}{\text{Tr}(\sum_{\mu \in \mathcal{P}_\nu} M_\mu \sigma M_\mu^\dagger)}, \frac{\sum_{\mu \in \mathcal{P}_\nu} M_\mu \rho M_\mu^\dagger}{\text{Tr}(\sum_{\mu \in \mathcal{P}_\nu} M_\mu \rho M_\mu^\dagger)} \right) \geq F^2(\sigma, \rho) \quad (5) \]
2 The standard case.

Take a realization of the Markov chain associated to the Kraus map $K$. Assume that we detect, for each $k$, the jump $\mu_k$ but that we do not know the initial state $\rho_0$. The objective is to propose at sampling $k$, an estimation $\hat{\rho}_k$ of $\rho_k$ based on the past detections $\mu_0, \ldots, \mu_{k-1}$. The simplest method consists in starting from an initial estimation $\hat{\rho}_0$ and at each sampling step to jump according to the detection. This leads to the following estimation scheme known as a quantum filter:

$$\hat{\rho}_{k+1} = M_{\mu_k}(\hat{\rho}_k)$$

with $p_\mu(\rho_k) = \text{Tr}(M_\mu \rho_k M_\mu)$ as probability of $\mu_k = \mu$. Notice that when $\text{Tr}(M_{\mu_k} \hat{\rho}_k M_{\mu_k}) = 0$, $M_{\mu_k}(\hat{\rho}_k)$ is not defined and should be replaced by $M_{\mu_k}(\xi)$ where $\xi$ is any density matrix such that $\text{Tr}(M_{\mu_k} \xi M_{\mu_k}) > 0$ (take, e.g., $\xi = \frac{1}{n} I_d$). The theorem here below is a first step to investigate the convergence of $\hat{\rho}_k$ towards $\rho_k$ as $k$ increases.

**Theorem 1.** Consider the Markov chain of state $(\rho_k, \hat{\rho}_k)$ satisfying (2) and (6). Then $F^2(\hat{\rho}_k, \rho_k)$ is a super-martingale: $\mathbb{E}(F^2(\hat{\rho}_{k+1}, \rho_{k+1})|(\hat{\rho}_k, \rho_k)) \geq F^2(\hat{\rho}_k, \rho_k)$.

When $\hat{\rho}_k$ or $\rho_k$ are pure states, $\hat{\rho}_{k+1}$ or $\rho_{k+1}$ remain also a pure states. Then, $F^2(\hat{\rho}_k, \rho_k) = \text{Tr}(\hat{\rho}_{k+1} \rho_{k+1}) = \text{Tr}(\hat{\rho}_{k+1} \rho_{k+1})$. In this case, theorem [11] has been proved in [2] using Cauchy-Schwartz inequalities for $m = 2$. The proof proposed here below deals with the general case when both $\rho_k$ and $\hat{\rho}_k$ can be mixed states. It relies on arguments similar to those used for the proof of theorem 9.6 in [3].

**Proof.** $\rho$ and $\hat{\rho}$ are associated to the Hilbert space $S = \mathbb{C}^n$: $\rho$ and $\hat{\rho}$ are operators from $S$ to $S$. Take a copy $Q = \mathbb{C}^n$ of $S$ and consider the composite system living on $S \otimes Q \equiv \mathbb{C}^{n^2}$. Then $\hat{\rho}$ and $\rho$ correspond to partial traces versus $Q$ of projectors $|\hat{\psi}\rangle\langle\hat{\psi}|$ and $|\psi\rangle\langle\psi|$ associated to pure states $|\hat{\psi}\rangle$ and $|\psi\rangle \in S \otimes Q$:

$$\hat{\rho} = \text{Tr}_Q(|\hat{\psi}\rangle\langle\hat{\psi}|), \quad \rho = \text{Tr}_Q(|\psi\rangle\langle\psi|)$$

$|\hat{\psi}\rangle$ and $|\psi\rangle$ are called purifications of $\hat{\rho}$ and $\rho$. They are not unique but one can always choose them such that $F(\hat{\rho}, \rho) = |\langle\hat{\psi}|\psi\rangle|$ (Uhlmann’s theorem).

Denote by $|\hat{\psi}_k\rangle$ and $|\psi_k\rangle$ such purifications of $\hat{\rho}_k$ and $\rho_k$ satisfying $F(\hat{\rho}_k, \rho_k) = |\langle\hat{\psi}_k|\psi_k\rangle|$. We have

$$\mathbb{E}(F^2(\hat{\rho}_{k+1}, \rho_{k+1})|(\hat{\rho}_k, \rho_k)) = \sum_{\mu=1}^{m} p_\mu(\rho_k) F^2(M_{\mu}(\hat{\rho}_k), M_{\mu}(\rho_k))$$

The matrices $M_{\mu}(\hat{\rho}_k)$ and $M_{\mu}(\rho_k)$ are also density matrices. Take the space $S \otimes Q \otimes E$ where $E$ is the Hilbert space of the environment appearing in the system-environment model of the Kraus map [11]. This model is recalled in appendix [A]. It introduced an unitary transformation $U$ on $S \otimes E$. This unitary transformation can be extended to $S \otimes Q \otimes E \equiv S \otimes E \otimes Q$ by setting $V = U \otimes I$ ($I$ is identity on $Q$). Then

$$M_{\mu}(\rho_k) = \text{Tr}_{Q \otimes E}(P_\mu V \langle \psi_k | \psi_k | \otimes | e_0 \rangle \langle e_0 |) V^\dagger P_\mu \rangle.$$
Set $|\phi_k\rangle = |\psi_k\rangle \otimes |e_0\rangle \in S \otimes Q \otimes E$ and $|\chi_k\rangle = V|\phi_k\rangle$. Using $P_{\mu}^2 = P_{\mu}$, we have

$$p_{\mu}(\rho_k) = \text{Tr}(M_{\mu}(\rho_k)) = \langle\phi_k|V^\dagger P_{\mu}V|\phi_k\rangle = \|P_{\mu}|\chi_k\rangle\|^2$$

For each $\mu$, the state $|\chi_{\mu}k\rangle = \frac{1}{\sqrt{p_{\mu}(\rho_k)}} P_{\mu}|\chi_k\rangle$ is a purification of $M_{\mu}(\rho_k)$:

$$M_{\mu}(\rho_k) = \text{Tr}_{Q \otimes E}(|\chi_{\mu}k\rangle \langle \chi_{\mu}k |).$$

Similarly set $|\hat{\phi}_k\rangle = |\hat{\psi}_k\rangle \otimes |e_0\rangle$ and $|\hat{\chi}_k\rangle = V|\hat{\phi}_k\rangle$. For each $\mu$, $|\hat{\chi}_{\mu}k\rangle = \frac{1}{\sqrt{p_{\mu}(\hat{\rho}_k)}} P_{\mu}|\hat{\chi}_k\rangle$ is also a purification of $M_{\mu}(\hat{\rho}_k)$. By Uhlmann’s theorem,

$$F^2(M_{\mu}(\hat{\rho}_k), M_{\mu}(\rho_k)) \geq |\langle \hat{\chi}_{\mu}k | \chi_{\mu}k \rangle|^2.$$ Thus we have

$$\mathbb{E}(F^2(\hat{\rho}_{k+1}, \rho_{k+1})/(\hat{\rho}_k, \rho_k)) \geq \sum_{\mu=1}^{m} p_{\mu}(\rho_k) \ |\langle \hat{\chi}_{\mu}k | \chi_{\mu}k \rangle|^2.$$

Since $V$ is unitary,

$$|\langle \hat{\chi}_{k} | \chi_{k} \rangle|^2 = |\langle \hat{\phi}_k | \phi_k \rangle|^2 = |\langle \hat{\psi}_k | \psi_k \rangle|^2 = F^2(\hat{\rho}_k, \rho_k).$$

Let us show that $\sum_{\mu=1}^{m} p_{\mu}(\rho_k) \ |\langle \hat{\chi}_{\mu}k | \chi_{\mu}k \rangle|^2 \geq |\langle \hat{\chi}_k | \chi_k \rangle|^2$. We have

$$p_{\mu}(\rho_k) \ |\langle \hat{\chi}_{\mu}k | \chi_{\mu}k \rangle|^2 = |\langle \hat{\chi}_{\mu}k | M_{\mu} \chi_{\mu}k \rangle|^2 = |\langle \hat{\chi}_k | M_{\mu} \chi_{\mu}k \rangle|^2,$$

thus it is enough to prove that $\sum_{\mu=1}^{m} |\langle \hat{\chi}_{\mu}k | \chi_{\mu}k \rangle|^2 \geq |\langle \hat{\chi}_k | \chi_k \rangle|^2$. Denote by $\hat{R} \subset S \otimes Q \otimes E$ the vector space spanned by the ortho-normal basis $\{ |\hat{\chi}_{\mu}k\rangle \}_{\mu \in \{1, \ldots, m\}}$ and by $\hat{P}$ the projector on $\hat{R}$. Since

$$|\hat{\chi}_k\rangle = \sum_{\mu=1}^{m} P_{\mu} |\hat{\chi}_k\rangle = \sum_{\mu=1}^{m} \sqrt{p_{\mu}(\hat{\rho}_k)} |\hat{\chi}_{\mu}k\rangle,$$

$|\hat{\chi}_k\rangle$ belongs to $\hat{R}$ and thus $|\langle \hat{\chi}_k | \chi_k \rangle|^2 = |\langle \hat{\chi}_k | \hat{P} | \chi_k \rangle \rangle|^2$. We conclude by Cauchy-Schwartz inequality

$$|\langle \hat{\chi}_k | \chi_k \rangle|^2 = |\langle \hat{\chi}_k | \hat{P} | \chi_k \rangle \rangle|^2 \leq ||\hat{\chi}_k||^2 ||\hat{P} | \chi_k \rangle \rangle^2 = ||\hat{P} | \chi_k \rangle \rangle^2 = \sum_{\mu=1}^{m} |\langle \hat{\chi}_{\mu}k | \chi_{\mu}k \rangle|^2.$$

$\square$

### 3 The aggregated case.

Let us consider another Markov chain attached to the same Kraus map (1) and associated to a partition of $\{1, \ldots, m\}$ into $p \geq 1$ sub-sets $P_\nu$ (aggregation of several quantum jumps via "partial Kraus maps"):

$$\rho_{k+1} = \frac{1}{\text{Tr}(\sum_{\mu \in P_\nu k} M_{\mu} \rho_k M_{\mu}^\dagger)} \left( \sum_{\mu \in P_\nu k} M_{\mu} \rho_k M_{\mu}^\dagger \right) \quad (7)$$
where \( \nu_k = \nu \) with probability \( \text{Tr} \left( \sum_{\mu \in P_{\nu}} M_\mu \rho_k M_\mu^\dagger \right) \). Consider the associated quantum filter

\[
\hat{\rho}_{k+1} = \frac{1}{\text{Tr} \left( \sum_{\mu \in P_{\nu}} M_\mu \hat{\rho}_k M_\mu^\dagger \right)} \left( \sum_{\mu \in P_{\nu}} M_\mu \hat{\rho}_k M_\mu^\dagger \right) \tag{8}
\]

where the jump index \( \nu_k \) coincides with the jump index \( \nu_k \) in (7). Then we have the following theorem.

**Theorem 2.** Consider the Markov chain of state \((\rho_k, \hat{\rho}_k)\) satisfying (7) and (8). Then \( F^2(\hat{\rho}_k, \rho_k) \) is a super-martingale: \( \mathbb{E} \left( F^2(\hat{\rho}_{k+1}, \rho_{k+1})/(\hat{\rho}_k, \rho_k) \right) \geq F^2(\hat{\rho}_k, \rho_k) \).

**Proof.** It is similar to the proof of theorem 1. We will just point out here the main changes using the same notations. We start from

\[
\mathbb{E} \left( F^2(\hat{\rho}_{k+1}, \rho_{k+1})/(\hat{\rho}_k, \rho_k) \right) = \sum_{\nu=1}^{p} \tilde{p}_\nu(\rho_k) F^2(\bar{M}_\nu(\hat{\rho}_k), \bar{M}_\nu(\rho_k)).
\]

where we have set

\[
\tilde{p}_\nu(\rho) = \text{Tr} \left( \sum_{\mu \in P_{\nu}} M_\mu \rho M_\mu^\dagger \right), \quad \bar{M}_\nu(\rho) = \frac{1}{\tilde{p}_\nu(\rho)} \left( \sum_{\mu \in P_{\nu}} M_\mu \rho M_\mu^\dagger \right).
\]

With \( \hat{P}_\nu \) the orthogonal projector on \( S \otimes Q \otimes \text{span}\{\ket{\mu}, \mu \in P_{\nu}\} \) and \( \bar{M}_\nu(\rho) = \sum_{\mu \in P_{\nu}} M_\mu \rho M_\mu^\dagger \), we have

\[
\bar{M}_\nu(\rho_k) = \text{Tr}_{Q \otimes E} \left( \hat{P}_\nu \left( \ket{\psi_k} \bra{\psi_k} \otimes \ket{e_0} \bra{e_0} \right) V^\dagger \hat{P}_\nu \right)
\]

and

\[
\tilde{p}_\nu(\rho_k) = \text{Tr} \left( \bar{M}_\nu(\rho_k) \right) = \langle \phi_k | V^\dagger \hat{P}_\nu V | \phi_k \rangle = \| \hat{P}_\nu | \chi_k \rangle \|^2
\]

For each \( \nu \), the state \( \ket{\tilde{x}_{k\nu}} = \frac{1}{\sqrt{\tilde{p}_\nu(\rho_k)}} \hat{P}_\nu \ket{\chi_k} \) is a purification of \( \bar{M}_\nu(\hat{\rho}_k) \):

\[
\bar{M}_\nu(\rho_k) = \text{Tr}_{Q \otimes E} \left( \ket{\tilde{x}_{k\nu}} \bra{\tilde{x}_{k\nu}} \right).
\]

Similarly \( \ket{\hat{x}_{k\nu}} = \frac{1}{\sqrt{\tilde{p}_\nu(\rho_k)}} \hat{P}_\nu \ket{\chi_k} \) is also a purification of \( \bar{M}_\nu(\hat{\rho}_k) \). By Uhlmann’s theorem,

\[
F^2(\bar{M}_\nu(\hat{\rho}_k), \bar{M}_\nu(\rho_k)) \geq \| \langle \hat{x}_{k\nu} | \tilde{x}_{k\nu} \rangle \|^2.
\]

Thus we have

\[
\mathbb{E} \left( F^2(\hat{\rho}_{k+1}, \rho_{k+1})/(\hat{\rho}_k, \rho_k) \right) \geq \sum_{\nu=1}^{p} \tilde{p}_\nu(\rho_k) \| \langle \hat{x}_{k\mu} | \tilde{x}_{k\mu} \rangle \|^2.
\]

Let us show that \( \sum_{\nu=1}^{p} \tilde{p}_\nu(\rho_k) \| \langle \hat{x}_{k\nu} | \tilde{x}_{k\nu} \rangle \|^2 \geq \| \hat{x}_k \ket{\chi_k} \|^2 = F^2(\hat{\rho}_k, \rho_k) \). We have

\[
\tilde{p}_\nu(\rho_k) \| \langle \hat{x}_{k\nu} | \tilde{x}_{k\nu} \rangle \|^2 = \| \langle \hat{x}_{k\nu} | \hat{P}_\nu \ket{\chi_k} \|^2 = \| \hat{P}_\nu \ket{\chi_k} \|^2,
\]
thus it is enough to prove that \(\sum_{\nu=1}^{p} |\langle \hat{\chi}_{k\nu} | \chi_k \rangle|^2 \geq |\langle \hat{\chi}_k | \chi_k \rangle|^2\). Denote \(\hat{R} \subset S \otimes Q \otimes E\) the vector space spanned by the ortho-normal basis \(\{|\hat{\chi}_{k\nu}\rangle\}_{\nu \in \{1,\ldots,p\}}\) and by \(\hat{P}\) the projector on \(\hat{R}\). Since

\[
|\hat{\chi}_k\rangle = \sum_{\nu=1}^{p} \hat{P}_\nu |\hat{\chi}_k\rangle = \sum_{\nu=1}^{p} \sqrt{\hat{p}_\nu} (\hat{\rho}_k) |\hat{\chi}_{k\nu}\rangle
\]

|\hat{\chi}_k\rangle \text{ belongs to } \hat{R} \text{ and thus } |\langle \hat{\chi}_k | \chi_k \rangle|^2 = |\langle \hat{\chi}_k | \hat{P} | \chi_k \rangle|^2\). We conclude by Cauchy-Schwartz inequality

\[
|\langle \hat{\chi}_k | \chi_k \rangle|^2 = |\langle \hat{\chi}_k | \hat{P} | \chi_k \rangle|^2 \leq \|\hat{\chi}_k\|^2 \|\hat{P} | \chi_k \rangle\|^2 = \sum_{\nu=1}^{p} |\langle \hat{\chi}_{k\nu} | \chi_k \rangle|^2.
\]

\[
|\langle \hat{\chi}_k | \chi_k \rangle|^2 = |\langle \hat{\chi}_k | \hat{P} | \chi_k \rangle|^2 = \sum_{\nu=1}^{p} |\langle \hat{\chi}_{k\nu} | \chi_k \rangle|^2.
\]

4 Concluding remarks

Theorems 1 and 2 are still valid if the Kraus operators \(M_\mu\) depend on \(k\). In particular, \(F(\hat{\rho}_k, \rho_k)\) remains a super-martingale even if the Kraus operators depend on \(\hat{\rho}_k\), i.e., in case of feedback.

When \(\sigma\) and \(\rho\) are pure states (projectors of rank one), \(D(\sigma, \rho) = \sqrt{1 - F^2(\sigma, \rho)}\). Consequently inequality (1) yields to

\[
\sum_{\mu=1}^{m} \text{Tr} \left( M_\mu \rho M_\mu^* \right) D \left( \frac{M_\mu \sigma M_\mu^*}{\text{Tr}(M_\mu \sigma M_\mu^*)} \right) \left( \frac{M_\mu \rho M_\mu^*}{\text{Tr}(M_\mu \rho M_\mu^*)} \right) \leq D(\sigma, \rho)
\]

for any pure states \(\sigma\) and \(\rho\) (use the fact that \([0, x] \ni x \mapsto \sqrt{1 - x}\) is decreasing and concave). We conjecture that such inequality hold also true for any mixed states and that \(D(\hat{\rho}_k, \rho_k) = \text{Tr} (|\hat{\rho}_k - \rho_k|)\) is a sub-martingale.

References


A  System-environment model

The quantum channel associated to the Kraus map (1) or the Markov chain (2) admits a system-environment model (see [1], chapter 4 entitled "The environment is watching"). Take the Hilbert space $E = \mathbb{C}^m$ associated to the environment and the composite system living on $S \otimes E$. Take a pure state $|\phi_k\rangle \in S$ and its density matrix $\rho_k = |\phi_k\rangle \langle \phi_k|$. Assume that before detection $\mu_k$ at step $k$, the composite system admits the pure state $|\phi_k\rangle \otimes |e_0\rangle$ where $|e_0\rangle$ is an environment pure state. Take $m$ states $|\mu\rangle$ forming an orthogonal base of $E$. Then exists a unitary transformation $U$ (not unique) of $S \otimes E$ such that, for all $|\phi\rangle \in S$,

$$U (|\phi\rangle \otimes |e_0\rangle) = \sum_{\mu=1}^{m} (M_\mu |\phi\rangle) \otimes |\mu\rangle.$$ 

This is a direct consequence of $\sum_{\mu=1}^{n} M_\mu^\dagger M_\mu = I$. For each $\mu$, denote by $P_\mu$ the orthogonal projector onto the subspace $S \otimes (\mathbb{C}|\mu\rangle)$. Then $P_\mu U (|\phi\rangle \otimes |e_0\rangle) = (M_\mu |\phi\rangle) \otimes |\mu\rangle$ and $\sum_{\mu} P_\mu = I$. We can then verify that for any density matrix $\rho$ associated to a state in $R$,

$$P_\mu U (\rho \otimes |e_0\rangle \langle e_0|) U^\dagger P_\mu = M_\mu \rho M_\mu^\dagger \otimes |\mu\rangle \langle \mu|$$

and thus

$$\text{Tr}_E (P_\mu U (\rho \otimes |e_0\rangle \langle e_0|) U^\dagger P_\mu) = M_\mu \rho M_\mu^\dagger.$$