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Feedback generation of quantum Fock states by discrete QND measures

Mazyar Mirrahimi, Igor Dotsenko and Pierre Rouchon

Abstract— A feedback scheme for preparation of photon number states in a microwave cavity is proposed. Quantum Non Demolition (QND) measurement of the cavity field provides information on its actual state. The control consists in injecting into the cavity mode a microwave pulse adjusted to increase the population of the desired target photon number. In the ideal case (perfect cavity and measures), we present the feedback scheme and its detailed convergence proof through stochastic Lyapunov techniques based on super-martingales and other probabilistic arguments. Quantum Monte-Carlo simulations performed with experimental parameters illustrate convergence and robustness of such feedback scheme.

I. INTRODUCTION

In [10], [5], [4] QND measures are exploited to detect and/or produce highly non-classical states of light trapped in a super-conducting cavity (see [6, chapter 5] for a description of such QED systems and [1] for detailed physical models with QND measures of light using atoms). For such experimental setups, we detail and analyze here a feedback scheme that stabilize the cavity field towards any photon-number states (Fock states). Such states are strongly non-classical since their photon numbers are perfectly defined. The control corresponds to a coherent light-pulse injected inside the cavity between atom passages. The overall structure of the proposed feedback scheme is inspired of [3] using a quantum adaptation of the observer/controller structure widely used for classical systems (see, e.g., [7, chapter 4]). The observer part of the proposed feedback scheme consists in a discrete-time quantum filter, based on the observed detector clicks, to estimate the quantum-state of the cavity field. This estimated state is then used in a state-feedback based on Lyapunov design, the controller part. In theorems 1 and 2 we prove the convergence almost surely of the closed-loop system towards the goal Fock-state in absence of modeling imperfections and measurement errors. Simulations illustrate this results and show that performance of the closed-loop system are not dramatically changed by false detections for 10% of the detector clicks. In [2] similar feedback schemes are also addressed with modified quantum filters in order to take into account additional physical effects and experimental imperfections. [2] focuses on physics and includes extensive

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closed-loop simulations whereas here we are interested by mathematical aspects and convergence proofs.

In section II, we describe very briefly the physical system and its quantum Monte-Carlo model. In section III the feedback is designed using Lyapunov techniques. Its convergence is proved in theorem 1. Section IV introduces a quantum filter to estimate the cavity state necessary for the feedback: convergence of the closed-loop system (quantum filter and feedback based on the estimate cavity state) is proved in theorem 2 assuming perfect model and detection. This section ends with Theorem 3 proving a contraction property of the quantum filter dynamics. Section V is devoted to closed-loop simulations where measurement imperfections are introduced for testing robustness.

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II. THE PHYSICAL SYSTEM AND ITS JUMP DYNAMICS

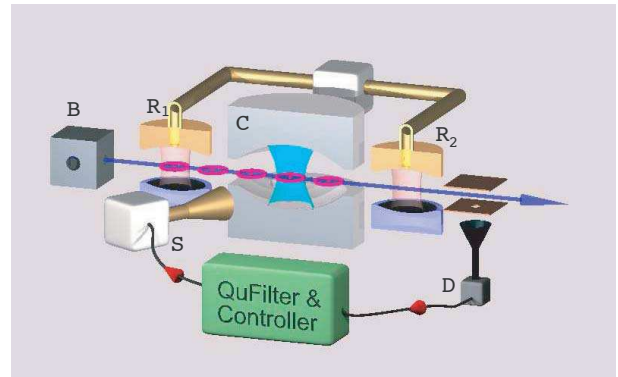


Fig. 1. The microwave cavity QED setup with its feedback scheme (in green).

As illustrated by figure 1, the system consists in C a high-Q microwave cavity, in B a box producing Rydberg atoms, in R_1 and R_2 two low-Q Ramsey cavities, in D an atom detector and in S a microwave source. The dynamics model is discrete in time and relies on quantum Monte-Carlo trajectories (see [6, chapter 4]). It takes into account the back-action of the measure. It is adapted from [5] where we have just added the control effect.

Each time-step indexed by the integer k corresponds to atom number k coming from B , submitted then to a first Ramsey $\pi/2$ -pulse in R_1 , crossing the cavity C and being entangled with it, submitted to a second $\pi/2$ -pulse in R_2 and finally being measured in D . The state of the cavity is described by the density operator ρ_k . Here the passage from the time step k to $k + 1$ corresponds to the projective

measurement of the cavity state, by detecting the state of the Rydberg atom number k after leaving R_2 . During this same step, an appropriate coherent pulse (the control) is injected into C . Denoting, as usual, by a the photon annihilation operator and by $N = a^\dagger a$ the photon number operator, the density matrix ρ_{k+1} is related to ρ_k through the following jump-relationships: $\rho_{k+1} = \frac{D(\alpha_k)M_k\rho_k M_k^\dagger D(-\alpha_k)}{\text{Tr}(M_k\rho_k M_k^\dagger)}$ where

- the measurement operator $M_k = M_g$ (resp. $M_k = M_e$), when the atom k is detected in the state $|g\rangle$ (resp. $|e\rangle$) with

$$M_g = \cos\left(\frac{\phi_R + \phi}{2} + N\phi\right), \quad M_e = \sin\left(\frac{\phi_R + \phi}{2} + N\phi\right). \quad (1)$$

Such measurement process corresponds to an off-resonant interaction between atom k and cavity where ϕ_R is the direction of the second Ramsey $\pi/2$ -pulse (R_2 in figure 1) and ϕ is the de-phasing-angle per photon.

- The probability $P_{g,k}$ (resp. $P_{e,k}$) of detecting the atom k in $|g\rangle$ (resp. $|e\rangle$) is given by $\text{Tr}(M_g\rho_k M_g)$ (resp. $\text{Tr}(M_e\rho_k M_e)$).
- $D(\alpha_k)$ is the displacement operator describing the coherent pulse injection, $D(\alpha_k) = \exp(\alpha_k(a^\dagger - a))$, and the scalar control α_k is a real parameter that can be adjusted at each time step k .

The time evolution of the step k to $k+1$, in fact, consists of two types of evolutions: a projective measurement and a coherent injection. For simplicity sakes, we will use the notation of $\rho_{k+\frac{1}{2}}$, to illustrate this intermediate step. Therefore,

$$\rho_{k+\frac{1}{2}} = \frac{M_k\rho_k M_k^\dagger}{\text{Tr}(M_k\rho_k M_k^\dagger)}, \quad \rho_{k+1} = D(\alpha_k)\rho_{k+\frac{1}{2}}D(-\alpha_k) \quad (2)$$

In the sequel, we consider the finite dimensional approximation defined by a maximum of photon number, n^{\max} . In the truncated Fock basis $(|n\rangle)_{0 \leq n \leq n^{\max}}$, N corresponds to the diagonal matrix $(\text{diag}(n))_{0 \leq n \leq n^{\max}}$, ρ is a $(n^{\max} + 1) \times (n^{\max} + 1)$ symmetric positive matrix with unit trace, and the annihilation operator a is an upper-triangular matrix with $(\sqrt{n})_{1 \leq n \leq n^{\max}}$ as upper diagonal, the remaining elements being 0. We restrict to real quantities since the phase of any Fock state is arbitrary. We set it here to 0.

III. FEEDBACK SCHEME AND CONVERGENCE PROOF

We aim to stabilize the Fock state with \bar{n} photons characterized by the density operator $\bar{\rho} = |\bar{n}\rangle\langle\bar{n}|$. To this end we choose the coherent feedback α_k such that the value of the Lyapunov function $V(\rho) = 1 - \text{Tr}(\rho\bar{\rho})$ decreases when passing from $\rho_{k+\frac{1}{2}}$ to ρ_{k+1} . Note that, for α small enough, the Baker-Campbell-Hausdorff formula yields the following approximation

$$D(\alpha)\rho D(-\alpha) \approx \rho - \alpha[\rho, a^\dagger - a] + \frac{\alpha^2}{2}[[\rho, a^\dagger - a], a^\dagger - a] \quad (3)$$

up to third order terms. Therefore, for α_k small enough, we have

$$\begin{aligned} \text{Tr}\left(D(\alpha_k)\rho_{k+\frac{1}{2}}D(-\alpha_k)\bar{\rho}\right) &= \\ &= \text{Tr}\left(\rho_{k+\frac{1}{2}}\bar{\rho}\right) - \alpha_k \text{Tr}\left([\rho_{k+\frac{1}{2}}, a^\dagger - a]\bar{\rho}\right) \\ &\quad + \frac{\alpha_k^2}{2} \text{Tr}\left([\rho_{k+\frac{1}{2}}, a^\dagger - a], a^\dagger - a]\bar{\rho}\right). \end{aligned}$$

Thus the feedback

$$\alpha_k = c_1 \text{Tr}\left([\bar{\rho}, a^\dagger - a]\rho_{k+\frac{1}{2}}\right) \quad (4)$$

with a gain $c_1 > 0$ small enough ensures that

$$\text{Tr}(\bar{\rho}\rho_{k+1}) - \text{Tr}(\bar{\rho}\rho_{k+\frac{1}{2}}) \geq \frac{c_1}{2} \left| \text{Tr}\left([\bar{\rho}, a^\dagger - a]\rho_{k+\frac{1}{2}}\right) \right|^2, \quad (5)$$

since $\text{Tr}\left([\rho_{k+\frac{1}{2}}, a^\dagger - a]\bar{\rho}\right) = -\text{Tr}\left([\bar{\rho}, a^\dagger - a]\rho_{k+\frac{1}{2}}\right)$. Furthermore, the conditional expectation of $\text{Tr}\left(\bar{\rho}\rho_{k+\frac{1}{2}}\right)$ knowing ρ_k is given by

$$\begin{aligned} \mathbb{E}\left(\text{Tr}\left(\bar{\rho}\rho_{k+\frac{1}{2}}\right) \mid \rho_k\right) &= P_{g,k} \text{Tr}\left(\frac{\bar{\rho}M_g\rho_k M_g^\dagger}{P_{g,k}}\right) \\ &\quad + P_{e,k} \text{Tr}\left(\frac{\bar{\rho}M_e\rho_k M_e^\dagger}{P_{e,k}}\right) = \text{Tr}(\bar{\rho}\rho_k) \end{aligned}$$

since $[\bar{\rho}, M_g] = [\bar{\rho}, M_e] = 0$ and $M_g^\dagger M_g + M_e^\dagger M_e = 1$. Thus

$$\mathbb{E}\left(\text{Tr}\left(\bar{\rho}\rho_{k+1}\right) \mid \rho_k\right) \geq \mathbb{E}\left(\text{Tr}\left(\bar{\rho}\rho_{k+\frac{1}{2}}\right) \mid \rho_k\right) = \text{Tr}(\bar{\rho}\rho_k)$$

and consequently, the expectation value of $V(\rho_k)$ decreases at each sampling time:

$$\mathbb{E}(V(\rho_{k+1})) \leq \mathbb{E}(V(\rho_k)). \quad (6)$$

Considering the Markov process ρ_k , we have therefore shown that $V(\rho_k)$ is a super-martingale bounded from below by 0. When V reaches its minimum 0, the Markov process ρ_k has converged to $\bar{\rho}$. However, one can easily see that this super-martingale has also the possibility to converge towards other attractors, for instance other Fock states which are all the stationary points of the closed-loop Markov process but with $V(\rho) = 1$ instead of 0. Following [9], we suggest the following modification of the feedback scheme:

$$\alpha_k = \begin{cases} c_1 \text{Tr}\left([\bar{\rho}, a^\dagger - a]\rho_{k+\frac{1}{2}}\right) & \text{if } V(\rho_k) \leq 1 - \varepsilon \\ \underset{\alpha \in [-\bar{\alpha}, \bar{\alpha}]}{\text{argmax}} \text{Tr}\left(\bar{\rho}D(\alpha)\rho_{k+\frac{1}{2}}D(-\alpha)\right) & \text{if } V(\rho_k) > 1 - \varepsilon \end{cases} \quad (7)$$

with $c_1, \varepsilon, \bar{\alpha} > 0$ constants.

Theorem 1: Consider (2) and assume that for all $n \in \{0, \dots, n^{\max}\}$ we have $\frac{\phi_R + \phi}{2} + n\phi \neq 0 \pmod{(\pi/2)}$ and that

$$\#\left\{\cos^2\left(\frac{\phi_R + \phi}{2} + n\phi\right) \mid n \in \{0, \dots, n^{\max}\}\right\} = n^{\max} + 1.$$

Take the switching feedback scheme (7) with $\bar{\alpha} > 0$. For small enough $c_1 > 0$ and $\varepsilon > 0$, the trajectories of (2) converge almost surely towards the target Fock state $\bar{\rho}$.

Remark 1: The second part of the feedback (7), dealing with states near the bad attractors, is not explicit and may seem hard to compute. Note that, this form has been particularly chosen to simplify the proof of the Theorem 1 and

in practice, one can take it to be any constant control field exciting the system around these bad attractors and ensuring a fast return to the inner set.

Remark 2: The controller gain c_1 can be tuned in order to maximize at each sampling time k , $\text{Tr}\left(D(\alpha_k)\rho_{k+\frac{1}{2}}D(-\alpha_k)\bar{\rho}\right)$ for $\rho_{k+\frac{1}{2}}$ near $\bar{\rho}$. Up to third order term in $\rho_{k+\frac{1}{2}} - \bar{\rho}$, (3) yields to

$$\begin{aligned} \text{Tr}\left(D(\alpha_k)\rho_{k+\frac{1}{2}}D(-\alpha_k)\bar{\rho}\right) &= \text{Tr}\left(\bar{\rho}\rho_{k+\frac{1}{2}}\right) + \\ &\left(\text{Tr}\left([\bar{\rho}, a^\dagger - a]\rho_{k+\frac{1}{2}}\right)\right)^2 \left(c_1 - \frac{c_1^2}{2}\text{Tr}\left([\bar{\rho}, a^\dagger - a][\bar{\rho}, a^\dagger - a]\right)\right). \end{aligned}$$

Thus $c_1 = 1/\text{Tr}\left([\bar{\rho}, a^\dagger - a][\bar{\rho}, a^\dagger - a]\right) \approx 1/(4\bar{n} + 2)$ for $n^{\max} \gg \bar{n}$ implies a maximum decrease at the sampling time, up to third-order terms in $\rho_k - \bar{\rho}$.

In order to prove the Theorem 1, we need some classical tools from stochastic processes namely the Doob's inequality and the Kushner's asymptotic invariance Theorem [8]. These results are been recalled in the Appendix.

Proof of Theorem 1. It is divided in 3 steps: in a first step, we show that for small enough ε and by applying the second part of the feedback scheme, the trajectories starting within the set $\{\rho \mid V(\rho) > 1 - \varepsilon\}$ reach in one step the set $\{\rho \mid V(\rho) \leq 1 - 2\varepsilon\}$ and this with probability 1; next, we show that trajectories starting within the set $\{\rho \mid V(\rho) \leq 1 - 2\varepsilon\}$, will never hit the set $\{\rho \mid V(\rho) > 1 - \varepsilon\}$ with a uniformly non-zero probability $p > 0$; finally, we will show that, the trajectories of the quantum filter converge towards $\bar{\rho}$ for almost all trajectories that never hit the set $\{\rho \mid V(\rho) > 1 - \varepsilon\}$. This is then an immediate conclusion of the Markov property that the trajectories of the quantum filter with the feedback scheme (7) will converge almost surely towards $\bar{\rho}$.

Step 1: We start by considering the process starting on the level set $\{\rho \mid V(\rho) = 1\}$. We have the following lemma:

Lemma 1: Consider ρ a well-defined density matrix such that $\text{Tr}(\rho\bar{\rho}) = 0$. We have

$$\min_{s \in \{g, e\}} \max_{\alpha \in [-\bar{\alpha}, \bar{\alpha}]} \frac{\text{Tr}(\bar{\rho}D(\alpha)M_s\rho M_s^\dagger D(-\alpha))}{\text{Tr}(M_s\rho M_s^\dagger)} > 0.$$

We denote any argument of the above min-max problem by $\bar{\alpha}(\rho) \in [-\bar{\alpha}, \bar{\alpha}]$.

Proof of Lemma 1: Define $\rho_s = \frac{M_s\rho M_s^\dagger}{\text{Tr}(M_s\rho M_s^\dagger)}$, $s \in \{g, e\}$. The matrices M_g and M_e being diagonal, we trivially have $\text{Tr}(\rho_s\bar{\rho}) = 0$. Let us fix s and assume that for all $\alpha \in [-\bar{\alpha}, \bar{\alpha}]$,

$$\text{Tr}(\bar{\rho}D(\alpha)\rho_s D(-\alpha)) = 0. \quad (8)$$

Decomposing ρ_s as a sum of projectors we have $\rho_s = \sum_{k=1}^m \lambda_{k,s} |\psi_{k,s}\rangle \langle \psi_{k,s}|$, where $\lambda_{k,s}$ are strictly positive eigenvalues and $\psi_{k,s}$ are the associated normalized eigenstates of ρ_s ($m = 1$ corresponds to the case where ρ_s is a projector). The equation (8), clearly, implies

$$\langle \psi_{k,s} \mid D(-\alpha)\bar{n} \rangle = 0, \quad \forall k \in \{1, \dots, m\}, \forall \alpha \in [-\bar{\alpha}, \bar{\alpha}]. \quad (9)$$

Fixing one $k \in \{1, \dots, m\}$ and taking $\psi = \psi_{k,s}$, noting that $D(-\alpha) = \exp(-\alpha(a^\dagger - a))$ and deriving j times versus α around 0 we get

$$\langle \psi \mid (a^\dagger - a)^j \bar{n} \rangle = 0, \quad \forall j = 0, \dots, n^{\max}. \quad (10)$$

But the family $((a^\dagger - a)^j \bar{n})_{0 \leq j \leq n^{\max}}$ is full rank. This is a direct consequence of [11, Theorem 4]. It is proved there that the truncated harmonic oscillator $\frac{d}{dt} |\phi\rangle_t = -(iN + v(t)(a^\dagger - a)) |\phi\rangle_t$, is completely controllable with the single scalar control $v(t)$. If the rank r of $((a^\dagger - a)^p |\bar{n}\rangle)_{0 \leq p \leq n^{\max}}$ is strictly less than $n^{\max} + 1$, then according to Cayley-Hamilton Theorem the rank of the infinite family $((a^\dagger - a)^p |\bar{n}\rangle)_{p \geq 0}$ is also r . Take $|\xi\rangle$, a state orthogonal to this family. For any control $v(t)$, the state $|\phi\rangle_t$ starting from $|\bar{n}\rangle$ remains orthogonal to $|\xi\rangle$. Thus it will be impossible to find a control $v(t)$ steering $|\phi\rangle_t$ from $|\bar{n}\rangle$ to $|\xi\rangle$.

Since the rank of $((a^\dagger - a)^p |\bar{n}\rangle)_{0 \leq p \leq n^{\max}}$ is maximum, (10) implies $|\psi_k\rangle = 0$ and leads to a contradiction. \square

Applying the compactness of the space of density matrices, we directly have the following corollary:

Corollary 1: There exists an $\varepsilon > 0$ such that

$$\inf_{\rho \in \{\text{Tr}(\rho\bar{\rho}) < \varepsilon\}} \frac{\text{Tr}(\bar{\rho}D(\bar{\alpha}(\rho))M_s\rho M_s^\dagger D(-\bar{\alpha}(\rho)))}{\text{Tr}(M_s\rho M_s^\dagger)} > 2\varepsilon \quad (11)$$

for $s = g, e$ and where $\bar{\alpha}(\rho)$ is defined in Lemma 1.

Proof of Corollary 1: We take

$$\delta = \inf_{\rho \in \{\text{Tr}(\rho\bar{\rho}) = 0\}} \min_{s \in \{g, e\}} \frac{\text{Tr}(\bar{\rho}D(\bar{\alpha}(\rho))M_s\rho M_s^\dagger D(-\bar{\alpha}(\rho)))}{\text{Tr}(M_s\rho M_s^\dagger)}.$$

By Lemma 1 and the compactness of the set $\{\rho \mid \text{Tr}(\rho\bar{\rho}) = 0\}$, we know that $\delta > 0$. By continuity of $\text{Tr}(\rho\bar{\rho})$ with respect to ρ and by compactness of the space of density matrices, there exists $\gamma > 0$ such that

$$\inf_{\rho \in \{\text{Tr}(\rho\bar{\rho}) < \gamma\}} \min_{s \in \{g, e\}} \frac{\text{Tr}(\bar{\rho}D(\bar{\alpha}(\rho))M_s\rho M_s^\dagger D(-\bar{\alpha}(\rho)))}{\text{Tr}(M_s\rho M_s^\dagger)} > \frac{\delta}{2}.$$

Therefore, by taking $\varepsilon = \min(\gamma, \delta/4)$, clearly, (11) holds true. \square

Through this corollary, we have shown that whenever the Markov process hits the set $\{\text{Tr}(\rho\bar{\rho}) < \varepsilon\}$, it is immediately rebounded to the set $\{\text{Tr}(\rho\bar{\rho}) > 2\varepsilon\}$ and this with probability 1.

Step 2: Let us assume that the process starts within the set $\{\text{Tr}(\rho\bar{\rho}) > 2\varepsilon\}$.

Lemma 2: Initializing the Markov process within the set $\{\rho \mid V(\rho) \leq 1 - 2\varepsilon\}$, ρ_k will never hit the set $\{\rho \mid V(\rho) > 1 - \varepsilon\}$ with a probability $p > \frac{\varepsilon}{1 - \varepsilon} > 0$.

Proof of Lemma 2: By (6), the process $V(\rho_k)$ is, clearly, a supermartingale. One only needs to use the Doobs inequality (cf. Appendix, Theorem 4) and we have

$$P\left(\sup_{0 \leq k < \infty} V(\rho_k) > 1 - \varepsilon\right) < \frac{V(\rho_0)}{1 - \varepsilon} \leq \frac{1 - 2\varepsilon}{1 - \varepsilon},$$

and thus $p > 1 - (1 - 2\varepsilon)/(1 - \varepsilon) = \varepsilon/(1 - \varepsilon)$. \square

We have shown that starting within the inner set $\{\text{Tr}(\rho\bar{\rho}) \geq 2\varepsilon\}$ there is a uniform non-zero probability of $\varepsilon/(1 - \varepsilon)$ for the process, to never hit the outer set $\{\text{Tr}(\rho\bar{\rho}) < \varepsilon\}$.

Step 3:

Lemma 3: The sample paths ρ_k remaining into the set $\{\text{Tr}(\rho\bar{\rho}) > \varepsilon\}$ converge in probability to $\bar{\rho}$ as $k \rightarrow \infty$.

Proof of Lemma 3: Consider the function $\mathcal{W}(\rho) = 1 - \text{Tr}(\rho\bar{\rho})^2$. For $s = g, e$, set $\rho_s = \frac{M_s\rho M_s^\dagger}{\text{Tr}(M_s\rho M_s^\dagger)}$. We have

$$\begin{aligned}\mathcal{W}(\rho_g) &= 1 - \frac{\text{Tr}(\rho M_g^\dagger \bar{\rho} M_g)^2}{\text{Tr}(M_g \rho M_g^\dagger)^2}, \\ &= 1 - \frac{|\cos\left(\frac{\phi_R + \phi}{2} + \bar{n}\phi\right)|^4}{\text{Tr}(M_g \rho M_g^\dagger)^2} \text{Tr}(\rho\bar{\rho})^2,\end{aligned}\quad (12)$$

and similarly

$$\mathcal{W}(\rho_e) = 1 - \frac{|\sin\left(\frac{\phi_R + \phi}{2} + \bar{n}\phi\right)|^4}{\text{Tr}(M_e \rho M_e^\dagger)^2} \text{Tr}(\rho\bar{\rho})^2.\quad (13)$$

Furthermore, whenever α is given by the first part of the feedback law, we have

$$\mathcal{W}(D(\alpha)\rho D(-\alpha)) - \mathcal{W}(\rho) \leq -2\epsilon c_1 \left| \text{Tr}\left([\bar{\rho}, a^\dagger - a]\rho\right) \right|^2, \quad (14)$$

where we have applied (5) together with the fact that

$$|\text{Tr}(D(\alpha)\rho D(-\alpha)\bar{\rho})| + |\text{Tr}(\rho\bar{\rho})| \geq 2\epsilon$$

since ρ is inside the set $\{\text{Tr}(\rho\bar{\rho}) > \epsilon\}$. Applying (2), (12), (13) and (14) for the paths never leaving the set $\{\text{Tr}(\rho\bar{\rho}) > \epsilon\}$, we have

$$\begin{aligned}\mathbb{E}(\mathcal{W}(\rho_{k+1}) | \rho_k) - \mathcal{W}(\rho_k) &\leq -2\epsilon c_1 \left| \text{Tr}\left([\bar{\rho}, a^\dagger - a]\rho_{k+\frac{1}{2}}\right) \right|^2 \\ &- \left(\frac{|\cos\left(\frac{\phi_R + \phi}{2} + \bar{n}\phi\right)|^4}{\text{Tr}(M_g \rho_k M_g^\dagger)} + \frac{|\sin\left(\frac{\phi_R + \phi}{2} + \bar{n}\phi\right)|^4}{\text{Tr}(M_e \rho_k M_e^\dagger)} - 1 \right) \text{Tr}(\rho_k \bar{\rho})^2.\end{aligned}$$

Noting that $\text{Tr}(M_g \rho_k M_g^\dagger) \geq 0$, $\text{Tr}(M_e \rho_k M_e^\dagger) \geq 0$, $\text{Tr}(M_g \rho_k M_g^\dagger) + \text{Tr}(M_e \rho_k M_e^\dagger) = 1$ and by Cauchy-Schwartz inequality, we have

$$\begin{aligned}\frac{|\cos\left(\frac{\phi_R + \phi}{2} + \bar{n}\phi\right)|^4}{\text{Tr}(M_g \rho_k M_g^\dagger)} + \frac{|\sin\left(\frac{\phi_R + \phi}{2} + \bar{n}\phi\right)|^4}{\text{Tr}(M_e \rho_k M_e^\dagger)} &= \\ \left(\frac{|\cos\left(\frac{\phi_R + \phi}{2} + \bar{n}\phi\right)|^4}{\text{Tr}(M_g \rho_k M_g^\dagger)} + \frac{|\sin\left(\frac{\phi_R + \phi}{2} + \bar{n}\phi\right)|^4}{\text{Tr}(M_e \rho_k M_e^\dagger)} \right) & \\ (\text{Tr}(M_g \rho_k M_g^\dagger) + \text{Tr}(M_e \rho_k M_e^\dagger)) \geq & \\ \left(\cos^2\left(\frac{\phi_R + \phi}{2} + \bar{n}\phi\right) + \sin^2\left(\frac{\phi_R + \phi}{2} + \bar{n}\phi\right) \right)^2 &= 1,\end{aligned}$$

with equality if and only if $\text{Tr}(M_g \rho_k M_g^\dagger) = \cos^2\left(\frac{\phi_R + \phi}{2} + \bar{n}\phi\right)$. We apply, now, the Kushner's invariance Theorem (cf. Appendix, Theorem 5) to the Markov process ρ_k with the Lyapunov function $\mathcal{W}(\rho_k)$. The process ρ_k converges in probability to the largest invariant set included in

$$\begin{aligned}\left\{ \rho \mid \text{Tr}(M_g \rho M_g^\dagger) = \cos^2\left(\frac{\phi_R + \phi}{2} + \bar{n}\phi\right) \right\} \\ \cap \left\{ \rho \mid \text{Tr}([\bar{\rho}, a^\dagger - a]M_s \rho M_s^\dagger) = 0, s = g, e \right\}.\end{aligned}$$

In particular, by invariance, ρ belonging to this limit set implies $\text{Tr}(M_g \rho M_g^\dagger) = \frac{\text{Tr}(M_g M_s \rho M_s^\dagger M_g^\dagger)}{\text{Tr}(M_s \rho M_s^\dagger)}$ for $s = g, e$.

Taking $s = g$, and noting that $M_g = M_g^\dagger$, this leads to $\text{Tr}(M_g^4 \rho) = \text{Tr}(M_g^2 \rho)^2$. However, by Cauchy-Schwartz inequality, and applying the fact that ρ is a positive matrix, we have $\text{Tr}(M_g^4 \rho) = \text{Tr}(M_g^4 \rho) \text{Tr}(\rho) \geq \text{Tr}(M_g^2 \rho)^2$, with equality if and only if $M_g^4 \rho$ and ρ are co-linear. Since M_g^4 has a non degenerate spectrum, ρ is necessarily a projector over one of the eigen-state of M_g^4 , i.e., a Fock state $|n\rangle$, for some $n \in \{0, \dots, n^{\max}\}$. Finally, as we have restricted ourselves to the paths never leaving the set $\{\rho \mid \text{Tr}(\rho\bar{\rho}) > \epsilon\}$, the only possibility for the invariant set is the isolated point $\{\bar{\rho}\}$. \square

Lemma 4: ρ_k converges to $\bar{\rho}$ for almost all paths remaining in the set $\{\text{Tr}(\rho\bar{\rho}) > \epsilon\}$.

Proof of Lemma 4: Define the event $P_{>\epsilon} = \{\omega \in \Omega \mid \rho_k \text{ never leaves the set } \{\text{Tr}(\rho\bar{\rho}) > \epsilon\}\}$. Through Lemma 3, we have shown that $\lim_{k \rightarrow \infty} \mathbb{P}(\|\rho_k - \bar{\rho}\| > \delta \mid P_{>\epsilon}) = 0, \forall \delta > 0$. By continuity of $V(\rho) = 1 - \text{Tr}(\rho\bar{\rho})$, this also implies that $\lim_{k \rightarrow \infty} \mathbb{P}(V(\rho_k) > \delta \mid P_{>\epsilon}) = 0, \forall \delta > 0$. As $V(\rho) \leq 1$, we have

$$\begin{aligned}\mathbb{E}(V(\rho_k) \mid P_{>\epsilon}) &\leq \mathbb{P}(V(\rho_k) > \delta \mid P_{>\epsilon}) \\ &+ \delta(1 - \mathbb{P}(V(\rho_k) > \delta \mid P_{>\epsilon})).\end{aligned}$$

Thus $\limsup_{k \rightarrow \infty} \mathbb{E}(V(\rho_k) \mid P_{>\epsilon}) \leq \delta, \forall \delta > 0$, and so $\lim_{k \rightarrow \infty} \mathbb{E}(V(\rho_k) \mid P_{>\epsilon}) = 0$. By Theorem 4, we know that $V(\rho_k)$ converges almost surely and therefore, as V is bounded, by dominated convergence, we obtain $\mathbb{E}(\lim_{k \rightarrow \infty} V(\rho_k) \mid P_{>\epsilon}) = 0. \square$

Now, we have all the elements to finish the proof of the Theorem 1. From Steps 1 and 2 and the Markov property, one deduces that for almost all paths ρ_k , there exists a \bar{K} such that ρ_k for $k \geq \bar{K}$ never leaves the set $\{\text{Tr}(\rho\bar{\rho}) > \epsilon\}$. This together with the step 3 finishes the proof of the Theorem. \square

IV. QUANTUM FILTERING FOR STATE ESTIMATION

The feedback law (7) requires the knowledge of $\rho_{k+\frac{1}{2}}$. When the measurement process is fully efficient and the jump model (2) admits no error, it actually represents a natural choice for quantum filter to estimate the value of ρ by ρ^{est} satisfying

$$\begin{aligned}\rho_{k+1}^{\text{est}} &= D(\alpha_k) \rho_{k+\frac{1}{2}}^{\text{est}} D(-\alpha_k) \\ \rho_{k+\frac{1}{2}}^{\text{est}} &= \frac{M_{s_k} \rho_k^{\text{est}} M_{s_k}^\dagger}{\text{Tr}(M_{s_k} \rho_k^{\text{est}} M_{s_k}^\dagger)}.\end{aligned}\quad (15)$$

where $s_k = g$ or e , depending on measure outcome k and on the control α_k .

Before passing to the parametric robustness of the feedback scheme, let us discuss the robustness with respect to the choice of the initial state for the filter equation when we replace $\rho_{k+\frac{1}{2}}$ by $\rho_{k+\frac{1}{2}}^{\text{est}}$ in the feedback (7). Note that, Theorem 1 shows that whenever the filter equation is initialized at the same state as the one which the physical system is prepared initially, the feedback law ensures the stabilization of the target state. The next theorem shows that as soon as the quantum filter is initialized at any arbitrary fully mixed initial state (not necessarily the same as the initial

state of the physical system (2)) and whenever the feedback scheme (7) is applied on the system, the state of the physical system will converge almost surely to the desired Fock state.

Theorem 2: Assume that the quantum filter (15) is initialized at a full-rank matrix ρ_0^{est} and that the feedback scheme (7) is applied to the physical system. The trajectories of the system (2), will then converge almost surely to the target Fock state $\bar{\rho}$.

Proof of Theorem 2: The initial state ρ_0^{est} being full-rank, there exists a $\gamma > 0$ such that $\rho_0^{\text{est}} = \gamma\rho_0 + (1 - \gamma)\rho_0^c$, where ρ_0 is the initial state of (2) at which the physical system is initially prepared and ρ_0^c is a well-defined density matrix. Indeed, ρ^{est} being positive and full-rank, for a small enough γ , $(\rho_0^{\text{est}} - \gamma\rho_0)/(1 - \gamma)$ remains non-negative, Hermitian and of unit trace.

Assume that, we prepare the initial state of another identical physical system as follows: we generate a random number r in the interval $(0, 1)$; if $r < \gamma$ we prepare the system in the state ρ_0 and otherwise we prepare it at ρ_0^c . Applying our quantum filter (15) (initialized at ρ_0^{est}) and the associated feedback scheme, almost all trajectories of this physical system converge to the Fock state $\bar{\rho}$. In particular, almost all trajectories that were initialized at the state ρ_0 converge to $\bar{\rho}$. This finishes the proof of the theorem. \square

The quantum filter (15) admits also some contraction properties confirming its robustness to experimental errors as shown by simulations of figures 3 and 4 where detection errors are introduced. We just provide here a first interesting inequality that will be used in future developments.

Theorem 3: Consider the process (2) and the associated filter (15) for any arbitrary control input $(\alpha_k)_{k=1}^{\infty}$. We have $\mathbb{E}(\text{Tr}(\rho_k \rho_k^{\text{est}})) \leq \mathbb{E}(\text{Tr}(\rho_{k+1} \rho_{k+1}^{\text{est}}))$, $\forall k$.

Proof Before anything, note that the coherent part of the evolution leaves the value of $\text{Tr}(\rho_k \rho_k^{\text{est}})$ unchanged:

$$\begin{aligned} \text{Tr}(\rho_{k+1} \rho_{k+1}^{\text{est}}) &= \text{Tr}\left(D(\alpha_k) \rho_{k+\frac{1}{2}} \rho_{k+\frac{1}{2}}^{\text{est}} D(-\alpha_k)\right) \\ &= \text{Tr}\left(\rho_{k+\frac{1}{2}} \rho_{k+\frac{1}{2}}^{\text{est}}\right). \end{aligned}$$

Concerning the projective part of the dynamics, we have

$$\begin{aligned} \mathbb{E}\left(\text{Tr}\left(\rho_{k+\frac{1}{2}} \rho_{k+\frac{1}{2}}^{\text{est}}\right) \mid \rho_k, \rho_k^{\text{est}}\right) &= \\ &= \sum_{s=g,e} \frac{\text{Tr}(M_s \rho_k M_s^\dagger M_s \rho_k^{\text{est}} M_s^\dagger)}{\text{Tr}(M_s \rho_k^{\text{est}} M_s^\dagger)}. \end{aligned} \quad (16)$$

Applying a Cauchy-Schwarz inequality as well as the identity $M_g^\dagger M_g + M_e^\dagger M_e = 1$, we have

$$\begin{aligned} \sum_{s=g,e} \frac{\text{Tr}(M_s^\dagger M_s \rho_k M_s^\dagger M_s \rho_k^{\text{est}})}{\text{Tr}(M_s \rho_k^{\text{est}} M_s^\dagger)} &= \\ \sum_{s=g,e} \text{Tr}(M_s \rho_k^{\text{est}} M_s^\dagger) \sum_{s=g,e} \frac{\text{Tr}(M_s^\dagger M_s \rho_k M_s^\dagger M_s \rho_k^{\text{est}})}{\text{Tr}(M_s \rho_k^{\text{est}} M_s^\dagger)} &\geq \\ \left(\sum_{s=g,e} \sqrt{\text{Tr}(M_s^\dagger M_s \rho_k M_s^\dagger M_s \rho_k^{\text{est}})}\right)^2 & \quad (17) \end{aligned}$$

Applying (16) and (17), we only need to show that

$$\left(\sum_{s=g,e} \sqrt{\text{Tr}(M_s^\dagger M_s \rho_k M_s^\dagger M_s \rho_k^{\text{est}})}\right)^2 \geq \text{Tr}(\rho_k \rho_k^{\text{est}}). \quad (18)$$

Noting, once again, that $M_g^\dagger M_g + M_e^\dagger M_e = 1$, we can write:

$$\text{Tr}(\rho_k \rho_k^{\text{est}}) = \sum_{s=g,e} \sum_{r=g,e} \text{Tr}(M_s^\dagger M_s \rho_k M_r^\dagger M_r \rho_k^{\text{est}}), \quad (19)$$

and therefore (18) is equivalent to

$$\begin{aligned} \sum_{s=g,e} \sqrt{\text{Tr}(M_s \rho_k M_s^\dagger M_s \rho_k^{\text{est}} M_s^\dagger)} \sum_{r=g,e} \sqrt{\text{Tr}(M_r \rho_k M_r^\dagger M_r \rho_k^{\text{est}} M_r^\dagger)} &\geq \\ \sum_{s=g,e} \sum_{r=g,e} \text{Tr}(M_s^\dagger M_s \rho_k M_r^\dagger M_r \rho_k^{\text{est}}). & \quad (20) \end{aligned}$$

Note that as ρ_k and ρ_k^{est} are positive Hermitian matrices, their square roots, $\sqrt{\rho_k}$ and $\sqrt{\rho_k^{\text{est}}}$, are well-defined. Once again by Cauchy-Schwarz inequality, we have

$$\begin{aligned} \text{Tr}(M_s^\dagger M_s \rho_k M_r^\dagger M_r \rho_k^{\text{est}}) &= \text{Tr}\left(\sqrt{\rho_k^{\text{est}}} M_s^\dagger M_s \sqrt{\rho_k} \sqrt{\rho_k} M_r^\dagger M_r \sqrt{\rho_k^{\text{est}}}\right) \\ &\leq \sqrt{\text{Tr}\left(\sqrt{\rho_k^{\text{est}}} M_s^\dagger M_s \sqrt{\rho_k} \sqrt{\rho_k} M_s^\dagger M_s \sqrt{\rho_k^{\text{est}}}\right)} \\ &\quad \sqrt{\text{Tr}\left(\sqrt{\rho_k^{\text{est}}} M_r^\dagger M_r \sqrt{\rho_k} \sqrt{\rho_k} M_r^\dagger M_r \sqrt{\rho_k^{\text{est}}}\right)} \\ &= \sqrt{\text{Tr}\left(M_s \rho_k M_s^\dagger M_s \rho_k^{\text{est}} M_s^\dagger\right)} \sqrt{\text{Tr}\left(M_r \rho_k M_r^\dagger M_r \rho_k^{\text{est}} M_r^\dagger\right)}. \end{aligned}$$

Summing over $s, r \in \{g, e\}$, we obtain the inequality (20) and therefore we finish the proof of the Theorem 3. \square

V. MONTE-CARLO SIMULATIONS

Figure (2) corresponds to a closed-loop simulation with a goal Fock state $\bar{n} = 3$ and a Hilbert space limited to $n^{\text{max}} = 15$ photons. ρ_0 and ρ_0^{est} are initialized at the same state, the coherent state $\exp(\sqrt{\bar{n}}(a^\dagger - a))|0\rangle$ of mean photon number \bar{n} . The number of iteration steps is fixed to 100. The dephasing per photon is $\phi = \frac{3}{10}$. The Ramsey phase ϕ_R is fixed to the mid-fringe setting, i.e. $\frac{\phi_R + \phi}{2} + \bar{n}\phi = \frac{\pi}{4}$. The feedback parameter ((7) with $\rho_{k+\frac{1}{2}}^{\text{est}}$ instead of $\rho_{k+\frac{1}{2}}$) are as follows: $c_1 = \frac{1}{4\bar{n}+1}$, $\epsilon = \frac{1}{10}$ and $\bar{\alpha} = \frac{1}{10}$.

Any real experimental setup includes imperfection and error. To test the robustness of the feedback scheme, a false detection probability $\eta_f = \frac{1}{10}$ is introduced. In case of false detection at step k , the atom is detected in g (resp. e) whereas it collapses effectively in e (resp. g). This means that in (15), $s_k = g$ (resp. $s_k = e$), whereas in (2), it is the converse $M_k = M_e$ (resp. $M_k = M_g$). Simulations of figure 3 differ from those of figure 2 by only $\eta_f = \frac{1}{10}$: we observe for this sample trajectory a longer convergence time. A much more significative impact of η_f is given by ensemble average. Figure 4 presents ensemble averages corresponding to the third sub-plot of figures 2 and 3. For $\eta_f = 0$ (left plot), we observe an average fidelity $\text{Tr}(\rho_k \bar{\rho})$ converging to 100%: it exceeds 90% after $k = 40$ steps. For $\eta_f = 1/10$, the asymptotic fidelity remains under 80% and reaches 70% after 30 iteration. The performance are reduced but not changed dramatically. The proposed feedback scheme appears to be robust to such experimental errors.

VI. CONCLUSION

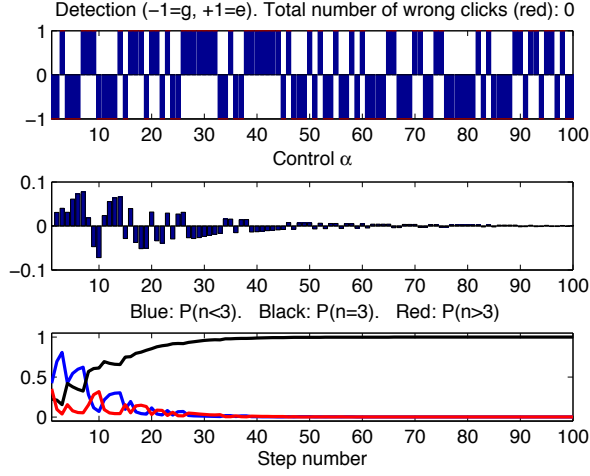


Fig. 2. A single closed-loop quantum trajectory in the ideal case ($\bar{n} = 3$).

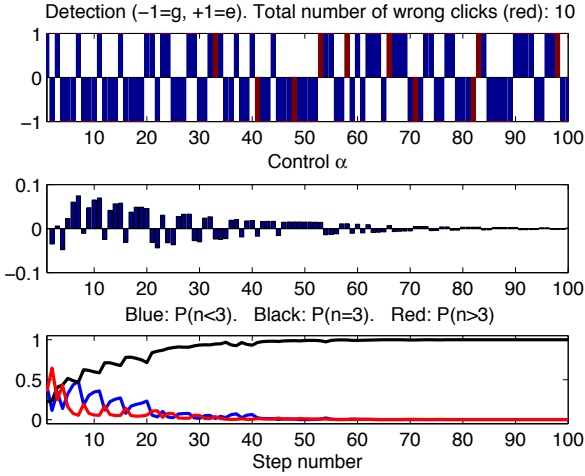


Fig. 3. A single closed-loop quantum trajectory with a false detection probability of $1/10$.

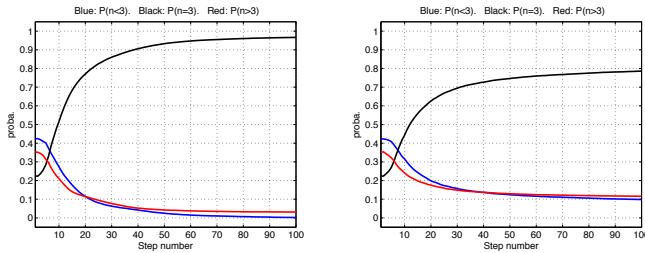


Fig. 4. Averages of 10^4 closed-loop quantum trajectories similar to the one of figure 2 (left, $\eta_f = 0$) and 3 (right, $\eta_f = \frac{1}{10}$).

In [2] more realistic simulations are reported. They include nonlinear shift per photon ($N\phi$ replaced by a non linear function $\Phi(N)$ in (1)) and additional experimental errors such as detector efficiency and delays. These simulations confirm the robustness of the feedback scheme, robustness that needs to be understood in a more theoretical way. In particular, it seems that the quantum filter (15) forgets its initial condition ρ_0^{est} almost surely and thus admits some strong contraction properties as indicated by Theorem 3.

With the truncation to n^{max} photons, convergence is proved only in the finite dimensional case. But feedback (7) and quantum filter (15) are still valid for $n^{\text{max}} = +\infty$. We conjecture that Theorems 1 and 2 remain valid in this case.

In the experimental results reported in [10], [5], [4] the time-interval corresponding to a sampling step is around $100\mu\text{s}$. Thus it is possible to implement, on a digital computer and in real-time, the Lyapunov feedback-law (7) where ρ is given by the quantum filter (15).

VII. APPENDIX: STABILITY THEORY FOR STOCHASTIC PROCESSES

We recall here Doob's inequality and Kushner's invariance theorem. For detailed discussions and proofs we refer to [8] (Sections 8.4 and 8.5).

Theorem 4 (Doob's Inequality): Let $\{X_n\}$ be a Markov chain on state space S . Suppose that there is a non-negative function $V(x)$ satisfying $\mathbb{E}(V(X_1) | X_0 = x) - V(x) = -k(x)$, where $k(x) \geq 0$ on the set $\{s : V(x) < \lambda\} \equiv Q_\lambda$. Then $\mathbb{P}\left(\sup_{\infty > n \geq 0} V(X_n) \geq \lambda | X_0 = x\right) \leq \frac{V(x)}{\lambda}$. Furthermore, there is some random $v \geq 0$, so that for paths never leaving Q_λ , $V(X_n) \rightarrow v \geq 0$ almost surely.

For the statement of the second Theorem, we need to use the language of probability measures rather than the random process. Therefore, we deal with the space \mathcal{M} of probability measures on the state space S . Let $\mu_0 = \varphi$ be the initial probability distribution (everywhere through this paper we have dealt with the case where μ_0 is a dirac on a state ρ_0 of the state space of density matrices). Then, the probability distribution of X_n , given initial distribution φ , is to be denoted by $\mu_n(\varphi)$. Note that for $m \geq 0$, the Markov property implies: $\mu_{n+m}(\varphi) = \mu_n(\mu_m(\varphi))$.

Theorem 5 (Kushner's invariance Theorem): Consider the same assumptions as that of the Theorem 4. Let $\mu_0 = \varphi$ be concentrated on a state $x_0 \in Q_\lambda$ (Q_λ being defined as in Theorem 4), i.e. $\varphi(x_0) = 1$. Assume that $0 \leq k(X_n) \rightarrow 0$ in Q_λ implies that $X_n \rightarrow \{x | k(x) = 0\} \cap Q_\lambda \equiv K_\lambda$. Under the conditions of Theorem 4, for trajectories never leaving Q_λ , X_n converges to K_λ almost surely. Also, the associated conditioned probability measures $\tilde{\mu}_n$ tend to the largest invariant set of measures M whose support set is in K_λ . Finally, for the trajectories never leaving Q_λ , X_n converges, in probability, to the support set of M .

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